Algebraic Bethe ansatz for the supersymmetric U model

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We present an algebraic Bethe ansatz for the supersymmetric U model for correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\otimes_{L}^{n=1}C^4$ (where L is the lattice length). The supersymmetry algebra of the model is the Lie superalgebra $gl(2|1)$ and contains one symmetry-preserving free real parameter which is the Hubbard interaction parameter U. The parameter U arises from the one-parameter family of inequivalent typical four-dimensional irreps of $gl(2|1)$. Eigenstates of the model are determined by the algebraic Bethe ansatz on a one-dimensional periodic lattice. [S0163-1829(96)03232-8]

I. INTRODUCTION

Solutions to the Yang-Baxter equation provide a well-known method for the construction of integrable models through the *quantum inverse scattering method* (QISM).[1,2] Supersymmetric generalizations have attracted considerable interest recently for their possible application to correlated electron systems. The supersymmetric Yang-Baxter equation was first studied in the work of Kulish and Sklyanin.[3] Using this approach Essler and Korepin[4] studied the *nested algebraic Bethe ansatz* (NABA) for the solution of the supersymmetric t-J model in one-dimension, using a $gl(2|1)$ invariant R matrix (see, also, Ref. 5). By adopting the QISM, models describing systems of correlated electrons have since been proposed. A q deformation of the supersymmetric t-J model has been studied in Refs. 6 and 7 using a $U_q[gl(2|1)]$ invariant R matrix. The *Bethe ansatz equations* (BAE) for the model on an open chain were obtained using the NABA. For the $gl(2|2)$ invariant case, Essler, Korepin, and Schoutens[8] derived a supersymmetric extended Hubbard model which was later shown to possess superconductive properties.[9–11] The q deformation of this model has recently been obtained.[12] Supersymmetric models based on the $osp(m|2n)$ algebras, which give rise to representations of the Birman-Wenzl-Murakami algebra, have been treated in Ref. 13. The BAE for the $osp(1|2)$ and $osp(2|2)$ cases have been obtained in Refs. 14 and 15, respectively.

The supersymmetric U model[18] is also an example of a correlated electron model which is integrable in one-dimension obtained through the QISM. It is a supersymmetric generalization of the Hubbard model with additional correlated hopping interaction terms. The supersymmetry algebra of the model is the Lie superalgebra $gl(2|1)$, which is also the supersymmetry algebra of the integrable t-J model. The solution of this model has been studied in Ref. 19 through the use of the coordinate Bethe ansatz. Here, we will derive the results of Ref. 19 by using the algebraic Bethe ansatz. This approach has been considered in Ref. 20 in terms of an abstract $gl(2|1)\cong osp(2|2)$ dynamical system. However, several technical aspects of the derivation of the results of Ref. 20 were not given. We will show that by employing the Yangian description of $gl(2|1)$ developed in Ref. 4 for the solution of the supersymmetric t-J model, we can obtain the Bethe ansatz equations for the supersymmetric U model. A q deformation for this model has also been studied in Refs. 16 and 17.

We now introduce some notation as in Ref. 18. Electrons on a lattice are described by canonical Fermi operators $c_{i,\sigma}$ and $c_{i,\sigma}^{\dagger}$ satisfying the anticommutation relations given by $\{c_{i,\sigma}^{\dagger},c_{j,\tau}\}=\delta_{ij}\delta_{\sigma\tau}$, where $i,j,=1,2,..,L$ and $\sigma,\tau=\uparrow,\downarrow$. The operator $c_{i,\sigma}$ annihilates an electron of spin σ at site i, which implies that the Fock vacuum $|0\rangle$ satisfies $c_{i,\sigma}|0\rangle=0$. At a given lattice site i, there are four possible electronic states:

$$|0\rangle,\ \ |\uparrow\rangle_i=c_{i,\uparrow}^{\dagger}|0\rangle,\ \ |\downarrow\rangle_i=c_{i,\downarrow}^{\dagger}|0\rangle,\ \ |\uparrow\downarrow\rangle_i=c_{i,\downarrow}^{\dagger}c_{i,\uparrow}^{\dagger}|0\rangle.$$

By $n_{i,\sigma}=c_{i,\sigma}^{\dagger}c_{i,\sigma}$, we denote the number operator for electrons with spin σ on site i, and we write $n_i=n_{i,\uparrow}+n_{i,\downarrow}$. The Hamiltonian for this model on a general d-dimensional lattice is given by

$$\begin{aligned}H=&-\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c_{i,\sigma}^{\dagger}c_{j,\sigma}+c_{j,\sigma}^{\dagger}c_{i,\sigma})\\&+U\sum_{\langle i,j\rangle}[(n_{i,\uparrow}-\tfrac{1}{2})(n_{i,\downarrow}-\tfrac{1}{2})+(n_{j,\uparrow}-\tfrac{1}{2})(n_{j,\downarrow}-\tfrac{1}{2})]\\&+\frac{U}{2}\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c_{i,\sigma}^{\dagger}c_{i,-\sigma}^{\dagger}c_{j,-\sigma}c_{j,\sigma}+\text{H.c.})\\&+(1+\sqrt{U+1})\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c_{i,\sigma}^{\dagger}c_{j,\sigma}+c_{j,\sigma}^{\dagger}c_{i,\sigma})\\&\times(n_{i,-\sigma}+n_{j,-\sigma})-(U+2+2\sqrt{U+1})\\&\times\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}(c_{i,\sigma}^{\dagger}c_{j,\sigma}+c_{j,\sigma}^{\dagger}c_{i,\sigma})n_{i,-\sigma}n_{j,-\sigma}\\&+\frac{2+U}{2}\sum_{\langle i,j\rangle}(n_i+n_j),\end{aligned}\tag{1}$$

where $\langle i,j\rangle$ denotes nearest-neighbor links on the lattice. The Hamiltonian contains the hopping term for electrons and an on-site interaction term for electron pairs (coupling U). The supersymmetry algebra underlying this model is $gl(2|1)$ and U, contained as a free parameter, does not affect the supersymmetry. Here, we restrict U to the range $U>-1$. The Hamiltonian is invariant under spin reflection.

