Exact boundary critical exponents and tunneling effects in integrable models for quantum wires

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Using the principles of the conformal quantum-field theory and the finite size corrections of the energy of the ground and various excited states, we calculate the boundary critical exponents of single- and multicomponent Bethe-Ansatz soluble models. The boundary critical exponents are given in terms of the dressed-charge matrix which has the same form as that of systems with periodic boundary conditions and is uniquely determined by the Bethe-ansatz equations. A Luttinger liquid with open boundaries is the effective low-energy theory of these models. As applications of the theory, the Friedel oscillations due to the boundaries and the tunneling conductance through a barrier are also calculated. The tunneling conductance is determined by a nonuniversal boundary exponent which governs its power law dependence on temperature and frequency. [S0163-1829(96)09635-X]

I. INTRODUCTION

Phase transitions take place in a different way on surfaces and in the bulk of a sample.[1] The exponents describing critical phenomena at surfaces differ from those of the bulk, and one may observe phenomena due to anisotropy and the breaking of translational invariance caused by the boundary, like oscillations in correlation functions which, in the bulk, are monotonous, coordinate dependences, and in particular Friedel oscillations, in local quantities, etc. There are many approximate methods for calculating the critical properties of bulk and surface phenomena.

Systems at a bulk critical point are not only scale invariant but also conformally invariant, a consequence of the combined rotational and scale invariance.[2,3] In two space dimensions or one space and one time dimension [thus including one-dimensional (1D) quantum systems], the constraints imposed by conformal invariance are much stronger than in higher dimensions because the conformal group is infinite dimensional, and these constraints strongly simplify the calculation of correlation functions. The conformal field theory is parametrized by a unique constant—the conformal anomaly or the central charge c of the corresponding Virasoro algebra.[4] All critical exponents of theories with $c<1$ are universal (independent of the interaction) and can be calculated exactly.[5] However, for $c\geqslant 1$, the conformal dimensions continuously depend on the coupling of the fields and there is no general theory to deduce the critical exponents exactly. Conformal field theory only predicts that the central charge and the conformal dimensions can be derived from the finite size corrections to the energy and momentum spectra.[6–8] For $c\geqslant 1$, it does not determine their actual numerical values. Still, this constitutes a powerful method to calculate the critical exponents of integrable 1D quantum models because their energy and momentum spectra are known exactly, and this strategy has been applied successfully for bulk properties.[9,10]

An entirely parallel and equivalent development has taken place in the theory of correlated fermions in 1D, using bosonization techniques and running under the name of ''Luttinger liquid.''[11,12] The main focus here was on non-Fermi liquid properties of strongly correlated fermions, an exciting topic of current research. Fermi liquid theory fails in 1D, and the Luttinger liquid provides the universal low-energy theory for gapless 1D quantum systems. Its salient properties are (i) absence of fermionic quasiparticle excitations, (ii) anomalous dimensions of various operators leading to nonuniversal power-law decay of correlation functions, and (iii) charge-spin separation. In terms of critical phenomena, the system is at a $T=0$ quantum critical point, is conformally invariant, and the central charge of the associated Virasoro algebra(s) is unity. The exponents of the correlation functions (critical exponents) are related to each other by scaling relations and depend on a single renormalized coupling constant K per degree of freedom, playing the role of the Landau parameters familiar from Fermi liquid theory. For the Luttinger liquid, there are constitutive relations between three velocities characterizing the low-energy sector of the spectrum of the Hamiltonian which determine the renormalized coupling constant and thus the critical exponents. For integrable models, the velocities and coupling constants have been determined from Bethe-ansatz (or other) solutions.[13] In some cases, conformal invariance has been used explicitly to determine the critical exponents.[14] For nonintegrable models, they can be obtained reliably by exact diagonalization of the Hamiltonian.

Recently, the problems of the one-dimensional systems with open boundaries have drawn much attention. There are several issues of both experimental and theoretical relevance. (i) As in higher-dimensional systems, the boundary critical exponents are expected to be different from the bulk ones.[1] (ii) Experimentally studied systems are finite, and the progress in microfabrication of semiconductor structures has provided us with quantum wires so small that the boundary effects could become relevant.[15] (iii) Here but also in the much bigger samples of quasi-1D metals, impurities can be relevant perturbations and effectively cut the systems into isolated strands of finite length. Specifically, a renormalization group analysis has shown that, for effectively repulsive interactions, the scattering potential due to an isolated impurity scales to infinitity in the low-energy regime and thus the problem is equivalent to an open boundary problem.[16] An attempt to test experimentally the predictions of this theory has been published recently.[17] Such effects have also been invoked in the interpretation of electron spin diffusion[18] and photoemission[19,20] experiments on quasi-1D organic conductors. (iv) Finally, some numerical methods such as the density-matrix renormalization group,[21] rely on the use of open boundaries and their results could, in principle, be affected by the chain ends.

Both conformal field theory[22] and, very recently, the theory of Luttinger liquids,[15] have been extended to systems with boundaries. What is missing to date, however, is an exact derivation of the boundary critical exponents from the Bethe-ansatz solution of integrable quantum systems bridging the gap between microscopic (often lattice) models containing both high- and low-energy physics, and the more effective theories for the low-energy properties. This gap has been bridged successfully for the periodic systems.[12–14] It is the purpose of this paper to present such an exact derivation of boundary critical exponents. Moreover, in the course of the study, we shall see that the same strategy can be applied to determine the exponents of nonintegrable systems by exact numerical diagonalization provided they satisfy the basic assumption of conformal invariance.

In this paper, we apply the method of the conformal field theory to the Bethe-ansatz soluble models with open boundaries. The layout of the present paper is the following. In the following section we briefly summarize some important results of boundary conformal field theory and of Luttinger liquid theory on bounded systems, in order to make the presentation self-contained and provide the basic tools. These are essentially the finite size corrections of the energy spectrum. In Sec. III we give a detailed calculation of the boundary critical exponents of two paradigmatic single-component Bethe-ansatz soluble models (δ-potential Bose gas and the antiferromagnetic Heisenberg chain). Section IV digresses to two important physical applications: the Friedel oscillation of the density distribution around the boundary (impurity) and the tunneling conductance through a barrier in a quantum wire. In Sec. V, we generalize the result to the multicomponent case with the Hubbard model as an example. The summary in Sec. VI attempts to provide a broader perspective on our results.

