Hall effect in moderately clean superconductors and the transverse force on a moving vortex

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For moderately clean superconductors with the mean free path approaching the coherence length from the clean side, the transverse force on a moving vortex starts to be dominated by variations in the pairing interaction produced by vortex motion. We calculate the force and the Hall conductivity using the microscopic theory of nonstationary superconductivity and find that they are modified, as compared to the effective-action result, by an electric potential induced due to charge neutrality. [S0163-1829(96)02634-3]

I. INTRODUCTION

The flux-flow Hall effect is studied theoretically both for dirty superconductors[1–3] with the mean free path l much shorter than the coherence length $\xi(T)$ and for clean superconductors with $l \gg \xi(T)$.[4–6] The Hall effect is small in the dirty case but it is expected to be quite substantial in clean superconductors.[7] The Ohmic and Hall conductivities of a clean superconductor in the low-field region $H \ll H_{c2}$ were obtained in Ref. 6. For temperatures not very close to T_c, they are, by the order of magnitude,

$$\sigma_O \sim \frac{Nec}{B}\frac{\omega_0\tau}{1+(\omega_0\tau)^2}, \quad \sigma_H \sim \frac{Nec}{B}\frac{(\omega_0\tau)^2}{1+(\omega_0\tau)^2}, \qquad (1)$$

where N is the density of carriers, $\omega_0 \sim \Delta^2/E_F$ is the characteristic distance between the energy levels with different angular momenta of the localized states within the vortex core, and τ is the mean free time, $l = v_F\tau$. (See Ref. 6 for more details.)

A rough estimate for dirty superconductors can be obtained from Eq. (1) with $\tau^{-1} \sim \Delta$: it gives $\sigma_H/\sigma_O \sim \Delta/E_F \ll 1$ with the Hall angle $\tan\Theta_H \sim 10^{-2}$, which is of the correct order of magnitude compared to the experimental data. Many mechanisms contribute to this general order-of-magnitude estimate. Microscopic theory[3] and the TDGL calculations[1,2] show that one of the most important effects is the feedback of vortex motion to the pairing interaction. Being proportional to $1/\lambda$, where λ is the BCS coupling constant, it can dominate for dirty superconductors in a weak-coupling limit $\lambda \ll 1$: the Hall conductivity is then $\sigma_H \sim \sigma_O(\Delta/E_F)(1/\lambda)$.

The flux-flow conductivities are coupled to the components of force from the heat bath, $\mathbf{F} = \mathbf{F}_\parallel + \mathbf{F}_\perp$, experienced by a vortex moving with the velocity $\mathbf{u}$:

$$\mathbf{F}_\parallel = -\frac{\phi_0 B\sigma_O}{c^2}\mathbf{u},$$

$$\mathbf{F}_\perp = -\frac{\phi_0 B\sigma_H}{c^2}[\mathbf{u}\times\hat{\mathbf{z}}]\mathrm{sgn}(e). \qquad (2)$$

Here $\hat{\mathbf{z}}$ is the unit vector along the vortex axis in the positive direction of its circulation and $\phi_0 = \pi c/|e|$ is the magnetic flux quantum. Note that the unit vector along the magnetic field $\hat{\mathbf{b}} = \hat{\mathbf{z}}\,\mathrm{sgn}(e)$ coincides with $\hat{\mathbf{z}}$ for positive charge of carriers. For further use, we write the total force as $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}'$ where the force $\mathbf{F}_0$ contains the conductivities as calculated in Ref. 6 while the additional force $\mathbf{F}'$ accounts for the effect of vortex motion on pairing interaction. The additional friction force $\mathbf{F}_\parallel$ results in a negligible correction to the dissipative Ohmic conductivity. The transverse component $\mathbf{F}_\perp$ is larger and affects a small Hall conductivity, thus its role is much more important.

In clean superconductors with $l \gg \xi(T)$, the change in pairing interaction gives only a small correction to the Hall conductivity of Eq. (1) and to the transverse force; it may become substantial, however, for a weak-coupling limit $\lambda \ll 1$ in a moderately clean case when l approaches $\xi(T)$.[8,9] In the present paper we consider this mechanism for clean superconductors and calculate the flux-flow Hall conductivity using the nonstationary microscopic theory. We assume that pinning is absent and concentrate on intrinsic mechanisms of vortex motion.

We restrict ourselves to an s-wave superconductor and assume a uniaxial symmetry of the crystal. The magnetic field is applied along the symmetry axis. We work in a low-field limit, $B \ll H_{c2}$ assuming an extreme type-II superconductor with $\kappa \gg 1$. We find that the additional transverse force is

$$\mathbf{F}'_\perp = -\pi\delta N_\Delta(\infty)[\mathbf{u}\times\hat{\mathbf{z}}]\beta(T), \qquad (3)$$

where δN_Δ is the difference between the total particle density of the system in the superconducting and in the normal state without Coulomb interaction; $\delta N_\Delta(\infty)$ denotes this difference at large distances from the vortex. The difference in densities is caused by a change in the electronic spectrum after transition into the superfluid state:

$$\delta N_\Delta = \left(\frac{1}{\lambda}\frac{d\nu}{d\zeta_p}\right)|\Delta|^2. \qquad (4)$$

The function $\beta(T)$ in Eq. (3) varies from $\beta = 1$ at $T \to T_c$ to $\beta \gg 1$ for $T \ll T_c$. The change in density is small, $\delta N_\Delta/N \sim (\Delta/E_F)^2$, however, the corresponding $1/\lambda$ contribution to the Hall conductivity

$$\sigma'_H = \beta(T)\,\delta N_\Delta(\infty)\,ec/B$$

becomes comparable to the Hall conductivity of Eq. (1) when the superconductor is still in a moderately clean regime with $l/\xi(T) \sim 1/\sqrt{\lambda}$ and dominates if l is further decreased. Relative magnitude of the correction increases both for low temperatures and for $T \to T_c$. Our calculations thus fill the gap between very dirty $l \ll \xi(T)$ and very clean $l \gg \xi(T)$ superconductors.

Our result differs from Ref. 9 which predicts $\beta = 1$ for any temperature. This very general prediction was obtained in Ref. 9 using an effective hydrodynamic action in the presence of vortices. A similar approach was used in Ref. 10 to derive the Magnus force on a moving vortex, which is only a part of the total transverse force of Eq. (2) (see Refs. 11 and 12). The prediction $\beta = 1$, however, does not comply with the microscopic calculations for dirty superconductors[3] and with the TDGL theory which give expressions of the type of Eq. (3) with a factor $\beta \neq 1$ depending on the specific relaxation processes in the superconductor. The present calculations identify the source of disagreement between the TDGL theory and the hydrodynamic approach: It consists in an additional contribution to Eq. (3) from the electric potential induced by moving vortices due to charge neutrality in superconductors. We show that the result of Ref. 9 is valid only in the limit of $T \to T_c$; it can be reproduced by the TDGL theory provided the pair-breaking rate is small.