The Hamiltonian may be obtained from the R matrix for the one-parameter family of the inequivalent typical four-dimensional irreps, which is afforded by the $gl(2|1)$ module W with the highest weight $(0,0|\alpha)$. For $\alpha>0$ or $\alpha<-1$, the module W is unitary and thus the tensor product $W\otimes W$ is completely reducible and $\alpha=U^{-1}$. We write $W\otimes W=W_1\oplus W_2\oplus W_3$, where W_1, W_2, and W_3 are $U_q[gl(2|1)]$ modules with the highest weights $(0,0|2\alpha)$, $(0,-1|2\alpha+1)$ and $(-1,-1|2\alpha+2)$, respectively. Let P_k, $k=1,2,3$ be the projection operator from $W\otimes W$ onto W_k. The rational R matrix, which satisfies the quantum Yang-Baxter equation, was given in Ref. 18 in the form

$$\check{R}(\theta)=\frac{\theta-2\alpha}{\theta+2\alpha}P_1+P_2+\frac{-(\theta+2\alpha+2)}{\theta-2\alpha-2}P_3.$$

Then the local Hamiltonian is given by[21]

$$H_{i,i+1}(\alpha)=\frac{d}{d\theta}\check{R}_{i,i+1}(\theta)\Big|_{\theta=0},$$

and the global Hamiltonian H is solvable by means of the QISM. The nature of $gl(2|1)$ allows us to replace the auxiliary space W with the vector representation space V, which is only three-dimensional and thus simplifies the calculation of the NABA.

The paper is set out as follows. The graded quantum inverse scattering method will be discussed in Sec. II. The use of the QISM enables us to obtain expressions for an infinite number of higher conservation laws at the quantum level. These conserved charges are of interest because physical interactions are not generally well approximated by interactions involving only nearest neighbors.[4] Section IV will be the construction of the algebraic Bethe ansatz for the model. We formulate a set of simultaneous eigenstates of the transfer matrix using a NABA. (See in Ref. 4 that due to the grading there are three choices of R matrix describing the same system, but these all lead to equivalent forms of the NABA.) The expression obtained for the BAE will be compared with those given in Ref. 19.

II. GRADED QUANTUM INVERSE SCATTERING METHOD

We will construct the eigenstates of the Hamiltonian of the one-dimensional supersymmetric model above, using the QISM. The supersymmetry of the model requires a modification of the QISM. We use the R matrix satisfying the graded Yang-Baxter equation and introduce an L operator constructed directly from the R matrix of the twisted representation.

The graded Yang-Baxter equation can be written as the operator equation:[3]

$$\begin{aligned}R_{\alpha_1\beta_1,\alpha_2\beta_2}(\theta-\theta')L(\theta)_{\beta_1\gamma_1ab}L(\theta')_{\beta_2\gamma_2bc}(-1)^{\beta_2(\beta_1+\gamma_1)}\\=L(\theta')_{\alpha_2\beta_2ab}L(\theta)_{\alpha_1\beta_1bc}(-1)^{\beta_2(\alpha_1+\beta_1)}\\\times R_{\beta_1\gamma_1,\beta_2\gamma_2}(\theta-\theta'),\end{aligned}\qquad(2)$$

acting on the spaces $V\otimes V\otimes W$, where V is the vector module and W is a four-dimensional module of inequivalent irreps. Greek indices are used to label the matrix spaces, that is the first two spaces and the Roman indices label the quantum space, which is the third space. The quantum space represents the Hilbert space over a site on the one-dimensional lattice. The R matrix acts in the matrix spaces and it is between the two matrix spaces that the graded tensor product acts.

The R matrix acts on $V\otimes V$ and is easily obtained following Refs. 4 and 22,

$$R(\theta)=b(\theta)P+a(\theta)I,$$

where $a(\theta)=-[\theta/(\theta-2)]$ and $b(\theta)=-[2/(\theta-2)]$, which can be seen to satisfy the Yang-Baxter equation. The L operator is constructed in the next section.

III. THE L OPERATOR

The L operator will be constructed from $V\otimes W$ representation where as before V denotes the vector module and W corresponds to the one-parameter family of the inequivalent typical four-dimensional irreps. The weights for module V are $(1,0|0)$, $(0,1|0)$, $(0,0|1)$, with corresponding weight basis $|1\rangle$, $|2\rangle$, and $|3\rangle$, respectively. On this module, the $gl(2|1)$ generators act as $E^i_j=e^i_j$. We choose the grading for module V to be

$$[1]=[2]=0,\quad [3]=1.$$

The weights for module W are $(0,0|\alpha)$, $(0,-1|\alpha+1)$, $(-1,0|\alpha+1)$, and $(-1,-1|\alpha+2)$, respectively, with basis vectors $|a\rangle$, $|b\rangle$, $|c\rangle$, $|d\rangle$. The $gl(2|1)$ generators act as

$$\begin{aligned}E^1_1&=-e_{cc}-e_{dd},\\E^2_2&=-e_{bb}-e_{dd},\\E^3_3&=\alpha e_{aa}+(\alpha+1)(e_{bb}+e_{cc})+(\alpha+2)e_{dd},\\E^1_2&=e_{bc},\\E^2_1&=e_{cb},\\E^1_3&=-\sqrt{\alpha}e_{ac}+\sqrt{\alpha+1}e_{bd},\\E^3_1&=-\sqrt{\alpha}e_{ca}+\sqrt{\alpha+1}e_{db},\\E^2_3&=\sqrt{\alpha}e_{ab}+\sqrt{\alpha+1}e_{cd},\\E^3_2&=\sqrt{\alpha}e_{ba}+\sqrt{\alpha+1}e_{dc}.\end{aligned}\qquad(3)$$