II. CONFORMAL FIELD THEORY AND LUTTINGER LIQUIDS IN BOUNDED SYSTEMS

Systems with open boundaries

$$\psi(x=0)=\psi(x=L)=0 \tag{1}$$

(where ψ is the wave function) are no longer space translational invariant but the time translational invariance is preserved. A two-point correlation function of a (primary) field at criticality then takes the general form[22]

$$G_b(\tau,x_1,x_2)=(x_1x_2)^{-d}\Phi\{[v^2\tau^2+x_1^2+x_2^2]/x_1x_2\}. \tag{2}$$

Equation (2) applies to a 1D quantum system, τ represents the imaginary time, and $x_{1,2}$ the spatial coordinates; d is the conformal dimension of the primary field in the bulk; v is the Fermi velocity. When x_1 and x_2 are near the boundary, and $\tau\to\infty$, G_b must behave as

$$G_b(\tau,x_1,x_2)\sim\frac{1}{\tau^{2x_b}}. \tag{3}$$

$2x_b$ is the boundary critical exponent. Equation (3) directly implies that $\lim_{y\to\infty}\Phi(y)\sim y^{-x_b}$. On the other hand, Cardy also showed that the n-point correlation function in a half plane (with one open boundary at $x=0$) is identical to the $2n$-point correlation function in the whole plane, provided only the z-dependent part is taken into account in the latter. In this way, the two-point correlation function can also be represented as

$$G_b(z_1,z_2,\overline{z_1},\overline{z_2})=[(z_1-\overline{z_1})(z_2-\overline{z_2})/(z_1-z_2)(\overline{z_1}-\overline{z_2})\times(z_1-\overline{z_2})(\overline{z_1}-z_2)]^{-2d}F_b(y), \tag{4}$$

where $F_b(y)$ is an unknown scaling function. Here, we have switched to a notation in terms of complex variables $z_j=v\tau_j+ix_j$, $\overline{z_j}=v\tau_j-ix_j$ and y is given by $y=(z_1-z_2)(\overline{z_1}-\overline{z_2})/(z_1-\overline{z_1})(z_2-\overline{z_2})$. For $y\to\infty$, $F_b(y)\to y^{-\alpha}$. Direct comparison to Eq. (2) gives

$$x_b=4d+\alpha. \tag{5}$$

In the following text, we shall use Eq. (2) and Eq. (4) alternatively.

The conformal dimensions or critical exponents can be calculated from the finite size corrections of the energy spectra. To see this, consider the transformation

$$\zeta=\frac{L}{\pi}\ln z,\quad \overline{\zeta}=\frac{L}{\pi}\ln\overline{z} \tag{6}$$

applied to the upper half-plane $x\geqslant 0$ only. Such a conformal transformation maps the system from the semi-infinite plane onto a strip of width L with open boundary conditions.[6] From the general transformation properties of the correlation functions of a (primary) conformal field $\phi(z,\overline{z})$

$$\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\rangle=\left(\frac{\partial\zeta_1}{\partial z_1}\right)^{\Delta}\left(\frac{\partial\overline{\zeta_1}}{\partial\overline{z_1}}\right)^{\overline{\Delta}}\left(\frac{\partial\zeta_2}{\partial z_2}\right)^{\Delta}\left(\frac{\partial\overline{\zeta_2}}{\partial\overline{z_2}}\right)^{\overline{\Delta}}\times\langle\phi(\zeta_1,\overline{\zeta_1})\phi(\zeta_2,\overline{\zeta_2})\rangle, \tag{7}$$

where $\Delta+\overline{\Delta}=d$ and ϕ and $\Delta-\overline{\Delta}=s$ give, respectively, the conformal dimension and spin of the field, one deduces for the correlation function on the strip

$$\langle \phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\rangle_L = \left[\frac{\frac{\pi}{4L}\sinh\frac{\pi(z_1-\overline{z_1})}{2L}\sinh\frac{\pi(z_2-\overline{z_2})}{2L}}{\sinh\frac{\pi(z_1-z_2)}{2L}\sinh\frac{\pi(\overline{z_1}-\overline{z_2})}{2L}\sinh\frac{\pi(z_1-\overline{z_2})}{2L}\sinh\frac{\pi(\overline{z_1}-z_2)}{2L}} \right]^{2d} \times F_b\left(\frac{\sinh\frac{\pi(z_1-\overline{z_1})}{2L}\sinh\frac{\pi(z_2-\overline{z_2})}{2L}}{\sinh\frac{\pi(z_1-z_2)}{2L}\sinh\frac{\pi(\overline{z_1}-\overline{z_2})}{2L}} \right). \tag{8}$$

There is also a spectral representation of the correlation function on the strip

$$\langle \phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\rangle_L = \sum_n \langle \phi(x_1,0)|n\rangle\langle n|\phi(x_2,0)\rangle \times \exp[-(\tau_1-\tau_2)(E_L^n-E_L^0)], \tag{9}$$

where $E_{b,L}^0$ is the energy of the ground state and $E_{b,L}^n$ are the energies of the excited states; $|n\rangle$ are the exact eigenstates of the Hamiltonian under consideration which form a complete set. Suppose E_L^1 (the energy of the first excited state with the form factor $\langle\phi|1\rangle\neq 0$) takes the form ($L\to\infty$),

$$E_L^1-E_L^0=\frac{\pi v}{L}x_b+o\left(\frac{1}{L}\right). \tag{10}$$

From the asymptotic form of the correlation functions Eq. (3) and comparing Eqs. (8) and (9) we get Eq. (5). Thus the finite size asymptotics of the low-lying levels determines the boundary critical exponents.

In general, the correlation function can oscillate, so that its asymptotics is not conformally invariant. In that case, however, one can decompose the field $\phi(z,\overline{z})$ into a sum of conformal fields $\phi_n(z,\overline{z})$ which then determine the power-law asymptotics.[9] Because of the reflection symmetry of the open boundary systems, the field ϕ must have definite parity. We can expand odd and even parity fields as

$$\phi(z,\overline{z})=\sum_n \phi_n(z,\overline{z})\sin(nk_Fx), \qquad \phi(z,\overline{z})=\sum_n \phi_n(z,\overline{z})\cos(nk_Fx) \tag{11}$$

respectively. n is an odd (even) integer for odd (even) parity fields.