II. FORCES ON A MOVING VORTEX

In the microscopic theory, the transport current in an array of moving vortices is determined by a balance of the Lorentz force from the transport current, $\mathbf{F}_L = \phi_0[\mathbf{j}_{\rm tr} \times \hat{\mathbf{z}}]/c$, and the force from the heat bath:[3,13]

$$\mathbf{F}_L - \int d^2r \int \frac{d\epsilon}{4\pi i}\,\frac{d^3p}{(2\pi)^3}\,\mathrm{Tr}[\mathcal{G}^{(\rm nst)}(\hat{\nabla}\mathcal{H} + e\nabla\varphi_0)] = 0. \tag{5}$$

The spatial integration is taken over the unit cell of the vortex lattice. The Green function

$$\mathcal{G}_{\epsilon_+,\epsilon_-}(\mathbf{p},\mathbf{r}) = \begin{pmatrix} G & F \\ -F^\dagger & \bar{G} \end{pmatrix}$$

and the ''effective force''

$$\hat{\nabla}\mathcal{H} = \begin{pmatrix} (e/c)[\mathbf{H}\times\mathbf{v}_F] & -\hat{\nabla}\Delta \\ \hat{\nabla}\Delta^* & -(e/c)[\mathbf{H}\times\mathbf{v}_F] \end{pmatrix}$$

are matrices in the Nambu space; $\hat{\nabla} = [\nabla \mp (2ie/c)\mathbf{A}]$ for Δ and Δ^*, respectively, and φ_0 is a small static potential which appears due to charge neutrality (see Sec. III). $\mathcal{G}^{\rm nst}$ is the nonstationary part of the total Green function[14]

$$\mathcal{G}^{(\rm nst)}_{\epsilon_+,\epsilon_-} = \mathcal{G}^{(\rm tot)}_{\epsilon_+,\epsilon_-} - (\mathcal{G}^R_\epsilon - \mathcal{G}^A_\epsilon) f^{(0)}(\epsilon)\, 2\pi\delta(\omega).$$

Here $\mathcal{G}^{R(A)}$ are the regular Green functions, and $\epsilon_\pm = \epsilon \pm \omega/2$. The equilibrium distribution function is $f^{(0)}(\epsilon) = \tanh(\epsilon/2T)$.

The nonstationary Green function contains the part which can be calculated quasiclassically by integrating over the energy variable $\zeta_p = \epsilon(\mathbf{p}) - E_F$ plus small corrections proportional to Δ/E_F. Among all numerous corrections of this order of magnitude, there is one which gives $1/\lambda$ after integration over frequencies. In a weak-coupling limit, $\lambda \ll 1$, this is the largest contribution; it can be calculated separately neglecting all other nonquasiclassical corrections.

The $1/\lambda$ correction to the conductivity comes from the regular Green functions $\mathcal{G}^{R(A)}_{\epsilon_+,\epsilon_-}$ due to the logarithmic divergence of the ϵ integration in Eq. (5) which is cut off at the limiting BCS frequency. Since the characteristic frequencies are large, one can expand the Green function in powers of the small ratio Δ/ϵ up to the leading term. For example, the correction to $F^{R(A)}$ proportional to ω is

$$F^{R(A)}_{\rm corr} = -\frac{\hat{\omega}\Delta_\omega}{2}\,\frac{\partial}{\partial\zeta_p}[G^{R(A)}_\epsilon \overline{G}^{R(A)}_\epsilon], \tag{6}$$

$F^{\dagger R(A)}_{\rm corr}$ is obtained by substituting $\hat{\omega}\Delta_\omega$ with $-\hat{\omega}\Delta^*_\omega$. We use the gauge invariant version of $\mathcal{G}$ introduced in Ref. 15. Here $\hat{\omega} = \omega \mp 2e\varphi$ with the upper sign for Δ and F, and the lower sign for Δ^* and $F^\dagger$. The potential φ does not include the static part φ_0 (see Sec. IV A). For the diagonal elements G and $\bar{G}$, the operator is $\hat{\omega} = \omega$.

The quasiclassical part of $\mathcal{G}^{(\rm nst)}$ is[13]

$$\begin{aligned} \hat{g}^{(\rm nst)} = {} & [(\hat{g}^R_{\epsilon_+,\epsilon_-} - \hat{g}^A_{\epsilon_+,\epsilon_-}) - (\hat{g}^R_\epsilon - \hat{g}^A_\epsilon)2\pi\delta(\omega)]f^{(0)} \\ & - \frac{\hat{\omega}}{2}(\hat{g}^R_{\epsilon_+,\epsilon_-} + \hat{g}^A_{\epsilon_+,\epsilon_-})\frac{\partial f^{(0)}}{\partial\epsilon} + (\hat{g}^R - \hat{g}^A)f_1 \\ & + (\hat{g}^R\hat{\sigma}_z - \hat{\sigma}_z\hat{g}^A)f_2. \end{aligned} \tag{7}$$

Here the matrix $\hat{g} = \int \mathcal{G}\,d\zeta_p/\pi i$ is constructed out of the functions g, f, etc., according to the same rule as the matrix $\mathcal{G}$ is made of G, F, etc., and f_1, f_2 are the nonequilibrium corrections to the distribution function.

The force balance of Eq. (5) becomes $\mathbf{F}_L + \mathbf{F} = 0$ where the total force can be separated into quasiclassical and nonquasiclassical parts: $\mathbf{F} = \mathbf{F}^{(\rm qc)} + \mathbf{F}^{(\rm nqc)}$. The quasiclassical force

$$\mathbf{F}^{(\rm qc)} = -\int d^2r \int \frac{d\epsilon}{4}\,\frac{dS_F}{(2\pi)^3 v_F}\,\mathrm{Tr}[\hat{g}^{(\rm nst)}(\hat{\nabla}\mathcal{H} + e\nabla\varphi_0)] \tag{8}$$

is calculated in Sec. IV A. We demonstrate that $\mathbf{F}^{(\rm qc)} = \mathbf{F}^{(0)} + \mathbf{F}'^{(\rm qc)}$ with the additional term

$$\mathbf{F}'^{(\rm qc)} = -\pi\,\delta N_\Delta(\infty)[\mathbf{u}\times\hat{\mathbf{z}}]\beta_1. \tag{9}$$

The nonquasiclassical force comes from Eq. (6):

$$\mathbf{F}^{(\rm nqc)} = \frac{i}{2}\left(\frac{1}{\lambda}\,\frac{d\nu}{d\zeta}\right)\int d^2r\left[\hat{\nabla}\Delta\,\frac{\hat{\partial}\Delta^*}{\partial t} - \hat{\nabla}\Delta^*\,\frac{\hat{\partial}\Delta}{\partial t}\right]. \tag{10}$$

Here $\hat{\partial}/\partial t = \partial/\partial t \pm 2ie\varphi$ with the upper sign for Δ and the lower sign for Δ^*. The density of states as a function of energy is $\nu(\zeta) = \int dS_\zeta/(2\pi)^3 v_\zeta$, where $v_\zeta = |\partial\epsilon(\mathbf{p})/\partial\mathbf{p}|$, and the integration is over the surface corresponding to the energy $\epsilon_p - E_F = \zeta$. Equation (10) is independent of impurity scattering. It is only the potential φ which depends on the real kinetics of a superconductor. We calculate it in Sec. IV B.