We choose the grading for module W to be

$$[a]=[d]=0,\quad [b]=[c]=1.$$

The tensor product decomposition is $V\otimes W=V_1\oplus V_2$, where V_1 has highest weight $(1,0|\alpha)$ and V_2 has highest weight $(0,0|\alpha+1)$. Applying the Baxterization procedure[23] gives the $\check{R}$ matrix for this $V\otimes W$ representation as

$$\check{R}(\theta)=\check{P}_1\frac{2-\theta+\alpha}{2+\theta+\alpha}+\check{P}_2.$$

In the above the $\check{P}_i$ are $gl(2|1)$-invariant operators $\check{P}_i:V\otimes W\to W\otimes V$. We define L operator as

$$L(\theta)=P\check{R}(\theta)=\frac{2-\theta+\alpha}{2+\theta+\alpha}P_1-P_2,$$

where P_1,P_2 are projectors and $P_1=P\check{P}_1$ and $P_2=-P\check{P}_2$. We construct these projectors in the following way.

The coproduct is defined by

$$\Delta(x)=x\otimes 1+1\otimes x,\quad \forall x\in gl(2|1).$$

Symmetry adapted orthonormal bases may be expressed in terms of $|\phi_\alpha^1\rangle$ and $|\phi_\beta^1\rangle$ for P_1 and P_2, respectively, with α=1,..8, β=1,..4. The basis for V_2 is given by

$$|\Phi_4^2\rangle=|3\rangle\otimes|d\rangle,$$

$$|\Phi_3^2\rangle=|2\rangle\otimes|d\rangle-\sqrt{\alpha+1}|3\rangle\otimes|c\rangle,$$

$$|\Phi_2^2\rangle=|1\rangle\otimes|d\rangle-\sqrt{\alpha+1}|3\rangle\otimes|b\rangle,$$

$$|\Phi_1^2\rangle=|1\rangle\otimes|c\rangle-|2\rangle\otimes|b\rangle+\sqrt{\alpha}|3\rangle\otimes|a\rangle. \tag{4}$$

So we may express

$$P_2=\sum|\Phi_\beta^2\rangle\langle\Phi_\beta^2|,$$

where β=1, . . . ,4. We find P_2 to be given by

$$\begin{aligned}(\alpha+2)P_2=&e_{11}\otimes(e_{dd}+e_{cc})+e_{22}\otimes(e_{bb}+e_{dd})\\&+e_{33}\otimes[(\alpha+1)(e_{bb}+e_{cc})+\alpha e_{aa}+(\alpha+2)e_{dd}]\\&-e_{12}\otimes e_{cb}-e_{21}\otimes e_{bc}\\&+e_{13}\otimes(\sqrt{\alpha+1}e_{db}-\sqrt{\alpha}e_{ca})\\&+e_{31}\otimes(\sqrt{\alpha}e_{ac}-\sqrt{\alpha+1}e_{bd})\\&+e_{23}\otimes(\sqrt{\alpha+1}e_{dc}+\sqrt{\alpha}e_{ba})\\&-e_{32}\otimes(\sqrt{\alpha}e_{ab}+\sqrt{\alpha+1}e_{cd}).\end{aligned} \tag{5}$$

This can be more easily read when expressed as

$$(\alpha+2)P_2=\begin{pmatrix}-E_1^1 & -E_1^2 & -E_1^3\\ -E_2^1 & -E_2^2 & -E_2^3\\ -E_3^1 & -E_3^2 & E_3^3\end{pmatrix}, \tag{6}$$

where to accommodate the grading, we make the transformation

$$e_j^i\rightarrow(-1)^{[j]([i]+[j])}e_j^i,$$

and here E^i_j is understood to denote the matrix representative acting on W. It can be easily seen that $P_1+P_2=I$. Then with $\lambda=(i/2)(\theta-2-\alpha)$, we have

$$L(\lambda)=\frac{1}{k}\begin{pmatrix}\lambda-iE_1^1 & -iE_1^2 & -iE_1^3\\ -E_2^1 & \lambda-iE_2^2 & -iE_2^3\\ -iE_3^1 & -iE_3^2 & \lambda+iE_3^3\end{pmatrix}$$

up to a normalization constant k. Setting $k=\lambda+i$, we have an L operator of a similar form (up to a change in grading convention) as in Ref. 4 for the integrable t-J model:

$$L(\lambda)=\begin{pmatrix}A(\lambda)-B(\lambda)E_1^1 & -B(\lambda)E_1^2 & -B(\lambda)E_1^3\\ -B(\lambda)E_2^1 & A(\lambda)-B(\lambda)E_2^2 & -B(\lambda)E_2^3\\ -B(\lambda)E_3^1 & -B(\lambda)E_3^2 & A(\lambda)+B(\lambda)E_3^3\end{pmatrix},$$

where

$$A(\lambda)=\frac{\lambda}{\lambda+i},$$

$$B(\lambda)=\frac{i}{\lambda+i}.$$

The similar form of the L operator for this model and the integrable t-J model stems from the fact that they share the same supersymmetry algebra $gl(2|1)$. We write

$$T_L(\theta)=L_L(\theta)L_{L-1}(\theta)\cdots L_1(\theta),$$

$$[T_L(\theta)^{ab}]_{\alpha_1,\beta_1,\ldots,\alpha_L,\beta_L}=L_L(\theta)^{ac_L}_{\alpha_L\beta_L}\cdots L_1(\theta)^{c_2b}_{\alpha_1\beta_1}\times(-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}.$$

We call $T(\theta)$ the monodromy matrix and by construction it fulfills the same intertwining relation as the L operators.