For systems with periodic boundary conditions, the Luttinger liquid phenomenology[11,12] provides a framework completely equivalent to conformal field theory but closer to the language of conventional solid state physics. It is based on the exactly solvable Luttinger model, and all physical properties can be described in terms of two parameters per degree of freedom ($\nu=\rho,\sigma$ for charge and spin), a renormalized sound velocity v_ν and an effective coupling constant K_ν which determines the decay of all correlation functions and thus the critical exponents. These parameters can be determined from the energies of the low-lying excited states of the Hamiltonian.[13]

This picture has been extended recently to systems with open boundaries.[15] Due to the boundary conditions (1), the right- and left-moving fermions commonly used in the Luttinger model are not independent, and a single species moving, say, to the right [$\Psi_{+,s}(x)$] is sufficient, and it is periodic on a length $2L$. The Fermi surface reduces to a single point $+k_F$ but the wavevectors $k=m\pi/L>0$ are quantized with twice the density of the periodic system. We then can rewrite the fermionic Hamiltonian

$$H_0=-iv_F\sum_s \int_{-L}^{L} dx\Psi_{+,s}^\dagger(x)\partial_x\Psi_{+,s}(x) \tag{12}$$

in an equivalent form involving the bosonic density fluctuations (particle-hole excitations) $\rho_{+,s}(x)$ and ''charge excitations'' ΔN_s corresponding to the addition of particles of spin s to the reference Fermi sea (i.e., $\Delta k_{Fs}=\Delta N_s\pi/L$)

$$H_0=\pi v_F\sum_s \int_{-L}^{L} dx{:}\rho_{+,s}(x)\rho_{+,s}(x){:}+\frac{\pi v_F}{2L}\sum_s (\Delta N_s)^2. \tag{13}$$

The Fourier transform $\rho_{+,s}(p)$ of the density operators do not contain the $p=0$ component which is represented explicitly by $\Delta N_s=\Sigma_k(c_{+,s,k}^\dagger c_{+,s,k}-\langle c_{+,s,k}^\dagger c_{+,s,k}\rangle_0)$ where $\langle\rangle_0$ denotes the (infinite) expectation value in the reference Fermi sea given by k_F^0 Unlike the periodic case, ''current excitations'' describing the difference of right- and left-moving fermion numbers, cannot be defined in the bounded system. The Hamiltonian including forward scattering can then be diagonalized by a Bogoliubov tranformation as in the periodic case, defining the renormalized velocities v_ν of the bosonic charge and spin density fluctuations $\rho_+(p)[\sigma_+(p)]=[\rho_{+,\uparrow}(p)\pm\rho_{+,\downarrow}(p)]/\sqrt{2}$, and coupling constants K_ν. The renormalized velocity of the charge excitations $\Delta N_{\rho(\sigma)}=\Delta N_\uparrow\pm\Delta N_\downarrow$ is given by v_ν/K_ν. The bosonization of this model is completed by an explicit representation of the Fermi operator $\Psi_{+,s}(x)$ in terms of the bosons $\rho_{+,s}(p)$ (Ref. 15) which allows to calculate all correlation functions of this model in terms of the v_ν and K_ν and thus defines its critical exponents.

v_ν and K_ν can now be found along the same lines as in the periodic systems:[13] (i) v_ν can be computed directly from

the spectrum of low-lying excitations; (ii) to get K_ρ, one calculates the compressibility κ

$$\frac{1}{\kappa}=\frac{1}{L}\frac{\partial^2 E_0(n)}{\partial n^2}=\frac{\pi v_e}{2K_\rho}. \tag{14}$$

The first equality gives the definition as the second derivative of the ground state energy with respect to particle density $n=N/L$ and can be computed in the integrable model, and the second equality gives the Luttinger liquid expression which can be solved for K_ρ. K_σ is required to be unity by spin-rotation invariance, but can be calculated in the same way in more general cases from the susceptibility. Our determination of boundary critical exponents below can also be viewed as exploring this strategy.

III. BETHE-ANSATZ SOLUBLE MODELS

We now compute the exact boundary critical exponents for Bethe-ansatz soluble models with open boundaries. We first consider single-component models before turning to multicomponent systems. In both cases, the critical exponents can be calculated explicitly. Two typical single-component models are the one dimensional δ-potential Bose gas model and the Heisenberg antiferromagnetic chain. Their Hamiltonians are

$$H_{\mathrm{BG}}=\int_0^L(\partial_x\Psi^\dagger\partial_x\Psi+c\Psi^\dagger\Psi^\dagger\Psi\Psi-h\Psi^\dagger\Psi)dx,$$
$$c>0,\quad h>0 \tag{15}$$

$$H_{XXZ}=\sum_{j=1}^{L-1}\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cos 2\eta\,\sigma_j^z\sigma_{j+1}^z-\frac{1}{2}\sigma_j^z\right),$$
$$0<2\eta<\pi,\quad 0<h<4(1-\cos 2\eta), \tag{16}$$

where h is the chemical potential for the Bose gas model and the magnetic field for the Heisenberg chain. The anisotropy of the exchange integrals for the Heisenberg model is $J_z=\cos(2\eta)$; critical behavior is obtained only for easy-plane-type anisotropy $|J_z|\leqslant 1$, and our definition of J_z restricts us to this range. η plays the role of a coupling constant, as is c for the Bose gas. $\sigma^{x,y,z}$ are the Pauli matrices. The open boundary conditions (1), for the Bose gas, translate into $\Psi(0)=\Psi(L)=0$ in terms of the boson operators $\Psi(x)$, while they are contained in our representation (16) of the Heisenberg chain because the sites 1 and N only couple to a single neighbor.

These models are solved by means of the Bethe-ansatz.[23] The N-particle wave function is parametrized by N numbers λ_j which satisfy the equations

$$2Lp_0(\lambda_j)=2\pi I_j-2\varphi(\lambda_j)-\sum_{l\neq j}[\Phi(\lambda_j-\lambda_l)+\Phi(\lambda_j+\lambda_l)], \tag{17}$$

where p_0 is the bare momentum and Φ is the bare scattering phase:

$$p_0^{\mathrm{BG}}(\lambda)=\lambda,\quad p_0^{XXZ}(\lambda)=i\ln\left(\frac{\cosh(\lambda-i\eta)}{\cosh(\lambda+i\eta)}\right),$$
$$\Phi^{\mathrm{BG}}(\lambda)=-2\arctan\frac{\lambda}{c}, \tag{18}$$
$$\Phi^{XXZ}(\lambda)=-\pi+i\ln\left(\frac{\sinh(\lambda+2i\eta)}{\sinh(\lambda-2i\eta)}\right).$$

The numbers I_j are positive integers, and the parity effects known from periodic systems are absent in models with open boundaries. The bare energy of each particle is

$$\epsilon_0^{\mathrm{BG}}(\lambda)=\lambda^2-h,$$
$$\epsilon_0^{XXZ}(\lambda)=h-2\sinh^2 2\eta/\cosh(\lambda+i\eta)\cosh(\lambda-i\eta). \tag{19}$$