The force $\mathbf{F}^{(\mathrm{nqc})}$ is perpendicular to the vortex velocity $\mathbf{u}$. Indeed, the scalar potential is proportional to $\mathbf{u}$; we shall see that $\varphi \propto u_\phi$, where (ρ,ϕ,z) are the coordinates in the cylindrical frame. We put $e\varphi = u_\phi \psi$ and $\partial/\partial t = -\mathbf{u}\cdot\nabla$ for a moving vortex and calculate the integral in Eq. (10). Neglecting the vector potential we obtain with help of Eq. (4)

$$\mathbf{F}^{(\mathrm{nqc})} = -\pi \delta N_\Delta(\infty)[\mathbf{u}\times\hat{\mathbf{z}}]\beta_2, \tag{11}$$

where

$$\beta_2 = 1 - |\Delta_\infty|^{-2}\int_0^\infty \psi \frac{d|\Delta|^2}{d\rho}\rho\, d\rho. \tag{12}$$

Thus, the full additional transverse force $\mathbf{F}'_\perp = \mathbf{F}'^{(\mathrm{qc})} + \mathbf{F}^{(\mathrm{nqc})}$ in Eq. (3) has $\beta = \beta_1 + \beta_2$.

III. EFFECTS OF CHARGE NEUTRALITY

Equation (10) shows that the nonquasiclassical transverse force is caused by the time derivative of the order-parameter phase coupled to the electric potential. Since these quantities are conjugated with density the force is related to the density variations. One can interpret the transverse force defined by Eqs. (9–12) in the following way. For a system without Coulomb interaction where $\varphi = \varphi_0 = 0$, the additional force $\mathbf{F}'_\perp$ comes from $\mathbf{F}^{(\mathrm{nqc})}$ only and is equal to the difference between two forces acting on a moving vortex: one is the sum of the Magnus forces from superfluid and normal components and is proportional to the total particle density of the system in the superfluid state N, while the other is the spectral-flow force proportional to the number of states $N_0(\mu)$ within the Fermi surface defined by the chemical potential μ.[11,12] The difference in densities δN_Δ is caused by a change in the electronic spectrum after transition into the superfluid state and is given by Eq. (4). The net force thus has the form of Eq. (3) with $\beta = 1$ in agreement with predictions of Ref. 9. For a Galilean invariant system such as superfluid ^{3}He, δN_Δ corresponds to the difference between N and $C_0 = p_F^3/3\pi^2$.

For metals, the Debye screening results in a charge neutrality $N = N_0(E_F) = N_{\mathrm{ion}}$ where E_F is the chemical potential of the normal state. The electron density is

$$N - N_0 = \delta N_\Delta - 2\nu e\varphi_0 + \delta N^{(\mathrm{nst})}, \tag{13}$$

where

$$\delta N^{(\mathrm{nst})} = -\int \frac{dS_F}{(2\pi)^3 v_F}\frac{d\epsilon}{4}\mathrm{Tr}\hat{g}^{(\mathrm{nst})}. \tag{14}$$

The constraint $N = N_0$ gives rise to a small static potential φ_0 which compensates the variation δN_Δ

$$e\varphi_0 = \delta N_\Delta / 2\nu. \tag{15}$$

However, the difference between the Magnus and spectral-flow forces remains the same. Indeed, the resulting change in chemical potential shifts the Fermi level and changes the function $N_0(E_F)$ to $N_0(E_F - e\varphi_0)$. As a result, the difference $N - N_0(E_F - e\varphi_0)$ remains to be δN_Δ,[16] though the real density of the system is unchanged: $N = N_0(E_F)$.

This consideration shows that the static shift in the chemical potential itself does not modify the additional force. However, the force is modified by kinetic effects associated with the charge neutrality. First, the quasiclassical force $\mathbf{F}'^{(\mathrm{qc})}$ appears because of an extra current induced by the time dependent potential $\varphi_0(\mathbf{r} - \mathbf{u}t)$ for a moving vortex to fulfill the incompressibility constraint $\mathrm{div}\mathbf{j} = 0$. Second, the density could change due to nonequilibrium variations in the distribution function f_2 in Eq. (14). (A similar effect of a moving vortex on density was discussed in Ref. 17.) To compensate for these variations, a potential $\varphi \propto u$ is induced through the condition $\delta N^{(\mathrm{nst})} = 0$. This potential changes the nonquasiclassical force $\mathbf{F}^{(\mathrm{nqc})}$.

IV. MICROSCOPIC CALCULATION OF FORCES

A. Quasiclassical force

The induced time dependence of the potential φ_0 for a moving vortex modifies the distribution function f_1. In addition, it contributes to the regular quasiclassical Green functions. If there were no static potential φ_0, the first line of Eq. (7) would give a contribution to Eq. (5) which is even in ω as it can be checked by inspecting the Eilenberger equations. Its expansion in ω would start with ω^2, thus it could be omitted in the linear-in-ω approximation for which the two last lines in Eq. (7) are only important. This fact has always been used in quasiclassical calculations of the flux-flow conductivity. However, the correction to the regular functions due to φ_0 is linear in ω and has to be included in Eq. (5).

To describe the kinetic effect of φ_0 we need to account for a time dependence of the chemical potential in the kinetic equations. The distribution function $f^{(0)} = \tanh(\epsilon/2T)$ refers to an equilibrium where the order parameter and the vector potential are constant in time, and the chemical potential is $E_F - e\varphi_0(\mathbf{r})$. There is a small electric field $-\nabla\varphi_0$, which prevents electrons from moving to places with higher magnitudes of the order parameter, thus there is no normal current in equilibrium. We disturb our system by putting vortices into motion with a small velocity $\mathbf{u}$. The deviation from equilibrium is described by nonequilibrium corrections f_1 and f_2 to the distribution function, in addition there appears a nonequilibrium electric field

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi,$$

where the potential φ is proportional to the vortex velocity, so that the electric field $\mathbf{E}$ is proportional to $\mathbf{u}$.

Using parity of the static quasiclassical Green functions

$$f_\epsilon^R(\chi,\mathbf{A}) = -f_{-\epsilon}^{R*}(-\chi,-\mathbf{A}), \quad g_\epsilon^R(\chi,\mathbf{A}) = g_{-\epsilon}^{R*}(-\chi,-\mathbf{A}),$$

etc., where χ is the phase of the order parameter, we can easily check that the contributions from the distribution functions to the quasiclassical force have the following parity with respect to the inversion of the vortex circulation $\hat{\mathbf{z}} \to -\hat{\mathbf{z}}$, equivalent to the inversion of the magnetic field plus complex conjugation of the order parameter.

(1) The distribution functions such that

$$f_1(\epsilon,\hat{\mathbf{z}})=-f_1(-\epsilon,-\hat{\mathbf{z}}),\quad f_2(\epsilon,\hat{\mathbf{z}})=-f_2(-\epsilon,-\hat{\mathbf{z}}) \tag{16}$$

give an even in $\hat{\mathbf{z}}\to-\hat{\mathbf{z}}$ contribution to the force, thus they give rise to a friction force or Ohmic conductivity.

(2) The distribution functions such that

$$f_1(\epsilon,\hat{\mathbf{z}})=f_1(-\epsilon,-\hat{\mathbf{z}}),\quad f_2(\epsilon,\hat{\mathbf{z}})=f_2(-\epsilon,-\hat{\mathbf{z}}) \tag{17}$$

give an odd in $\hat{\mathbf{z}}\to-\hat{\mathbf{z}}$ contribution to the force, thus they give rise to a reactive transverse force or Hall conductivity. With these parity considerations, one can easily identify the necessary corrections to distribution functions.