The transfer matrix of the integrable model is given as the supertrace of the monodromy matrix. This operator is given by

$$\tau(\theta)=\mathrm{str}[T(\theta)]=\sum_i(-1)^{[i]}T(\theta)_{ii}.$$

The $\tau(\theta)$ form a one-parameter family of commuting operators. The transfer matrix may be taken as integrals of the motion and we can obtain an infinite number of higher conservation laws of the model. It can be employed to construct exactly solvable models in the usual way.

IV. ALGEBRAIC BETHE ANSATZ WITH BBF GRADING

We use the matrix from the vector representation as our R matrix and the L operator given above for obtaining the defining equations for the algebra constructed from (2). We represent the monodromy matrix in the following way:

$$T_L(\theta)=L_L(\theta)L_{L-1}(\theta)\cdots L_1(\theta)=\begin{bmatrix}T_{11}(\theta) & T_{12}(\theta) & T_{13}(\theta)\\ T_{21}(\theta) & T_{22}(\theta) & T_{23}(\theta)\\ T_{31}(\theta) & T_{32}(\theta) & T_{33}(\theta)\end{bmatrix}. \tag{7}$$

The transfer matrix is given by

$$\tau(\theta')=\mathrm{str}[T_L(\theta')]=T_{11}(\theta')+T_{22}(\theta')-T_{33}(\theta').$$

We take the lowest weight state as the reference state in W and for convenience take out a factor of $[\lambda+i(\alpha+2)]/(\lambda+i)$ from the L matrix. Then the action of $L_k(\theta)$ on the reference state on the kth site is

$$L(\lambda)|0\rangle_k=\begin{pmatrix} \frac{\lambda+i}{\lambda+i(\alpha+2)} & 0 & 0 \\ 0 & \frac{\lambda+i}{\lambda+i(\alpha+2)} & 0 \\ * & ** & 1 \end{pmatrix}|0\rangle_k .$$

* and ** represent complicated values that are not necessary to evaluate. Substituting $\lambda=(i/2)(\theta-\alpha-2)$, we find that the action of the monodromy matrix on the reference state is given by

$$T_L(\theta)|0\rangle=\begin{pmatrix} \left[\frac{\theta-\alpha}{\theta+2+\alpha}\right]^L & 0 & 0 \\ 0 & \left[\frac{\theta-\alpha}{\theta+2+\alpha}\right]^L & 0 \\ T_{31}(\theta) & T_{32}(\theta) & 1 \end{pmatrix}|0\rangle. \quad (8)$$

We construct a set of eigenstates of the transfer matrix using the technique of the NABA. The creation operators are $T_{31}(\theta)$, $T_{32}(\theta)$ due to the choice of reference state. Thus, we use the following for the ansatz for the eigenstates of $\tau(\theta')$:

$$|\theta_1,\ldots,\theta_n|F\rangle=T_{3a_1}(\theta_1)T_{3a_2}(\theta_2)\ldots T_{3a_n}(\theta_n)|0\rangle F^{a_n\cdots a_1}, \quad (9)$$

where indices a_i have values 1 or 2 and $F^{a_n\cdots a_1}$ is a function of the spectral parameters θ_j. The action of these states is determined by the monodromy matrix and the relations (2), which in essence determine the Yangian $Y[gl(2|1)]$. The relations necessary for the construction of the NABA are

$$T_{33}(\theta')T_{3a}(\theta)=-\frac{1}{a(\theta-\theta')}T_{3a}(\theta)T_{33}(\theta')+\frac{b(\theta-\theta')}{a(\theta-\theta')}T_{3a}(\theta')T_{33}(\theta), \quad (10)$$

$$T_{ab}(\theta')T_{3c}(\theta)=\frac{r_{pc,db}(\theta'-\theta)}{a(\theta'-\theta)}T_{3p}(\theta)T_{ad}(\theta')+\frac{b(\theta'-\theta)}{a(\theta'-\theta)}T_{3b}(\theta')T_{ac}(\theta), \quad (11)$$

$$T_{3a_1}(\theta_1)T_{3a_2}(\theta_2)=r_{b_2a_2,b_1a_1}(\theta_1-\theta_2)T_{3b_2}(\theta_2)T_{3b_1}(\theta_1), \quad (12)$$

where

$$r(\theta')=b(\theta')P+a(\theta')I.$$

Since $[1]=[2]=0$, this R matrix is essentially not graded and it can be seen that $r(\theta')$ fulfills a Yang-Baxter equation and can be identified with the R matrix of the spin $\frac{1}{2}$ Heisenberg (XXX) model. The diagonal elements of the monodromy matrix act on the states in the following way:

$$T_{33}(\theta')|\theta_1\ldots\theta_n|F\rangle=(-1)^n\prod_{i=1}^{n}\frac{1}{a(\theta_i-\theta')}|\theta_1\ldots\theta_n|F\rangle+\sum_{k=1}^{n}(\check{\Lambda}_k)^{b_1\ldots b_n}_{a_1\ldots a_n}T_{3b_k}(\theta')\times\prod_{j=1,j\neq k}^{n}T_{3b_j}(\theta_j)|0\rangle F^{a_n\cdots a_1}, \quad (13)$$

$$[T_{11}(\theta')+T_{22}(\theta')]|\theta_1\ldots\theta_n|F\rangle=I(\theta')^L\prod_{j=1}^{n}\frac{1}{a(\theta'-\theta_j)}\prod_{l=1}^{n}T_{3b_l}(\theta_l)|0\rangle\times\tau^{(1)}(\theta')^{b_1\ldots b_n}_{a_1\ldots a_n}F^{a_n\cdots a_1}+\sum_{k=1}^{n}(\Lambda_k)^{b_1\ldots b_n}_{a_1\ldots a_n}T_{3b_k}(\theta')\prod_{j=1,j\neq k}^{n}T_{3b_j}(\theta_j)|0\rangle F^{a_n\cdots a_1}, \quad (14)$$

where

$$I^L(\theta')=\frac{\theta'-\alpha}{\theta'+\alpha+2}$$

and

$$\tau^{(1)}(\theta')^{b_1\ldots b_n}_{a_1\ldots a_n}=\mathrm{str}[T^{(1)}_n(\theta')]. \quad (15)$$

That is,

$$\tau^{(1)}(\theta')^{b_1\ldots b_n}_{a_1\ldots a_n}=\mathrm{str}[L^{(1)}_n(\theta'-\theta_n)L^{(1)}_{n-1}(\theta'-\theta_{n-1})\ldots\times L^{(1)}_2(\theta'-\theta_2)L^{(1)}_1(\theta'-\theta_1)].$$

We have

$$[L^{(1)}_k(\theta)]_{ij}=\sum r_{ik,jl}e_{kl}=\begin{bmatrix} a(\theta)+b(\theta)e^{11}_k & b(\theta)e^{21}_k \\ b(\theta)e^{12}_k & a(\theta)+b(\theta)e^{22}_k \end{bmatrix}. \quad (16)$$

So the operators $L^{(1)}$ and $r(\theta')$ can be interpreted as the L operator and the R matrix of the XXX model. Hence $T^{(1)}_n(\theta')$ and $\tau^{(1)}(\theta')$ are the monodromy and transfer matrices for the corresponding model with inhomogeneities θ_i, $i=1,\ldots,n$. The eigenvalue condition

$$\tau(\theta')|\theta_1\ldots\theta_n|F\rangle=\mu(\theta',\{\theta_j\},F)|\theta_1\ldots\theta_n|F\rangle$$

leads to the requirement that F be an eigenvector of the nested transfer matrix $\tau^{(1)}(\theta')$, and that the unwanted terms, Λ_k, $\check{\Lambda}_k$ cancel. That is,

$$[(\Lambda_k)^{b_1\ldots b_n}_{a_1\ldots a_n}-(\check{\Lambda}_k)^{b_1\ldots b_n}_{a_1\ldots a_n}]F^{a_n\cdots a_1}=0.$$

These values are computed in Appendix A. This leads us to the conditions on the spectral parameters θ_j and coefficients F, necessary for the eigenvalue condition to hold. That is

$$[I(\theta_k)]^{-L}(-1)^n\prod_{i=1,i\neq k}^{n}\frac{a(\theta_k-\theta_i)}{a(\theta_i-\theta_k)}F^{b_n\cdots b_1}=\tau^{(1)}(\theta_k)^{b_1\ldots b_n}_{a_1\ldots a_n}F^{a_n\cdots a_1}, \quad k=1,\ldots,n. \quad (17)$$

The next step in the NABA is to solve the nesting. The condition that F be an eigenvector of $\tau^{(1)}(\theta')$ requires the diagonalization of $\tau^{(1)}(\theta')$, which can be achieved by performing a second, nested Bethe ansatz. We write the monodromy and transfer matrices as follows:

$$T_n^{(1)}(\theta')=\begin{bmatrix} T_{11}^{(1)}(\theta') & T_{12}^{(1)}(\theta') \\ T_{21}^{(1)}(\theta') & T_{22}^{(1)}(\theta') \end{bmatrix},$$

$$\tau^{(1)}(\theta')=T_{11}^{(1)}(\theta')+T_{22}^{(1)}(\theta'). \tag{18}$$

Obtaining as before the equations from the relation (2) necessary for the NABA, we have

$$T_{11}^{(1)}(\theta')T_{21}^{(1)}(\theta)=\frac{a(\theta'-\theta)+b(\theta'-\theta)}{a(\theta'-\theta)}\,T_{21}^{(1)}(\theta)T_{11}^{(1)}(\theta') - \frac{b(\theta'-\theta)}{a(\theta'-\theta)}\,T_{21}^{(1)}(\theta')T_{11}^{(1)}(\theta), \tag{19}$$

$$T_{22}^{(1)}(\theta')T_{21}^{(1)}(\theta)=\frac{a(\theta-\theta')+b(\theta-\theta')}{a(\theta-\theta')}\,T_{21}^{(1)}(\theta)T_{22}^{(1)}(\theta') - \frac{b(\theta-\theta')}{a(\theta-\theta')}\,T_{21}^{(1)}(\theta')T_{22}^{(1)}(\theta), \tag{20}$$

$$T_{21}^{(1)}(\theta)T_{21}^{(1)}(\theta')=T_{21}^{(1)}(\theta')T_{21}^{(1)}(\theta).$$