The phase $\varphi(\lambda_j)=0$ for the Bose gas and $\varphi(\lambda_j)=p_0(\lambda_j)+\Phi(2\lambda_j)$ for the XXZ chain. The eigenvalue of the Hamiltonian is equal to the sum of the bare energies of the particles

$$E_L=\sum_{j=1}^{N}\epsilon_0(\lambda_j). \tag{20}$$

Carefully checking the wave functions with the boundary conditions Eq. (1) we find that $\{\pm\lambda\}$ correspond to the same state. This is not surprising because of the reflection symmetry of the system, and corresponds to standing-wave-like solutions. The Bethe-ansatz equation (17) thus only allows solutions with $\lambda_j\neq\pm\lambda_l$ for $j\neq l$ so that all $\lambda_j\geqslant 0$. The system therefore has only one Fermi point k_F. This is very different from the case of the cyclic systems. A similar feature, however, occurs in Luttinger liquids in bounded systems (Sec. II). The appearence of a single Fermi point identifies the system as chiral, and 1D quantum systems with boundaries therefore appear to be special cases of ''chiral Luttinger liquids,'' a notion that has appeared previously in the superficially quite unrelated area of edge states in the fractional quantum Hall effect.[24]

With open boundaries, the total momentum is no longer a good quantum number. However, the quantity

$$P=\frac{\pi}{L}\sum_{j=1}^{N}|I_j| \tag{21}$$

is conserved. This follows from an argument of adiabatic continuity similar to one originally given by Yang and Yang.[25] We shall call P the ''momentum'' of the models. In the ground state, I_j takes consecutive integer values from 1 to N ($I_j=0$ is not allowed). We can then define a dressed momentum

$$p_b(\lambda_j)=p_0(\lambda_j)+\frac{1}{2L}\sum_{l\neq j}[\Phi(\lambda_j-\lambda_l)+\Phi(\lambda_j+\lambda_l)]+\frac{\varphi(\lambda_j)}{L}$$
$$=\frac{\pi I_j}{L}. \tag{22}$$

The Fermi momentum $k_F=\pi N/L=\pi n$ has the same value as in the periodic system.

In the thermodynamic limit $L\to\infty$, $N\to\infty$ keeping n finite, the ground state solves the following integral equations:

$$\rho_b(\lambda)=\frac{p_0'(\lambda)}{\pi}+\int_0^\Lambda K_b(\lambda,\mu)\rho_b(\mu)d\mu, \tag{23}$$

$$\epsilon_b(\lambda)=\epsilon_0(\lambda)+\int_0^\Lambda K_b(\lambda,\mu)\epsilon_b(\mu)d\mu, \tag{24}$$

$$K_b(\lambda,\mu)=\frac{1}{2\pi}\frac{\partial}{\partial\lambda}[\Phi(\lambda-\mu)+\Phi(\lambda+\mu)].$$

The cutoff parameter Λ is defined by the requirement $\epsilon_b(\Lambda)=0$; $\rho_b(\lambda)$ is the density of λ per unit length. Equation (22) becomes

$$p_b(\lambda)=p_0(\lambda)+\frac{1}{2}\int_0^\Lambda[\Phi(\lambda-\mu)+\Phi(\lambda+\mu)]\rho_b(\mu)d\mu. \tag{25}$$

$p_b'(\lambda)=\pi\rho_b(\lambda)$ follows from a combination of (23) and (25), and $p_b(\Lambda)=k_F$ ensures that the Fermi surface is the same as in the finite system with the same electron density.

We now compare equivalent quantities of the bounded and periodic systems and use the same symbols but without the subscript b for the latter. Directly comparing Eqs. (23)–(25) with their periodic equivalents[9] we find

$$\rho_b(\lambda)=2\rho(\lambda),$$
$$\epsilon_b(\lambda)=\epsilon(\lambda), \tag{26}$$
$$p_b(\lambda)=p(\lambda),$$

if $n=n_b$ (thus $\Lambda_b=\Lambda$). There is no restriction on sign(λ) for the periodic systems but $\lambda>0$ for the bounded ones. In the vicinity of the Fermi surface, the excitation energy is

$$e(\lambda)=v|p_b(\lambda)-k_F|, \tag{27}$$

where the Fermi velocity is given by

$$v=\frac{\epsilon'(\Lambda)}{p'(\Lambda)}=\frac{\epsilon_b'(\Lambda)}{\pi\rho_b(\Lambda)}. \tag{28}$$

It takes the same value of that of the system with the periodic boundary condition as it must be for the Fermi velocity can be determined by the leading term of the free energy which should not depend on the boundary conditions in the thermodynamic limit.

Unlike the cyclic systems, the systems with open boundaries have only two kinds of elementary excitations: (i) Particle-hole (soundlike) excitations at the Fermi point k_F. Their finite size corrections give the boundary critical exponent of the current-current correlation function for the Bose gas model and or the boundary critical exponent of the S_z component of the spin correlation function for the Heisenberg spin chain. (ii) The change of the free energy induced by the variation of the particle number (termed ''charge excitations'' above). Its finite size correction gives the boundary critical exponent of the single particle correlation function for the Bose gas model and the critical exponent of the transverse spin-spin correlation function of the Heisenberg chain. These features are reproduced precisely in the Luttinger liquid theory of systems with open boundary conditions.

We consider first the particle-hole excitations. To construct the lowest excitation state, we must put $I_N\to I_N+1$ in (17) keeping the other I_j unchanged. The change of the momentum is thus

$$\Delta P=\frac{\pi}{L}, \tag{29}$$

and the excitation energy is

$$\Delta E_{b,L}=\frac{\pi v x_b^{\parallel}}{L},\quad x_b^{\parallel}=1. \tag{30}$$

To obtain the second kind of excitations (charge excitations), we define the dressed-charge function as

$$Z_b(\lambda)=1+\int_0^\Lambda K_b(\lambda,\mu)Z_b(\mu)d\mu. \tag{31}$$

Obviously, $Z_b(\lambda)=Z(\lambda)$ for $\lambda>0$. This is a consequence of (i) the symmetry $Z(\lambda)=Z(-\lambda)$ of the dressed-charge of the periodic systems, and (ii) the fact that, using this symmetry, the equation satisfied by $Z(\lambda)$ in the periodic system can be transformed into Eq. (31) for the open system for $\lambda>0$. The change of the free energy by ΔN additional particles is[9]

$$\Delta E_{b,L}=L\left[f_0\left(n+\frac{\Delta N}{L}\right)-\Delta N\frac{h}{L}-f_0(n)\right]=\frac{(\Delta N)^2}{2L}\frac{\partial h}{\partial n}, \tag{32}$$

where f_0 is the free energy density of the ground state. This gives

$$\Delta E_{b,L}=\frac{\pi v(\Delta N)^2}{2LZ^2(\Lambda)},\quad x_b^{\perp}=\frac{(\Delta N)^2}{2Z^2(\Lambda)}. \tag{33}$$

For the general case, suppose a conformal field ϕ induce the momentum shift relative to the ground state as $\pi\Delta I/L$ and the change of the particle number ΔN. The energy change relative to the ground state is then

$$\Delta E_{b,L}=\frac{\pi v}{L}\left[\Delta I+\frac{(\Delta N)^2}{2Z^2(\Lambda)}\right]+o\left(\frac{1}{L}\right). \tag{34}$$

The boundary critical exponent takes the form

$$2x_b(\Delta I,\Delta N)=2\Delta I+\frac{(\Delta N)^2}{Z^2(\Lambda)}, \tag{35}$$

where ΔI and ΔN are non-negative integers. The above relation was suggested by Alcaraz *et al.*[26] for integrable spin chains from numerical simulations.