The kinetic equations for clean superconductors were derived in Ref. 15. They can be easily generalized to the case when the equilibrium chemical potential has a time dependence:

$$\left(e\mathbf{v}_F\mathbf{E}g_-+\frac{1}{2}\left[f_-\frac{\hat{\partial}\Delta^*}{\partial t}+f_-^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon}-\left(\frac{1}{2}[(\hat{\boldsymbol{\nabla}}\Delta)f_-^\dagger+(\hat{\nabla}\Delta^*)f_-]-\frac{e}{c}[\mathbf{v}_F\times\mathbf{H}]g_-\right)\frac{\partial f_1}{\partial\mathbf{p}}+g_-\frac{\partial f_1}{\partial t}+\mathbf{v}_F\nabla(f_2g_-)=J^{(1)}, \tag{18}$$

and

$$\frac{1}{2}\left[f_+\frac{\hat{\partial}\Delta^*}{\partial t}-f_+^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\frac{\partial f^{(0)}}{\partial\epsilon}+i[\Delta^*f_++\Delta f_+^\dagger]f_2-\frac{1}{2}[(\hat{\nabla}\Delta^*)f_+-(\hat{\nabla}\Delta)f_+^\dagger]\frac{\partial f_1}{\partial\mathbf{p}}-\left(\frac{1}{2}[\Delta^*\hat{\nabla}f_-+\Delta\hat{\nabla}f_-^\dagger]-\frac{e}{c}[\mathbf{v}_F\times\mathbf{H}]g_-\right)\frac{\partial f_2}{\partial\mathbf{p}}$$
$$+g_-\mathbf{v}_F\nabla f_1+g_-\frac{\partial(e\varphi_0)}{\partial t}\frac{\partial f^{(0)}}{\partial\epsilon}=J^{(2)}. \tag{19}$$

Here $J^{(1)}$ and $J^{(2)}$ are the collision integrals,[15] and $\hat{g}_\pm=(\hat{g}^R\pm\hat{g}^A)/2$. We stress that the operator $\hat{\partial}/\partial t=\partial/\partial t\pm 2ie\varphi$ does not contain the static potential φ_0.

For clean superconductors, the static Green function of an electron with the momentum projection on the vortex axis p_z and the impact parameter b can be written as the spectral sum

$$\hat{g}_-=\sum_n \hat{g}_n\delta[\epsilon-E_n(p_z,b)],$$

where $E_n(p_z,b)$ are the energy levels of the localized electrons in the vortex core characterized by the radial quantum number n, the impact parameter b, and momentum p_z.

We restrict ourselves to a moderately clean case where $l\gg\xi(T)$ but $\omega_0\tau\ll 1$. The main terms in Eq. (19) give

$$g_-\mathbf{v}_F\nabla f_1+g_-\frac{\partial(e\varphi_0)}{\partial t}\frac{\partial f^{(0)}}{\partial\epsilon}=0. \tag{20}$$

From Eq. (20) we have $f_1=f_1'+f_1''$ where f_1' is constant along the trajectory and

$$f_1''=\frac{1}{v_\perp}\frac{\partial f^{(0)}}{\partial\epsilon}\left[(\mathbf{u}\hat{\mathbf{v}}_\perp)[e\varphi_0(\rho)-e\varphi_0(\infty)]\right.$$
$$\left.-([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})b\int_0^s\frac{1}{\rho'}\frac{\partial(e\varphi_0)}{\partial\rho'}ds'\right]. \tag{21}$$

We denote the distance along the trajectory by $s=(\hat{\mathbf{v}}_\perp\mathbf{r})$ where $\hat{\mathbf{v}}_\perp$ is the unit vector along the projection of $\mathbf{v}_F$ on the (x,y) plane perpendicular to the vortex axis, and introduce the impact parameter b. We see that f_1'' has the parity of Eq. (17) and contributes to the transverse reactive force.

For $|\epsilon|>\Delta_\infty$, the boundary conditions[6] $f_1=\pm g_-f_2$ for $s\to\pm\infty$ together with Eq. (18) define the constants

$$g_-f_2'=-\frac{([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})b}{v_\perp}\frac{\partial f^{(0)}}{\partial\epsilon}\int_0^\infty\frac{1}{\rho}\frac{\partial(e\varphi_0)}{\partial\rho}ds \tag{22}$$

and $f_1'=0$. For $|\epsilon|<\Delta_\infty$, the integration constant in Eq. (21) is included in f_1'; it can be found by integrating Eq. (18) along the trajectory.

With the identity

$$\int_{-\infty}^{\infty}\mathrm{Tr}[(\hat{\nabla}\mathcal{H})\hat{g}_-]ds=2\pi[\hat{\mathbf{z}}\times\mathbf{v}_\perp]\sum_n\frac{\partial E_n(b)}{\partial b}\delta[\epsilon-E_n(b)]$$

derived in Ref. 6, we obtain for $|\epsilon|<\Delta_\infty$

$$\pi([\mathbf{v}_\perp\times\mathbf{u}]\hat{\mathbf{z}})\frac{\partial f^{(0)}}{\partial\epsilon}\frac{\partial E_n}{\partial b}+\pi\left(\left[\mathbf{v}_\perp\times\frac{\partial f_1'}{\partial\mathbf{p}_\alpha}\right]\hat{\mathbf{z}}\right)\frac{\partial E_n}{\partial b}$$
$$+\int_{-\infty}^{\infty}J_n\{f_1'\}ds+\int_{-\infty}^{\infty}J_n\{f_1''\}ds=0, \tag{23}$$

where

$$J^{(1)}=\sum_n J_n\delta(\epsilon-E_n).$$

As in Ref. 6, we adopt the τ approximation for the collision integral

$$\int_0^\infty J_n\{f_1'\}ds=-\frac{\pi v_\perp}{2\tau_n}f_1'. \tag{24}$$

For $\omega_0\tau\ll 1$, the distribution function is

$$f_1' = -\frac{\partial f^{(0)}}{\partial \epsilon} p_\perp (([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})\tau_n\omega_n \pm (\mathbf{u}\hat{\mathbf{v}}_\perp)(\tau_n\omega_n)^2) + \frac{\tau_n}{\pi v_\perp}\int_{-\infty}^{\infty} J_n\{f_1''\} ds. \quad (25)$$

Here $\omega_n = p_\perp^{-1}(\partial E_n/\partial b)$ is the distance between the localized levels $E_n(b)$ with different angular momenta $m = p_\perp b$ of an electron in the vortex core; the $\pm$ signs refer to electrons and holes, respectively.

From Eqs. (21) and (25) we see that the correction to the distribution function produced by φ_0 is small compared to the dissipative component of f_1 [the term with $([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})$], but can be of the same order of magnitude as the reactive part with $(\mathbf{u}\hat{\mathbf{v}}_\perp)$ when $\Delta\tau \sim 1/\sqrt{\lambda}$.