For the reference states, we choose

$$|0\rangle_k^{(1)}=\begin{bmatrix}0\\1\end{bmatrix}, \quad |0\rangle^{(1)}=\otimes_{k=1}^{n}|0\rangle_k^{(1)}.$$

The action of the nested monodromy matrix $T^{(1)}(\theta')$ on the reference state is

$$T_{11}^{(1)}(\theta')|0\rangle^{(1)}=\prod_{j=1}^{n} a(\theta'-\theta_j)|0\rangle^{(1)},$$

$$T_{22}^{(1)}(\theta')|0\rangle^{(1)}=\prod_{j=1}^{n}[a(\theta'-\theta_j)+b(\theta'-\theta_j)]|0\rangle^{(1)} = -\prod_{j=1}^{n}\frac{a(\theta'-\theta_j)}{a(\theta_j-\theta')}\,|0\rangle^{(1)}. \tag{21}$$

We choose the following ansatz for the eigenstates of $\tau^{(1)}(\theta')$:

$$|\theta_1^{(1)},\ldots,\theta_{n_1}^{(1)}\rangle=T_{21}^{(1)}(\theta_1^{(1)}),\ldots,T_{21}^{(1)}(\theta_{n_1}^{(1)})|0\rangle^{(1)}.$$

These states can be related to the coefficients $F^{a_n\cdots a_1}$ by noting that the state $|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle$ exists on a lattice of n sites and is thus an element of a direct product over n two-dimensional Hilbert spaces. The action of $\tau^{(1)}(\theta')$ on the states is computed as before from the relations (2). We obtain

$$T_{22}^{(1)}(\theta')|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle=\prod_{i=1}^{n_1}\frac{[a(\theta_i^{(1)}-\theta')+b(\theta_i^{(1)}-\theta')]}{a(\theta_i^{(1)}-\theta')}\times\prod_{j=1}^{n}[a(\theta'-\theta_j)+b(\theta'-\theta_j)]\times|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle+\sum_{k=1}^{n_1}(\check{\Lambda}_k)^{(1)}\times T_{21}^{(1)}(\theta')\prod_{j=1,j\neq k}^{n_1}T_{21}^{(1)}(\theta_j)|0\rangle^{(1)}, \tag{22}$$

$$T_{11}^{(1)}(\theta')|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle=\prod_{i=1}^{n_1}\frac{[a(\theta'-\theta_i^{(1)})+b(\theta'-\theta_i^{(1)})]}{a(\theta'-\theta_i^{(1)})}\times\prod_{j=1}^{n}a(\theta'-\theta_j)|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle+\sum_{k=1}^{n_1}(\Lambda_k)^{(1)}T_{21}^{(1)}(\theta')\times\prod_{j=1,j\neq k}^{n_1}T_{21}^{(1)}(\theta_j)|0\rangle^{(1)}. \tag{23}$$

The eigenvalues for $\tau^{(1)}(\theta')$ are found to be

$$\tau^{(1)}(\theta')|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle = \left[\prod_{i=1}^{n_1}\frac{[a(\theta'-\theta_i^{(1)})+b(\theta'-\theta_i^{(1)})]}{a(\theta'-\theta_i^{(1)})}\prod_{j=1}^{n}a(\theta'-\theta_j) + \prod_{i=1}^{n_1}\frac{[a(\theta_i^{(1)}-\theta')+b(\theta_i^{(1)}-\theta')]}{a(\theta_i^{(1)}-\theta')}\prod_{j=1}^{n}[a(\theta'-\theta_j) + b(\theta'-\theta_j)]\right]|\theta_1^{(1)}\ldots\theta_{n_1}^{(1)}\rangle. \tag{24}$$

Inserting this into Eq. (17) for $\theta'=\theta_k$, we have the first of the Bethe equations,

$$I(\theta_k)^L=\prod_{i=1}^{n_1}-a(\theta_k-\theta_i^{(1)}), \quad k=1,\ldots,n.$$

To ensure that we have the eigenstates of the transfer matrix for the nesting, we must have the unwanted terms $\check{\Lambda}_k^{(1)}$, $\Lambda_k^{(1)}$ cancelling and these values are computed in Appendix A. The resulting equation after some simplification is the set of Bethe equations for the nesting as follows:

$$\prod_{j=1,j\neq p}^{n_1}\frac{a(\theta_p^{(1)}-\theta_j^{(1)})}{a(\theta_j^{(1)}-\theta_p^{(1)})}=\prod_{i=1}^{n}\frac{-1}{a(\theta_i-\theta_p^{(1)})}, \quad p=1,\ldots,n_1. \tag{25}$$

n and n_1 can be identified as the total number of electrons (N_e) and the number of spin-down electrons ($N_\downarrow$), respectively. After substitution and simplification, the Bethe equations are as follows:

$$\left[\frac{-\alpha+\theta_k}{\alpha+2+\theta_k}\right]^L=\prod_{i=1}^{N_\downarrow}\frac{-\theta_k+\theta_i^{(1)}}{2-\theta_k+\theta_i^{(1)}},\quad k=1,\ldots,N_e,\quad(26)$$

$$\prod_{j=1,j\neq p}^{N_\downarrow}\frac{-\theta_p^{(1)}+\theta_j^{(1)}-2}{\theta_j^{(1)}-\theta_p^{(1)}+2}=\prod_{k=1}^{N_e}\frac{\theta_p^{(1)}-\theta_k+2}{\theta_p^{(1)}-\theta_k},$$