For example, we consider the Heisenberg chain in a zero magnetic field. In this case,[27] $\Lambda=\infty$ and $2Z^2(\infty)=\pi/2\eta$. For n_1, n_2 near the boundary and $t\to\infty$, the correlation functions take the following asymptotic forms:

$$\langle\sigma_{n_1}^z(t)\sigma_{n_2}^z(0)\rangle\sim\frac{1}{t^{\eta_\parallel}},$$
$$\Delta I=1,\quad \Delta N=0,\quad \eta_\parallel=2x_b^{\parallel}=2; \tag{36}$$
$$\langle\sigma_{n_1}^-(t)\sigma_{n_2}^+(0)\rangle\sim\frac{1}{t^{\eta_\perp}},$$

$$\Delta I=0, \quad \Delta N=1, \quad \eta_\perp=2x_b^\perp=\frac{4\eta}{\pi}. \tag{37}$$

On the other hand, for $t=0$, $n_1=1$ and $n_2=n\gg 1$, from Eq. (2) we have

$$\langle\sigma_1^z(0)\sigma_n^z(0)\rangle\sim\frac{(-1)^{n-1}}{n^{\theta_\parallel}},$$

$$\theta_\parallel=1+\frac{\pi}{4\eta} \quad \text{for} \quad 2\eta>\frac{\pi}{2} \tag{38}$$

$$\theta_\parallel=2 \quad \text{for} \quad 2\eta\leqslant\frac{\pi}{2}$$

and

$$\langle\sigma_1^-(0)\sigma_n^+(0)\rangle\sim\frac{(-1)^{n-1}}{n^{\theta_\perp}},$$

$$\theta_\perp=\frac{3\eta}{\pi}. \tag{39}$$

Notice that above we have used the bulk conformal dimensions $d_\parallel=\pi/4\eta$ for $4\eta>\pi$, $d_\parallel=1$ for $4\eta\leqslant\pi$ and $d_\perp=\eta/\pi$. When $2\eta=\pi$, the coupling is isotropic and $\eta_\parallel=\eta_\perp=2$, $\theta_\parallel=\theta_\perp=\frac{3}{2}$ as they should be for the spin SU(2) symmetry. The boundary critical exponents $\eta_{\parallel,\perp}$ measuring the decay of correlations with time, are two times larger than those of the bulk. On the other hand, there are new exponents $\theta_{\parallel,\perp}$ for the decay of spatial correlations. In periodic systems, the two sets are identical as a consequence of conformal invariance. In bounded systems, the breaking of translational invariance along x by the boundary conditions while maintaining it along t, generates a new set of critical exponents.

We now consider the finite size correction to the ground state energy

$$E_{b,L}^0=\sum_{j=1}^{N}\epsilon_0(\lambda[j/L]). \tag{40}$$

Using the Euler-Maclaurin formula we have[28]

$$\begin{aligned}E_{b,L}^0&=L\int_0^n\epsilon_0(\lambda[x])dx+f_b-\frac{1}{24L}\frac{\partial\epsilon_0(x)}{\partial x}\bigg|_{x=n}+\frac{1}{24L}\frac{\partial\epsilon_0(x)}{\partial x}\bigg|_{x=0}+o\left(\frac{1}{L}\right)\\&=Lf_0(n)+f_b-\frac{\pi}{24L}\left[\frac{\epsilon_b'(\Lambda)}{\pi\rho_b(\Lambda)}-\frac{\epsilon_b'(0)}{\pi\rho_b(0)}\right]+o\left(\frac{1}{L}\right).\end{aligned} \tag{41}$$

From Eqs. (19) and (24) we know that $\epsilon_b'(0)=0$. Thus we have

$$E_{b,L}^0-Lf_0(n)\approx f_b-\frac{\pi v}{24L}. \tag{42}$$

Here f_b is the boundary energy which was extensively studied.[23,29–31] Equation (42) agrees with the predictions of conformal field theory,[8] and determines the central charge as $c=1$.

IV. FRIEDEL OSCILLATION AND TUNNELING CONDUCTANCE

A. Friedel oscillation

Since the systems under consideration are obviously not translationally invariant, the density distribution is no longer homogeneous. Therfore, the ground state will exhibit Friedel oscillations. For a 1D free fermion system of length L with open boundaries (1), the single-particle wave functions take the form

$$\Psi_m^L(x)=\sqrt{\frac{2}{L}}\sin\frac{m\pi x}{L}, \tag{43}$$

with m positive integers. For the ground state, the density distribution in the box can be easily calculated as

$$\begin{aligned}\langle n(x)\rangle&=\sum_{m=1}^{N}|\Psi_m^L(x)|^2=\frac{2}{L}\sum_{m=1}^{N}\sin^2\frac{m\pi x}{L}\\&\approx n-\frac{\sin(2k_Fx)}{2\pi x} \quad \text{for} \quad x\ll L.\end{aligned} \tag{44}$$

For the interacting systems, we expect the density distribution to have a similar form

$$\langle n(x)\rangle_b\approx n-\frac{A\sin(2k_Fx-\phi)}{x^\gamma}, \quad 0\ll x\ll L, \tag{45}$$

where A and ϕ are two unknown constants and γ is the exponent dominating the decay.

For definiteness, we consider the following spinless fermion model

$$\begin{aligned}H=&-\sum_{r=\pm}\sum_{j=1}^{N_r-1}\{(C_{r,j}^\dagger C_{r,j+1}+\text{H.c.})-Un_{r,j}n_{r,j+1}\}\\&-h\sum_{r=\pm}\sum_{j=1}^{N_r}n_{r,j},\end{aligned} \tag{46}$$

where $r=\pm$ labels two different chains which in the next subsection, we will couple by a tunneling matrix element. Here, we just consider one of the chains. $C_{r,j}^\dagger$ ($C_{r,j}$) are the creation (annihilation) operators of the spinless fermions;

$n_{r,j}=C^{\dagger}_{r,j}C_{r,j}$ and h here denotes the chemical potential. The Hamiltonian is equivalent to an XXZ chain (16) via a Jordan-Wigner transformation with $U=\cos(2\eta)$. The dressed-charge of this model is given by Eq. (31).