Using charge neutrality $\delta N^{(\mathrm{nst})} = 0$ with Eq. (14), the quasiclassical force from Eqs. (7) and (8) becomes $\mathbf{F}^{(\mathrm{qc})} = \mathbf{F}_0 + \mathbf{F}'^{(\mathrm{qc})}$ where the additional quasiclassical force is

$$\mathbf{F}'^{(\mathrm{qc})} = -\sum_n \int db \int \frac{dS_F}{(2\pi)^3 v_F}[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]\frac{\partial E_n}{\partial b}\tau_n \int_{-\infty}^{\infty} J_n\{f_1''\} ds - \int d^2r \int \frac{d\epsilon}{4}\frac{dS_F}{(2\pi)^3 v_F}\mathrm{Tr}[(\hat{\nabla}\mathcal{H})(\hat{g}^{(\mathrm{nst})} - 2\hat{g}_- f_1')]. \quad (26)$$

The quasiclassical force $\mathbf{F}_0$ is calculated with the distribution function f_1 without the φ_0 correction and contains only the first line of Eq. (25). Through Eq. (2), it is expressed in terms of conductivities $\sigma_O^{(0)}$ and $\sigma_H^{(0)}$ obtained in Ref. 6.

B. Nonquasiclassical force

To calculate $\mathbf{F}^{(\mathrm{nqc})}$ we need the dynamic potential φ. It appears due to the distribution function f_2 in zero approximation with respect to φ_0. To the leading approximation in $\omega_0\tau \ll 1$, the kinetic equation (18) becomes

$$\left(e\mathbf{v}_F\mathbf{E}g_- + \frac{1}{2}\left[f_-\frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon} + \mathbf{v}_F\nabla(f_2 g_-) = J^{(1)}. \quad (27)$$

The charge neutrality reads $\delta N^{(\mathrm{nst})} = 0$ where

$$\delta N^{(\mathrm{nst})} = -2\nu\left(e\varphi + \frac{1}{2}\frac{\partial\chi}{\partial t}\right) - \int g_-\tilde{f}_2 \frac{dS_F}{(2\pi)^3 v_F} d\epsilon. \quad (28)$$

Here we put

$$f_2 = \tilde{f}_2 + [e\varphi + \tfrac{1}{2}(\partial\chi/\partial t)](\partial f^{(0)}/\partial\epsilon)$$

and took into account corrections to $g_- + \overline{g_-}$ proportional to $e\varphi + (1/2)(\partial\chi/\partial t)$. Equations (27) and (28) determine the potential $e\varphi \sim u\Delta/v_F$ which appears in Eqs. (10)–(12).

To find f_2 we take f_1 from Eq. (25) in the leading approximation in $\omega_0\tau$. For $|\epsilon| < \Delta_\infty$ it is

$$f_1 = -\frac{\partial f^{(0)}}{\partial\epsilon} p_\perp \omega_n \tau_n([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}}) \quad (29)$$

and $f_1 = 0$ for $|\epsilon| > \Delta_\infty$.

The full expressions for the forces can be found analytically at least for three limiting cases: (1) low temperature limit, $T \ll T_c$; (2) high temperature limit, $T \to T_c$; and (3) arbitrary temperatures for a model vortex with the core size larger than ξ_0.

V. RESULTS: LOW TEMPERATURES

At low temperatures, the low-energy Caroli–de Gennes-Matricon levels[18] on the chiral branch of the bound-state spectrum are only important. Remember that, at low energies $\epsilon \ll \Delta$, and for $b \ll \xi$ and $\rho \gg b$, the regular functions are[19] $g_- = g_0\delta[\epsilon - E_0(p_z, b)]$, $f_- = f_0\delta[\epsilon - E_0(p_z, b)]$, $f_-^\dagger = f_-^*$ where

$$g_0 = (\pi v_\perp e^{-K}/2C), \quad f_0 = g_0 e^{i\phi}(b + is)/\rho \quad (30)$$

with the phase $\chi = \phi$. Here $b = \rho\sin(\phi - \alpha)$ is the impact parameter, $s = \rho\sin(\phi - \alpha)$, ϕ is the azimuthal angle, and α is the angle between $\mathbf{v}_\perp$ and the x axis:

$$K(s) = \frac{2}{v_\perp}\int_0^\rho |\Delta| d\rho', \quad C = \int_0^\infty \exp(-K) d\rho.$$

The bound-state energy is[18]

$$E_0(p_z, b) = C^{-1}\int_0^\infty \frac{b|\Delta|}{\rho}\exp(-K) d\rho. \quad (31)$$

The energy $E_0(b)$ forms the anomalous branch which crosses zero of energy as a function of the quasicontinuous impact parameter.

The density of states at $\epsilon = 0$ averaged over the Fermi surface diverges at small distances from the vortex axis:

$$\langle g_- \rangle = \left\langle \frac{v_\perp e^{-K}}{2C\rho|\partial E_0/\partial b|} \right\rangle. \quad (32)$$

(Angular brackets denote averaging over the Fermi surface.) It gives rise to the Kramer-Pesch effect,[19] i.e., to an increase in the order-parameter derivative for $\rho \to 0$: $d|\Delta|/d\rho \to \Delta_\infty/\xi_1$ where

$$\xi_1 \sim (T/\Delta_\infty)\xi_0. \quad (33)$$

To incorporate this effect into the order-parameter coordinate dependence, one can use an approximate expression

$$|\Delta(\rho)| = \begin{cases} \Delta_\infty\rho/\xi_1 & \text{for } \rho < \xi_1, \\ \Delta_\infty & \text{for } \rho > \xi_1. \end{cases} \quad (34)$$

For such an order parameter, the bound-state energy is

$$E_0(p_z,b)=\frac{2b\Delta_\infty^2}{v_\perp}\ln\left(\frac{\xi_0}{b}\right). \tag{35}$$

It is actually independent of the real structure of $|\Delta(\rho)|$ within the logarithmic accuracy. For $E_0\sim T\ll\Delta_\infty$, the characteristic value of the impact parameter is

$$b\sim\xi_1/\ln(\Delta_\infty/T).$$

The $1/\rho$ divergence of the density of states is cut off by impurity scattering. Indeed, the impurity self-energy in the Eilenberger equations which define the quasiclassical Green functions $g^{R(A)}$ and $f^{R(A)}$ becomes important when ϵ is of the order $(1/\tau)\langle g^{R(A)}\rangle$. Since $\epsilon\sim T$ we find from Eq. (32) that the Kramer and Pesch result is valid for

$$\rho\gg\frac{\xi_0}{\tau T\,\ln(T_c/T)}. \tag{36}$$

We shall see that the growth of the zero-energy density of states at small distances gives rise to a large increase in the additional transverse force at low temperatures.

A. Quasiclassical force

For $|\epsilon|\ll\Delta_\infty$ the distribution function f_2 is small. The first term in the second line in Eq. (7) gives only a small correction to the dissipative component of the force. Therefore,

$$\hat g^{(\mathrm{nst})}-2\hat g_-f_1'=2\hat g_-f_1''+2\hat g_-'f^{(0)}, \tag{37}$$

where $\hat g_-'$ is the correction to $\hat g_-$ due to $\partial\varphi_0/\partial t$. It is determined by the Eilenberger equations

$$-i\mathbf{v}\nabla f'-2\epsilon f'+2\Delta g'-\frac{2e}{c}\mathbf{vA}f'-ie\frac{\partial\varphi_0}{\partial t}\frac{\partial f}{\partial\epsilon}=0,$$

$$i\mathbf{v}\nabla f^{\dagger\prime}-2\epsilon f^{\dagger\prime}+2\Delta^*g'-\frac{2e}{c}\mathbf{vA}f^{\dagger\prime}+ie\frac{\partial\varphi_0}{\partial t}\frac{\partial f^\dagger}{\partial\epsilon}=0,$$

and the normalization condition $2gg'-f^\dagger f'-ff^{\dagger\prime}=0$. We put $f'e^{-i\phi}=\tilde f'$, $f^{\dagger\prime}e^{i\phi}=f^{\dagger\prime}$. Using Eq. (30), for small b and ϵ we get $\tilde f'=-\tilde f^{\dagger\prime}=ig'$, and

$$v_\perp\frac{\partial\tilde f'}{\partial s}+2|\Delta|\tilde f'+ie\frac{\partial\varphi_0}{\partial t}\frac{\partial g}{\partial\epsilon}=0.$$