$$p=1,\ldots,N_\downarrow.\quad(27)$$

The eigenvalues of the transfer matrix are given by

$$\mu(\theta',\theta_j,F)=\left[\frac{-\alpha+\theta'}{2+\alpha+\theta'}\right]^L\prod_{j=1}^{N_e}\frac{2-\theta'+\theta_j}{\theta'-\theta_j}\,\mu^{(1)}(\theta')-\prod_{i=1}^{N_e}\frac{-2+\theta_i-\theta'}{\theta_i-\theta'},\quad(28)$$

$$\mu^{(1)}(\theta')=\prod_{i=1}^{N_\downarrow}\frac{\theta'-\theta_i^{(1)}+2}{\theta'-\theta_i^{(1)}}\prod_{j=1}^{N_e}\frac{\theta'-\theta_j}{2-\theta'+\theta_j}+\prod_{i=1}^{N_\downarrow}\frac{\theta_i^{(1)}-\theta'+2}{\theta_i^{(1)}-\theta'}\prod_{j=1}^{N_e}\frac{\theta'-\theta_j+2}{2-\theta'+\theta_j}.\quad(29)$$

V. CONCLUSION

This model has been solved previously by Ref. 19 with the use of the coordinate Bethe ansatz. The resulting BAE's in generic form were

$$\left[\frac{v_j-\frac{i}{2}}{v_j+\frac{i}{2}}\right]^L=\prod_{\alpha=1}^{M}\frac{v_j-\lambda_\alpha+\frac{ic}{2}}{v_j-\lambda_\alpha-\frac{ic}{2}},\quad j=1,\ldots,N_e$$

$$\prod_{j=1}^{N_e}\frac{\lambda_\alpha-v_j+\frac{ic}{2}}{\lambda_\alpha-v_j-\frac{ic}{2}}=-\prod_{\beta=1}^{M}\frac{\lambda_\alpha-\lambda_\beta+ic}{\lambda_\alpha-\lambda_\beta-ic},\quad \alpha=1,\ldots,M.$$

This agrees with our solution above by noting the following substitutions:

$$c=\frac{-1}{\alpha+1},\quad v_j=\frac{-ic}{2}(\theta_k+1),\quad \lambda_\alpha=-\frac{ic}{2}(\theta_p^{(1)}+2).$$

The quantum version of this solution will be presented in a future publication on the closed chain, and it will be shown that the above solution can be obtained from the quantum solution by taking the limit $q\to 1$. These models depending upon two generic complex parameters are associated with type-I quantum superalgebras, which admit nontrivial one-parameter families of representations and provide solutions to the Yang-Baxter equation with additional parameters. These solutions give us integrable models depending on two independent parameters.[16] The quantum version on the open chain will also be presented in a future publication.

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APPENDIX

Here, we calculate the unwanted terms following the method set out in Ref. 4. The unwanted terms are identified by containing a creation operator with spectral parameter θ'. The cancellation of the unwanted terms ensures that the states (9) are eigenstates of the transfer matrix $\tau(\theta')$. To determine $\breve{\Lambda}_k$, it is convenient to commute the first creation operator with spectral parameter λ_k to the first place in the ansatz using the commutation rule extracted from the relations arising from (2). That is, we write

$$\prod_{i=1}^{n}T_{3a_i}(\theta_i)=T_{3b_k}(\theta_k)\prod_{i=1}^{k-1}T_{3b_i}(\theta_i)\prod_{j=k+1}^{n}T_{3a_j}(\theta_j)\times S(\theta_k)_{a_1\ldots a_k}^{b_1\ldots b_k},$$

$$S(\theta_k)_{a_1\ldots a_k}^{b_1\ldots b_k}=r(\theta_{k-1}-\theta_k)_{b_{k-1}a_{k-1}}^{c_{k-1}a_k}\times r(\theta_{k-2}-\theta_k)_{b_{k-2}a_{k-2}}^{c_{k-2}c_{k-1}}\ldots r(\theta_1-\theta_k)_{b_1a_1}^{b_kc_2}.\quad(A1)$$

To obtain an unwanted term, we commute $T_{33}(\theta)$ past $T_{3b_k}(\theta_k)$, using the second term in (10) then we use the first term to commute $T_{33}(\theta_k)$ with the other terms in the ansatz until it acts on the vacuum according to (8). The resulting equation for $\breve{\Lambda}_k$ is

$$\breve{\Lambda}_kF^{b_1\ldots b_n}=S(\theta_k)_{a_1\ldots a_k}^{b_1\ldots b_k}F^{b_n\ldots b_{k+1}a_k\ldots a_1}\left[\frac{b(\theta_k-\theta')}{a(\theta_k-\theta')}\right]\times(-1)^{n-1}\prod_{j=1,j\neq k}^{n}\frac{1}{a(\theta_j-\theta_k)}.$$