As pointed out in Ref. 22, the n-point correlation functions of the open boundary systems are directly related to the $2n$-point correlation functions of the periodic boundary systems. Thus

$$\langle n(x)\rangle_b=\langle n(z)n(\overline{z})\rangle. \tag{47}$$

The right hand side of Eq. (47) is to be understood in the sense that only the z-dependent part of the density-density correlation function $\langle n(z_1,\overline{z_1})n(z_2,\overline{z_2})\rangle$ is taken into account, and that we have to set $z_2=\overline{z_1}$. In addition, the oscillating term only originates from the current-current correlation $\langle n(z_1,\overline{z_1})n(z_2,\overline{z_2})\rangle_J$ (in another language,[12] this object is designed as the $2k_F$-charge density wave correlation function) which has been calculated in Ref. 9 as

$$\langle n(z_1,\overline{z_1})n(z_2,\overline{z_2})\rangle_J\approx-\frac{Be^{i2k_F(x_1-x_2)}+\overline{B}e^{i2k_F(x_2-x_1)}}{(z_1-z_2)^{Z^2(\Lambda)}(\overline{z_1}-\overline{z_2})^{Z^2(\Lambda)}}, \tag{48}$$

where B is a constant. Choosing the z-dependent part in Eq. (48) and putting $z_2=\overline{z_1}$, we obtain Eq. (45) with

$$\gamma=Z^2(\Lambda). \tag{49}$$

This result agrees with Ref. 15 where the Friedel oscillation for the Luttinger model with open boundaries was calculated, and determines the exact value of γ from the Bethe-ansatz equations through the dressed-charge which can be calculated easily. Specifically, repeating the Luttinger liquid analysis[15] for spinless fermions, one finds $\gamma=K$ thereby suggesting the identification $K=Z^2(\Lambda)$ between the Luttinger stiffness constant K and the Bethe-ansatz dressed-charge $Z^2(\Lambda)$.

B. Tunneling conductance

The boundary critical exponents are very important to study the tunneling effect in quantum wires. A strong barrier cuts the chain into two half-chains which, in first order, behave as two independent subsystems with an open boundary.

We add a tunneling term to the Hamiltonian (46)

$$\mathbf{T}=-V[C^{\dagger}_{+,1}C_{-,1}+C^{\dagger}_{-,1}C_{+,1}],\quad V\ll 1. \tag{50}$$

The tunneling current is thus

$$J=-iV[C^{\dagger}_{+,1}C_{-,1}-C^{\dagger}_{-,1}C_{+,1}]. \tag{51}$$

From linear response theory we know that the tunneling conductance up to order V^2 is given by

$$G(\omega)=i\int dte^{i\omega t}\int dt'\,\theta(t-t')\int dt''\,\theta(t-t'')\times\langle[J(t),J(t'')]\rangle. \tag{52}$$

Since the average $\langle\rangle$ is taken at $\mathbf{T}=0$, the current correlation function may be separated into

$$\langle[J(t),J(t'')]\rangle\sim\prod_{r=\pm}\langle C_{r,1}(t)C^{\dagger}_{r,1}(t'')\rangle\sim(t-t'')^{-2\eta_{\perp}}, \tag{53}$$

where $\eta_{\perp}$ is the boundary critical exponent of the single particle correlation function. Substituting the above relation into (52) we readily obtain, using (33) and (37),

$$G(\omega)\sim\omega^{\theta},\quad \theta=2(\eta_{\perp}-1)=\frac{2}{Z^2(\Lambda)}-2. \tag{54}$$

This equation describes tunneling through an impurity between two quantum wires. To compare again to Fabrizio and Gogolin, we also can consider tunneling between a normal metal and a quantum wire. The current-current correlation function and the tunneling exponent θ then are given by

$$\langle[J(t),J(t'')\rangle\sim(t-t'')^{-\eta_{\perp}-1} \tag{55}$$

and

$$\theta=\eta_{\perp}-1=\frac{1}{Z^2(\Lambda)}-1, \tag{56}$$

respectively. The spinless Luttinger liquid[15] has $\theta=K^{-1}-1$ which is again consistent with the identification $K=Z^2(\Lambda)$ suggested above via the Friedel oscillation exponent. At finite but very low temperatures $T\sim 0$, the conductance behaves as

$$G(T)=G_0T^{\theta}, \tag{57}$$

where G_0 is a constant.

At $U=0$, H describes free fermions with a barrier. In this case, $\eta_{\perp}=1$ and $G(T)$ is independent of temperature and finite. The system is marginal. For $U>0$, $\eta_{\perp}>1$ and the conductance tends to zero as $T\to 0$. The fermion-barrier scattering is relevant and the ''Coulomb blockade'' behavior arises — a result consistent with the observations of Kane and Fisher.[16] For $U<0$, $\eta_{\perp}<1$ and the tunneling conductance diverges as $T\to 0$. This is a consequence of the divergent superconducting fluctuations found in that situation.

V. MULTICOMPONENT INTEGRABLE MODELS

Recently much attention has been focused on the open boundary problem for integrable models with multicomponent fields.[32–34] Typical models are the one-dimensional δ-potential Fermi gas model,[35] the Hubbard chain,[36] and the supersymmetric $t-J$ model with open boundaries.[32,33] The above discussion can also be generalized to these models. In these cases, the reflection Bethe-ansatz equations take the general form

$$2Lp^{\alpha}_0(\lambda^{\alpha}_j)=2\pi I^{\alpha}_j-\sum_{\beta=1}^{M}{\sum_{l=1}^{N_\beta}}'\,[\Phi_{\alpha\beta}(\lambda^{\alpha}_j-\lambda^{\beta}_l)+\Phi_{\alpha\beta}(\lambda^{\alpha}_j+\lambda^{\beta}_l)], \tag{58}$$

where $\Phi_{\alpha\beta}(\lambda^{\alpha}_j-\lambda^{\beta}_l)$ are the bare scattering phases and odd functions of their arguments, M is the number of the components, and the prime after the sums means that when $\alpha=\beta$, $j\neq l$. The eigenvalue of the Hamiltonian is

$$E=\sum_{\alpha=1}^{M}\sum_{j=1}^{N_\alpha}\epsilon_0^\alpha(\lambda_j^\alpha). \tag{59}$$