The solution is

$$\tilde f'=i\frac{(\mathbf{u}\hat{\mathbf{v}}_\perp)}{v_\perp}(e\varphi_0(\rho)-e\varphi_0(\infty))\frac{\partial g}{\partial\epsilon}.$$

Being inserted into Eq. (26), this term gives

$$2\,\mathrm{Tr}[\hat\nabla\mathcal{H}\hat g_-']f^{(0)}=-\frac{[\hat{\mathbf{v}}_\perp\times\hat{\mathbf{z}}](\mathbf{u}\mathbf{v}_\perp)}{v_\perp}\frac{4|\Delta|}{\rho}\frac{\partial g_-}{\partial\epsilon}f^{(0)}$$
$$\times(e\varphi_0(\rho)-e\varphi_0(\infty))$$

which cancels the contribution from the first term on the RHS of Eq. (37). Therefore, the additional quasiclassical force becomes

$$\mathbf{F}'^{(\mathrm{qc})}=-\int db\int\frac{dS_F}{(2\pi)^3v_F}[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]\frac{\partial E_0}{\partial b}\tau_0\int_{-\infty}^{\infty}J_0\{f_1''\}ds. \tag{38}$$

It was shown in Ref. 6 that the main contribution to the collision integral within the logarithmic accuracy comes from distances $s\sim b\ll\xi_1$. Therefore, one can write

$$\int_{-\infty}^{\infty}J_0\{f_1''\}=-\frac{\pi v_\perp}{\tau_0}f_1''(\rho\sim b)$$
$$=-\frac{\pi}{\tau_0}\frac{\partial f^{(0)}}{\partial\epsilon}[(\mathbf{u}\hat{\mathbf{v}}_\perp)][e\varphi_0(\rho\sim b)-e\varphi_0(\infty)].$$

Since $\varphi_0(\rho\sim b)/\varphi_0(\infty)\sim b^2/\xi_1^2\ll1$ we obtain $\beta_1=-1/2$ for the additional transverse force in Eq. (9).

B. Nonquasiclassical force

Equation (27) gives

$$g_-\tilde f_2=\int_{-\infty}^{s}\frac{ds}{2v_\perp}\left\{2J^{(1)}-\left[v_\perp g_-\left(\frac{\partial^2\phi}{\partial s\partial t}-\frac{2e}{c}\frac{\partial A_s}{\partial t}\right)\right.\right.$$
$$\left.\left.+(\tilde f_-+\tilde f_-^\dagger)\frac{\partial|\Delta|}{\partial t}\right]\frac{\partial f^{(0)}}{\partial\epsilon}\right\}, \tag{39}$$

where $f_-=\tilde f_-e^{i\phi}$, $f_-^\dagger=\tilde f_-^\dagger e^{-i\phi}$. Note that, for localized electrons with $|\epsilon|<\Delta_\infty$, Eq. (39) determines the function f_1 in the limit $s\to\infty$ when $g_-\to0$.

For low energies, we get

$$g_-\tilde f_2=-\frac12\frac{\partial\phi}{\partial t}\frac{\partial f^{(0)}}{\partial\epsilon}g_-$$
$$+\int_0^sJ^{(1)}\frac{ds}{v_\perp}-\frac{([\mathbf{z}\times\hat{\mathbf{v}}_\perp]\mathbf{u})}{v_\perp}\frac{\partial f^{(0)}}{\partial\epsilon}\int_0^s\frac{|\Delta|g_-}{\rho}ds. \tag{40}$$

Due to the Kramer-Pesch effect, Eq. (33), we can simplify this expression. We shall see that the distances of the order of $\rho\sim\xi_1=(T/\Delta_\infty)\xi_0$ give the main contribution to the quasiclassical force. For such distances, the last line in Eq. (40) is of the order of g_-/ξ_0 while the first is of the order of g_-/ξ_1, therefore

$$\tilde f_2=-\frac12\frac{\partial\phi}{\partial t}\frac{\partial f^{(0)}}{\partial\epsilon}.$$

From Eqs. (28) and (35) the dynamic potential becomes

$$e\varphi=\frac{u_\phi}{2\rho}(1-\langle g_-\rangle)=\frac{u_\phi}{2\rho}\left(1-\frac{\langle v_\perp\rangle}{2\rho\Delta_\infty\ln(\Delta_\infty/T)}\right).$$

Here we assumed the order parameter in the form of Eq. (34). The potential grows for a decreasing ρ which is a consequence of the peak in the density of states at zero energy. However, one cannot extrapolate the increase in φ down to $\rho=0$ because Eq. (30) used for the Green functions is valid only for $\rho\gg b$ with the characteristic impact parameter such that $E_0(b)\sim T$. One expects that the divergence of the potential is cut off either at distances $\rho\ll b$ or by the impurity scattering. We discuss this again in the next section in connection with the model vortex.

We get for the nonquasiclassical force of Eq. (11)

$$\beta_2=\frac{1}{2}+\frac{\langle v_\perp\rangle}{4\Delta_\infty^3\ln(\Delta_\infty/T)}\int_0^\infty\frac{d|\Delta|^2}{d\rho}\frac{d\rho}{\rho}.$$

Finally, the total additional transverse force $\mathbf{F}_\perp'$ in Eq. (3) has $\beta=\beta_1+\beta_2$ where

$$\beta=\frac{\langle v_\perp\rangle}{4\Delta_\infty^3\ln(\Delta_\infty/T)}\int_0^\infty\frac{d|\Delta|^2}{d\rho}\frac{d\rho}{\rho}. \tag{41}$$

Here the distances $\rho\sim\xi_1$ give the main contribution so that β increases with lowering the temperature. With Eq. (34) we obtain

$$\beta=\frac{\langle v_\perp\rangle}{2\Delta_\infty\xi_1\ln(\Delta_\infty/T)}\sim\frac{\Delta_\infty}{T\,\ln(\Delta_\infty/T)}. \tag{42}$$

The increase in β is cut off by impurity scattering. The lowest temperature for which Eq. (42) is still valid is found by putting $\rho\sim\xi_1$ in Eq. (36) which gives $T/T_c\sim1/\sqrt{(T_c\tau)\ln(T_c\tau)}$. Therefore, the maximum value of β is

$$\beta_{\max}\sim\sqrt{\frac{T_c\tau}{\ln(T_c\tau)}}.$$

VI. RESULTS: MODEL VORTEX AND HIGH TEMPERATURES

It is instructive to consider also a continuous model in which the vortex has a core size much larger than the coherence length ξ_0. We assume that the order parameter is $\Delta=|\Delta(\rho)|e^{i\phi}$; its magnitude varies smoothly from $|\Delta|=\Delta_\infty$ at large distances from the core with the radius $R\gg\xi_0$ to $|\Delta|=0$ at $\rho=0$. It is the opposite extreme as compared to a low-temperature vortex with a small core discussed in the previous section. This simple model offers a possibility to get an analytical solution for the whole temperature range and to demonstrate that the additional force depends crucially on the vortex core structure.