Also, $\lambda_j^\alpha>0$ is supposed. In complete analogy to Eqs. (23)–(25) and (31), dressed quantities are defined as

$$\rho_b^\alpha(\lambda^\alpha)=\frac{p_0^{\alpha\prime}(\lambda^\alpha)}{\pi}+\sum_{\beta=1}^{M}\int_0^{\Lambda_\beta}K_{\alpha\beta}(\lambda^\alpha,\lambda^\beta)\rho_b^\beta(\lambda^\beta)d\lambda^\beta, \tag{60}$$

$$p_b^\alpha(\lambda^\alpha)=p_0^\alpha(\lambda^\alpha)+\frac{1}{2}\sum_{\beta=1}^{M}\int_0^{\Lambda_\beta}[\Phi_{\alpha\beta}(\lambda^\alpha-\lambda^\beta)+\Phi_{\alpha\beta}(\lambda^\alpha+\lambda^\beta)]\rho_b^\beta(\lambda^\beta)d\lambda^\beta, \tag{61}$$

$$\epsilon_b^\alpha(\lambda^\alpha)=\epsilon_0^\alpha(\lambda^\alpha)+\sum_{\beta=1}^{M}\int_0^{\Lambda_\beta}K_{\alpha\beta}(\lambda^\alpha,\lambda^\beta)\epsilon_b^\beta(\lambda^\beta)d\lambda^\beta, \tag{62}$$

$$Z_{\alpha\beta}^b(\lambda^\beta)=\delta_{\alpha\beta}+\sum_{\gamma=1}^{M}\int_0^{\Lambda_\gamma}Z_{\alpha\gamma}^b(\lambda^\gamma)K_{\gamma\beta}(\lambda^\gamma,\lambda^\beta)d\lambda^\gamma, \tag{63}$$

where $K_{\alpha\beta}(\lambda^\alpha,\lambda^\beta)=(1/2\pi)[\Phi_{\alpha\beta}{}'(\lambda^\alpha-\lambda^\beta)+\Phi_{\alpha\beta}{}'(\lambda^\alpha+\lambda^\beta)]$ is an even function. The relations

$$\frac{\partial}{\partial\lambda^\alpha}p_b^\alpha(\lambda^\alpha)=\pi\rho_b^\alpha(\lambda^\alpha)=2\pi\rho_\alpha(\lambda^\alpha), \tag{64}$$

$$\epsilon_b^\alpha(\lambda^\alpha)=\epsilon_\alpha(\lambda^\alpha), \tag{65}$$

$$Z_{\alpha\beta}^b(\lambda^\beta)=Z_{\alpha\beta}(\lambda^\beta), \tag{66}$$

for $\lambda^\alpha>0$, compare bounded and periodic systems at equal density and generalize (26). The finite size correction to the energies of the excited states is then

$$E_{b,L}-E_{b,L}^0=\frac{\pi}{L}\sum_{\alpha=1}^{M}v_\alpha\left\{\frac{1}{2}[(\mathbf{Z^{-1}}\Delta N)_\alpha]^2+\Delta I_\alpha\right\}+o\left(\frac{1}{L}\right), \tag{67}$$

where $Z_{\alpha\beta}=Z_{\alpha\beta}(\Lambda_\beta)$, $\mathbf{\Delta N}=\{\Delta N_1,\ldots,\Delta N_M\}$ are M-dimensional vectors with integer components. The number ΔN_α gives the change of N_α, the number of pseudoparticles of type α (pseudoparticles refers to the particlelike excitations in the Bethe-ansatz and not necessarily to physical particles), in the excited state with respect to the ground state (i.e., the charge excitations). The non-negative integers ΔI_α describe pseudoparticle-pseudohole excitations [more precisely, a change of $\Sigma_{j=1}^{N\alpha}p_b^\alpha(\lambda_j^\alpha)$ in units of (π/L)] in the vicinity k_F^α [Fermi momenta k_F^α of the pseudoparticles are defined as $k_F^\alpha=p_b^\alpha(\Lambda_\alpha)=\pi n_\alpha$]. The Fermi velocity is

$$v_\alpha=\frac{\epsilon_\alpha{}'(\Lambda_\alpha)}{2\pi\rho_\alpha(\Lambda_\alpha)}$$

as usual.

The finite size correction of the ground state energy is given by

$$E_{b,L}^0=\frac{L}{\pi}\sum_{\alpha=1}^{M}\int_0^{\Lambda_\alpha}p_0^{\alpha\prime}(\lambda^\alpha)\epsilon_b^\alpha(\lambda^\alpha)d\lambda^\alpha+f_b-\frac{\pi}{24L}\sum_{\alpha=1}^{M}v_\alpha+o\left(\frac{1}{L}\right). \tag{68}$$

The Fermi velocities v_α are arbitrary in principle and quantitatively depend on details of the interactions in practice. As a consequence, the system is described by a sum of M conformal algebras, each with a central charge 1. Their contributions to the boundary critical exponents

$$2x_b^\alpha=[(\mathbf{Z^{-1}}\Delta N)_\alpha]^2+2\Delta I_\alpha, \tag{69}$$

are additive, and the total boundary critical exponent is thus

$$2x_b=2\sum_{\alpha=1}^{M}x_b^\alpha=2\sum_{\alpha=1}^{M}\Delta I_\alpha+(\mathbf{Z^{-1}}\Delta N)^T(\mathbf{Z^{-1}}\Delta N). \tag{70}$$

The same structure is found in periodic systems.

As an example, we give some leading boundary critical exponents of the Hubbard chain with open boundaries. The bulk critical exponents of this model were determined by Frahm and Korepin.[14] The Hamiltonian reads

$$H=-\sum_{i=1}^{N-1}\sum_{\sigma=\pm}C_{i\sigma}^\dagger C_{i+1\sigma}+4U\sum_{i=1}^{N}n_{i\uparrow}n_{i\downarrow}-\mu\sum_{i=1}^{N}\sum_{\sigma=\pm}n_{i\sigma}-\frac{h}{2}\sum_{i=1}^{N}(n_{i\uparrow}-n_{i\downarrow}), \tag{71}$$

where $C_{i\sigma}$ ($C_{i\sigma}^\dagger$) is the electron annihilation (creation) operator; μ denotes the chemical potential and h is the external magnetic field. The wave functions are parametrized by two sets of parameters $\{k\}$ and $\{\lambda\}$, the rapidities of the charges and spins, respectively. The following set of integral equations determine their bare ($\epsilon_0^{c,s}$) and dressed ($\epsilon_b^{c,s}$) energies and distribution functions ($\rho_b^{c,s}$)

$$\epsilon_0^c(k)=-2\cos k+\mu-\frac{h}{2},\quad \epsilon_0^s(\lambda)=\frac{h}{2},$$

$$\epsilon_b^c(k)=\epsilon_0^c(k)+\int_0^{\Lambda_s}K_1(\sin k,\lambda)\epsilon_b^s(\lambda)d\lambda,$$

$$\epsilon_b^s(\lambda)=\epsilon_0^s(\lambda)+\int_0^{\Lambda_c}\cos kK_1(\lambda,\sin k)\epsilon_b^c(k)dk-\int_0^{\Lambda_s}K_2(\lambda,\mu)\epsilon_b^s(\mu)d\mu, \tag{72}$$

$$\rho_b^c(k)=\frac{1}{\pi}+\cos k\int_0^{\Lambda_s}K_1(\sin k,\lambda)\rho_b^s(\lambda)d\lambda,$$

$$\rho_b^s(\lambda)=\int_0^{\Lambda_c}K_1(\lambda,\sin k)\rho_b^c(k)dk-\int_0^{\Lambda_s}K_2(\lambda,\mu)\rho_b^s(\mu)d\mu. \tag{73}$$

The dressed-charge (31), in the multicomponent problem, takes a matrix structure, with elements

$$Z^b_{cc}(k)=1+\int_0^{\Lambda_s} Z^b_{cs}(\lambda)K_1(\lambda,\sin k)d\lambda,$$

$$Z^b_{cs}(\lambda)=\int_0^{\Lambda_c}\cos k Z^b_{cc}(k)K_1(\sin k,\lambda)dk - \int_0^{\Lambda_s} Z^b_{cs}(\mu)K_2(\mu,\lambda)d\mu,$$

$$Z^b_{sc}(k)=\int_0^{\Lambda_s} Z^b_{ss}(\lambda)K_1(\lambda,\sin k)d\lambda, \tag{74}$$

$$Z^b_{ss}(\lambda)=1+\int_0^{\Lambda_c}\cos k Z_{sc}(k)K_1(\sin k,\lambda)dk - \int_0^{\Lambda_s} Z^b_{ss}(\mu)K_2(\mu,\lambda)d\mu,$$

with the kernel

$$K_n(\lambda,\mu)=\frac{1}{\pi}\left[\frac{nU}{(nU)^2+(\lambda-\mu)^2}+\frac{nU}{(nU)^2+(\lambda+\mu)^2}\right], \quad n=1,2. \tag{75}$$

As in the single-component case, the dressed-charge matrices for open and periodic boundary conditions are identical, $Z^b_{\alpha,\beta}=Z_{\alpha,\beta}$. The Fermi velocities are given by

$$v_c=\frac{\epsilon_b^{c\prime}(\Lambda_c)}{\pi\rho_b^c(\Lambda_c)}, \quad v_s=\frac{\epsilon_b^{s\prime}(\Lambda_s)}{\pi\rho_b^s(\Lambda_s)}, \tag{76}$$

and $\Lambda_{c,s}$ are defined by $\epsilon_b^{c,s}(\Lambda_{c,s})=0$.

Below we list some correlation functions and the $\Delta N_{c,s}$, $\Delta I_{c,s}$ which must be used in Eqs. (69) and (70) in order to determine the leading critical exponents:

(1) The field correlator

$$G_{\Psi\Psi}(x_1,x_2,t)=\langle C_{x_1\uparrow}(t)C^\dagger_{x_2\uparrow}(0)\rangle,$$

$$\Delta N_c=1, \quad \Delta N_s=0, \quad \Delta I_c=\Delta I_s=0. \tag{77}$$

(2) The density-density correlator

$$G_{nn}(x_1,x_2,t)=\langle n_{x_1}(t)n_{x_2}(0)\rangle, \tag{78}$$

$$\Delta N_c=\Delta N_s=0, \quad \Delta I_c=1,\Delta I_s=0$$

or

$$\Delta I_c=0,\Delta I_s=1.$$

(3) The spin-spin correlators

$$G^z_{\sigma\sigma}(x_1,x_2,t)=\langle S^z(x_1,t)S^z(x_2,0)\rangle, \tag{79}$$

$$S^z(x,t)=\tfrac{1}{2}[n_{x\uparrow}-n_{x\downarrow}],$$

$$\Delta N_c=\Delta N_s=0, \quad \Delta I_c=1, \quad \Delta I_s=0$$

or

$$\Delta I_c=0, \quad \Delta I_s=1,$$

$$G^\perp_{\sigma\sigma}(x_1,x_2,t)=\langle S^-(x_1,t)S^+(x_2,0)\rangle, \tag{80}$$

$$S^+(x,t)=C^\dagger_{x\uparrow}(t)C_{x\downarrow}(t),$$

$$\Delta N_c=0, \quad \Delta N_s=1, \quad \Delta I_c=\Delta I_s=0.$$

(4) The triplet pair correlator

$$G^{(1)}_p(x_1,x_2,t)=\langle C_{x_1+1\uparrow}(t)C_{x_1\uparrow}(t)C^\dagger_{x_2\uparrow}(0)C^\dagger_{x_2+1\uparrow}(0)\rangle, \tag{81}$$

$$\Delta N_c=2, \quad \Delta N_s=0, \quad \Delta I_c=\Delta I_s=0.$$

(5) The singlet pair correlator

$$G^{(0)}_p(x_1,x_2,t)=\langle C_{x_1\uparrow}(t)C_{x_1\downarrow}(t)C^\dagger_{x_2\downarrow}(0)C^\dagger_{x_2\uparrow}(0)\rangle, \tag{82}$$

$$\Delta N_c=2, \quad \Delta N_s=1, \quad \Delta I_c=\Delta I_s=0.$$

Precise values for the critical exponents then follow immediately, via (69) and (70), once the dressed-charge matrix (74) is calculated. This is a matter of routine, and due to the equality of this matrix for open and periodic systems (cf. above), the published results for the periodic Hubbard model[14] can be used directly to evaluate the boundary critical expoenents.

VI. SUMMARY

We have derived explicitly the boundary critical exponents of both single-component and multicomponent Bethe-ansatz soluble models of interacting bosons and fermions. Our results imply that the descendant fields (particle-hole excitations) contribute the same (integer) amount to the boundary and the bulk critical exponents. However, the contribution from charge excitations (additional particles) to the boundary critical exponents is twice as big as to the bulk exponents. The current excitations are completely depressed for open boundaries and thus contribute nothing to the boundary critical exponents. Apparently, this statement is valid much beyond the the Bethe-ansatz soluble models and applies in general to Luttinger liquids with open boundaries.[15] The critical exponents are determined by the dressed-charge matrix which we have shown to be independent of the boundary conditions. Moreover, our method of calculation relies only on the determination of energies which can be performed accurately by numerical methods in models which cannot be solved by Bethe-ansatz. Therefore, one can determine, at least numerically, the boundary critical exponents for all 1D quantum systems, provided they are conformally invariant, by the method described in this paper.

Recently, we have learned that Affleck, Eggert, and Sorensen have compared conformal field theory predicitions with numerical results for the $S=1/2$ Heisenberg chain with

general boundary conditions[37] and with Bethe-ansatz results for an $S=1$-impurity in a $S=1/2$ Heisenberg chain.[38] Moreover, Fujimoto and Kawakami have studied the boundary critical exponents of the Kondo problem[39] and produced a similar solution of the boundary critical exponents of multicomponent fermion models.[40]

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