A. Quasiclassical force

Within the continuous vortex model, the Green functions can be found by expanding the Eilenberger equations in small gradients. Neglecting the vector potential, we have

$$\mathrm{Tr}(\hat{\nabla}\mathcal{H}\hat{g}_-)=2\nabla|\Delta|f+[\mathbf{v}_\perp\times\hat{\mathbf{z}}]\frac{1}{\rho}\frac{\partial g}{\partial\rho}, \tag{43}$$

$$\mathrm{Tr}(\hat{\nabla}\mathcal{H}\hat{g}'_-)=(\nabla\phi)(\mathbf{u}\nabla e\varphi_0)\frac{\partial g}{\partial\epsilon}, \tag{44}$$

where $\hat{g}_-$ is the static part and $\hat{g}'_-$ is the correction due to $\partial\varphi_0/\partial t$. Here $g=(\epsilon/\sqrt{\epsilon^2-|\Delta|^2})\Theta(\epsilon^2-|\Delta|^2)$ and $f=(|\Delta|/\sqrt{\epsilon^2-|\Delta|^2})\Theta(\epsilon^2-|\Delta|^2)$.

The combination $\hat{g}_+=0$ for $|\epsilon|>|\Delta|$, therefore, the function f_2' drops out of $\hat{g}^{(\mathrm{nst})}$, and we again arrive at Eq. (37). To calculate the quasiclassical force we use Eq. (26) with f_1'' from Eq. (21). The component of f_1'' with $([\mathbf{u}\times\mathbf{v}_\perp]\hat{\mathbf{z}})$ is odd in both s and b, and vanishes after integration over ds in the collision integral J_n and after averaging over $\mathbf{v}_\perp$ in the last term in Eq. (26). Since the component with $(\mathbf{u}\mathbf{v}_\perp)$ in f_1'' is an even function of the impact parameter b, all the terms with $n\neq0$ in the sum in Eq. (26) vanish. This is because $\partial E_n/\partial b$ is even in b for $n=0$ and odd for all other $n\neq0$ (see, for example, Ref. 6).

Using the parity with respect to s and b, with help of Eqs. (21), (43), and (44) we find that the last term in Eq. (26) vanishes and we again obtain Eq. (38). The collision integral now can be transformed using the fact that the Green functions of the zero level E_0 for the model vortex are localized near the point $\rho=b$ at distances $s\sim\sqrt{R\xi}$ much shorter than the core size.[6] Therefore, we can write

$$\int_{-\infty}^{\infty}J_0\{f_1''\}ds=-\frac{\pi v_\perp}{\tau_0}f_1''(\rho=b).$$

The β factor for the additional quasiclassical force becomes

$$\beta_1=-\frac{1}{2\Delta_\infty^2}\int_0^\infty\left\langle\tanh\left(\frac{E_0(\rho)}{2T}\right)\right\rangle\frac{\partial|\Delta|^2}{\partial\rho}d\rho. \tag{45}$$

Here we used Eq. (15) for φ_0.

Note that Eq. (45) gives $\beta_1=-1/2$ for $T\to0$, in agreement with the result of the previous section. We see that the factor β_1 is not very sensitive to the core structure.

B. Nonquasiclassical force

For our model of a large-core vortex, Eq. (27) gives

$$g\left(\tilde{f}_2+\frac{1}{2}\frac{\partial\phi}{\partial t}\frac{\partial f^{(0)}}{\partial\epsilon}\right)=\frac{\partial f^{(0)}}{\partial\epsilon}\left[-\frac{(\mathbf{u}\hat{\mathbf{v}}_\perp)}{v_\perp}\eta_\epsilon(\rho)+([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})\int_0^s\frac{ds'}{v_\perp}\left(\frac{v_\perp}{2\rho'}\frac{\partial g}{\partial\rho'}-f\frac{\partial|\Delta|}{\partial\rho'}\frac{b}{\rho'}\right)\right]+\int_0^s J^{(1)}\frac{ds'}{v_\perp}, \tag{46}$$

where $\eta_\epsilon(\rho)=\sqrt{\epsilon^2-|\Delta|^2}\Theta(\epsilon^2-|\Delta|^2)$. Note that the RHS of Eq. (46) vanishes for $s\to\infty$ if $|\epsilon|<\Delta_\infty$; this condition determines f_1 for localized electrons. For $|\epsilon|>\Delta_\infty$ the function $\eta_\epsilon(\infty)$ determines, through the boundary conditions, a small correction to f_1. From the symmetry of the function f_2 of the type of Eq. (16) we observe that it only would give a small correction to the quasiclassical friction force.

To calculate φ we average Eq. (46) over the Fermi surface. Using $([\mathbf{u}\times\hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})=-(u_\rho b+u_\phi s)/\rho$ and $(\mathbf{u}\hat{\mathbf{v}}_\perp)=(u_\rho s-u_\phi b)\rho$, we find that the term with η on the RHS of Eq. (46) vanishes after integration over α. In the second term of Eq. (46), a nonvanishing contribution comes from the first term under the integral which is even both in s and b. In the collision integral, the Green functions of the zero branch are localized near $\rho=b$, therefore, for the level $n=0$ we can extend the integration over ds to infinity if $s\gg\sqrt{R\xi}$, and write

$$\int_0^s J^{(1)} ds' = \mathrm{sgn}(s) \int_0^\infty J_0 ds\, \delta[\epsilon - E_0(b)] + \text{terms odd in } b$$

since $\partial E_n / \partial b$ in Eq. (29) are odd functions of b for $n \neq 0$. Therefore, after averaging we obtain from charge neutrality Eq. (28)

$$e\varphi = -\frac{1}{2}\frac{\partial \chi}{\partial t}\left[1 - \int g \frac{\partial f^{(0)}}{\partial \epsilon}\frac{d\epsilon}{2}\right] - \left\langle \int \frac{d\epsilon}{2}\frac{\partial f^{(0)}}{\partial \epsilon}([\mathbf{u}\times \hat{\mathbf{v}}_\perp]\hat{\mathbf{z}})\left[\int_0^s ds' \frac{1}{2\rho'}\frac{\partial g}{\partial \rho'}\right]\right\rangle - \left\langle \int \frac{d\epsilon}{2}\delta[\epsilon - E_0(b)]\frac{\mathrm{sgn}(s)}{v_\perp}\int_0^\infty J_0 ds\right\rangle .$$

Because of parity, the component with u_ϕ only remains here after averaging, and we obtain, with help of Eqs. (24) and (29),

$$e\varphi = \frac{u_\phi}{2\rho}\left\langle \int \frac{\partial f^{(0)}}{\partial \epsilon} d\epsilon \int_{-\rho}^{+\rho}\frac{db}{2}\frac{\partial E_0(b)}{\partial b}\delta[\epsilon - E_0(b)]\right\rangle = \frac{u_\phi}{2\rho}\left\langle \tanh\left(\frac{E_0(\rho)}{2T}\right)\right\rangle . \tag{47}$$

The potential remains finite as $\rho \to 0$ since $E_0(b) \to 0$ for small $b \ll R$. This is in contrast to the result of the previous section for a low-temperature small-core vortex where an increase in the potential was found at small distances from the vortex axis.

Collecting the results of Eqs. (11), (45), and (47) we obtain the full additional transverse force $\mathbf{F}'_\perp$ with $\beta = \beta_1 + \beta_2$:

$$\beta = 1 - |\Delta_\infty|^{-2}\int_0^\infty \left\langle \tanh\left(\frac{E_0(\rho)}{2T}\right)\right\rangle \frac{d|\Delta|^2}{d\rho} d\rho .$$

The energy of the the anomalous branch for the model vortex with a large core is[6] $E_0(b) = \mathrm{sgn}(b)|\Delta(b)|$. Therefore, we have

$$\beta(x) = 1 - \tanh x + \frac{1}{x^2}\int_0^x \frac{x'^2 dx'}{(\cosh x')^2} \tag{48}$$

with $x = \Delta_\infty / 2T$. The factor β vanishes for $T \to 0$, and it is $\beta = 1$ for $T \to T_c$.

We see that, in the continuous vortex model, the static, φ_0, and dynamic, φ, potentials contribute equally to modification of the additional transverse force: one-half comes from a relaxation process involved in the formation of the dynamic potential φ, and the other from relaxation associated with the time dependence of $\varphi_0(\mathbf{r} - \mathbf{u}t)$.

Note that, for the continuous vortex model, β vanishes at low temperatures according to Eq. (48). This differs drastically from its behavior for a real vortex where β diverges at $T \to 0$, Eqs. (41) and (42). The reason is that the zero-energy density of states in a real vortex, $\langle g_- \rangle$, diverges for $\rho \to 0$ according to Eq. (32), while that in a model vortex remains finite and transforms into the normal-state value, $g_- \to 1$.

C. High temperatures

The conclusion that $\beta \to 1$ for $T \to T_c$ does not depend on the model of a vortex and holds in a general case: Indeed, for $\Delta \ll T_c$, the gradient expansion for the Green functions works in general for high energies $\epsilon \sim T \gg \Delta$, while low energies $\epsilon \sim \Delta$ give a small contribution of the relative order of Δ / T. This can be easily checked for the quasiclassical force by inspection of Eq. (26). At the same time, for $\Delta \ll T$, Eq. (27) results in the distribution function $\tilde{f}_2 = -(1/2)(\partial \phi / \partial t)(\partial f^{(0)} / \partial \epsilon)$ and the dynamic potential φ vanishes since $g_- = 1$ in the leading approximation in Δ / T. Therefore, we arrive at $\beta = 1$ for $T \to T_c$.

For these temperatures our result agrees both with Ref. 9 and with the microscopic[3] and TDGL calculations[2] in the limit of a small pair-breaking rate. Within the TDGL theory, a weak pair breaking is modeled by a small coefficient in front of the time derivative of the order parameter in the TDGL equation.

VII. DISCUSSION

The additional transverse force Eq. (3) is thus essentially modified as compared to the effective-action result by kinetic effects caused by charge neutrality. The modification is more pronounced at low temperatures where the additional transverse force increases as T^{-1} according to Eqs. (41) and (42). The charge neutrality effects are to ensure the incompressibility of the electron liquid div$\mathbf{j} = 0$ and are closely related with the charge imbalance relaxation in superconductors. In clean superconductors, delocalized excitations are almost in a full equilibrium with the heat bath while those localized in vortex cores are highly involved in the vortex motion and thus in the relaxation.[6] This is why the force is mainly affected by localized excitations. In a moderately clean case, when the relaxation is very effective, the additional force is strongly modified with respect to the effective-action result for $T \to 0$ when all excitations are localized, and restores its relaxation-free form for $T \to T_c$ when the number of localized electrons is small.

Our results for the force exerted on a moving vortex can be used to deduce the Hall conductivity from Eq. (2):

$$\sigma_H = \sigma_H^{(0)} + \frac{ec}{B}\left(\frac{1}{\lambda}\frac{\partial \nu}{\partial \zeta}\right)\Delta_\infty^2 \beta(T). \tag{49}$$

It is interesting to note that Eq. (49) for the Hall conductivity as a function of temperature allows for multiple sign reversals. The possibility of a double sign reversal was predicted in Ref. 9 and agrees with our calculations. Indeed, the Hall conductivity $\sigma_H^{(0)}$ for a moderately clean case with $l/\xi(T) \gg 1/\sqrt{\lambda}$ is determined[6] by the integral over the Fermi surface which contains the bound-sate spectrum of electrons and the relaxation time. The sign of the Hall effect in this regime may differ from that in the normal state. However, the parameter $\omega_0 \tau \sim \Delta^2 \tau / E_F$ is small near T_c due to a small Δ. According to Ref. 6 the Hall conductivity

$$\sigma_H^{(0)} \sim \frac{Nec}{B}\left(\frac{T_c^2\tau}{E_F}\right)^2\left(\frac{\Delta_\infty}{T_c}\right)^5$$

for $T \to T_c$. Therefore, when the sample is cooled down from the normal into the superconducting state, the superconducting Hall effect is first determined by the correction term until l/ξ_0 becomes of the order of $(1/\sqrt{\lambda})(1-T/T_c)^{-3/4}$. The sign of the Hall angle in this high-temperature regime is determined by another characteristic of the Fermi surface, viz., the energy derivative of the density of states, and may be different from either the sign in the normal state or the sign in the low-temperature regime. This may result in a double sign reversal if the latter signs are the same. A double sign reversal was observed experimentally.[20] Here we point out the possibility of a third sign reversal at still lower temperatures. Indeed, the correction term increases for $T \to 0$ and can again exceed $\sigma_H^{(0)}$ if $l/\xi_0 < 1/[\lambda^{2/3}\ln^{1/3}(T_c\tau)]$ which can result in another change of sign of the Hall effect.

In the present paper we assumed that pinning is absent. There are generally no doubts that pinning can bring about more interesting features to the Hall behavior of superconductors. Our results for conductivities can be used as the input parameters when constructing a more general theory which would take into account possible effects of pinning and fluctuations. A theory of such kind was suggested in Ref. 21. It was shown that a double sign reversal in presence of pinning can appear already within a simple Bardeen-Stephen-type model of vortex dynamics. However, we emphasize here that there may exist another reason for multiple sign reversals based on entirely intrinsic mechanisms of vortex motion.

Here we consider only an s-wave superconductor. An increasing amount of evidence appears now in favor of a d-wave symmetry of the order parameter. The specifics of vortex dynamics in a d-wave superconductor is associated with the gap nodes at the Fermi surface. An example how the gap nodes influence the vortex motion can be found for a p-wave superfluid system, namely, for a phase of superfluid ^{3}He. It was demonstrated[22] that the mechanism of vortex motion in a p-wave system remains qualitatively the same and is governed by the localized states in the vortex core. We thus believe that the general physical picture and results of the present paper can be applied qualitatively to a d-wave superconductor since the most important effect considered here is due to the charge neutrality of metals.

In conclusion, we have calculated the Hall conductivity in moderately clean superconductors and found an additional transverse force on a moving vortex. We show that the effective-action result of Ref. 9 is strongly modified by an electric potential generated by moving vortices due to charge neutrality in superconductors. The effective-action result is recovered near T_c where the microscopic theory agrees both with the effective action formalism and with the TDGL theory in the limit of a small pair-breaking rate.

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