For Λ_k, we have two terms involved. So we write $\Lambda_k=\Lambda_{k,1}+\Lambda_{k,2}$, these terms arise from $T_{11}(\theta')$ and $T_{22}(\theta')$, respectively. With a similar working to that for $\breve{\Lambda}_k$, we find the contribution from the $T_{11}(\theta')$ terms to be

$$\Lambda_{k,1}F^{b_1\ldots b_n}=S(\theta_k)_{a_1\ldots a_k}^{c_1\ldots c_k}F^{a_n\ldots a_1}\left[\frac{b(\theta'-\theta_k)}{a(\theta'-\theta_k)}\right]\delta_{b_k,1}\delta_{d_{n-1},1}\times I(\theta_k)^L\prod_{j=1,j\neq k}^{n}\frac{1}{a(\theta_k-\theta_j)}\times r(\theta_k-\theta_1)_{d_1c_k}^{b_1c_1}r(\theta_k-\theta_2)_{d_2d_1}^{b_2c_2}\times\ldots r(\theta_k-\theta_{k-1})_{d_{k-1}d_{k-2}}^{b_{k-1}c_{k-1}}\times r(\theta_k-\theta_{k+1})_{d_kd_{k-1}}^{b_{k+1}a_{k+1}}\times r(\theta_k-\theta_{k+2})_{d_{k+1}d_k}^{b_{k+2}a_k}\ldots r(\theta_k-\theta_n)_{d_{n-1}d_{n-2}}^{b_na_n}.\quad(A2)$$

The δ functions appearing in this equation arise in the following way. Commuting $T_{11}(\theta_k)$ past the terms of the ansatz to the vacuum leads us to the term $\delta_{d_{n-1},1}$. $\delta_{b_k,1}$ being necessary as in (13), we need to identify the $T_{3b_k}(\theta')$ term with the $T_{11}(\theta')$ contributions. The contribution from the $T_{22}(\theta')$ are obtained similarly with the factors $\delta_{b_k,2}, \delta_{d_{n-1},2}$ being the only difference between $\Lambda_{k,1}$ and $\Lambda_{k,2}$. Then, with $\Lambda_k=\Lambda_{k,1}+\Lambda_{k,2}$, we have

$$\Lambda_k F^{b_1\dots b_n}=S(\theta_k)^{c_1\dots c_k}_{a_1\dots a_k}F^{a_n\dots a_1}\left[\frac{b(\theta'-\theta_k)}{a(\theta'-\theta_k)}\right]$$
$$\times I(\theta_k)^L\prod_{j=1,j\neq k}^{n}\frac{1}{a(\theta_k-\theta_j)}\,r(\theta_k-\theta_1)^{b_1c_1}_{d_1c_k}$$
$$\times r(\theta_k-\theta_2)^{b_2c_2}_{d_2d_1}\dots r(\theta_k-\theta_{k-1})^{b_{k-1}c_{k-1}}_{d_{k-1}d_{k-2}}$$
$$\times r(\theta_k-\theta_{k+1})^{b_{k+1}a_{k+1}}_{d_kd_{k-1}}r(\theta_k-\theta_{k+2})^{b_{k+2}a_{k_2}}_{d_{k+1}d_k}\dots$$
$$\times r(\theta_k-\theta_n)^{b_na_n}_{b_kd_{n-2}}. \tag{A3}$$

We may simplify this equation by contracting the $c_1\dots c_n$ indices using the unitarity of the r matrix. That is,

$$r(\theta_1-\theta_2)r^T(\theta_2-\theta_1)=I,$$

or in component form

$$r(\theta_1-\theta_2)^{b_1a_1}_{b_2a_2}r(\theta_2-\theta_1)^{c_1b_1}_{c_2b_2}=\delta_{a_1c_1}\delta_{a_2c_2}.$$

As a result, the following terms in (A3) may be simplified,

$$S(\theta_k)^{c_1\dots c_k}_{a_1\dots a_k}r(\theta_k-\theta_1)^{b_1c_1}_{d_1c_k}\dots r(\theta_k-\theta_{k-1})^{b_{k-1}c_{k-1}}_{d_{k-1}d_{k-2}}$$
$$=\prod_{i=1}^{k-1}\delta_{a_i,d_i}\delta_{b_{k-1},a_k}.$$

We convert the remaining r matrices into L operators according to

$$[L^{(1)}(\theta)]_{ij}=\sum r(\theta)_{ik,jl}e_{kl},$$

then the unwanted terms are written as

$$\Lambda_k F^{b_1\dots b_n}=\left[\frac{b(\theta'-\theta_k)}{a(\theta'-\theta_k)}\right]I(\theta_k)^L$$
$$\times\prod_{i=1,i\neq k}^{n}\frac{1}{a(\theta_k-\theta_i)}F^{a_n\dots a_kb_{k-1}\dots b_1}$$
$$\times L_n^{(1)}(\theta_k-\theta_n)^{b_na_n}_{b_kd_{n-2}}L^{(1)}_{n-1}$$
$$\times(\theta_k-\theta_{n-1})^{b_{n-1}a_{n-1}}_{d_{n-2}d_{n-3}}\dots$$
$$\times L^{(1)}_{k+1}(\theta_k-\theta_{k+1})^{b_{k+1}a_{k+1}}_{d_ka_k}. \tag{A4}$$

We now insert the equations for $\breve{\Lambda}_k$ and Λ_k into the equation for the cancellation of unwanted terms and multiply throughout by $S^{(-1)}(\theta_k)$. We note once again using the unitarity of $r(\theta')$ that we have

$$S^{(-1)}(\theta_k)^{p_1\dots p_k}_{b_1\dots b_k}S(\theta_k)^{b_1\dots b_k}_{a_1\dots a_k}=\prod_{i=1}^{k}\delta_{a_i,p_i}.$$

The result after some simplification is

$$I(\theta_k)^{(-L)}(-1)^n\prod_{i=1,i\neq k}^{n}\frac{a(\theta_k-\theta_i)}{a(\theta_i-\theta_k)}F^{b_n\dots b_{k+1}p_k\dots p_1}$$
$$=[\tau^{(1)}(\theta_k)F]^{b_n\dots b_{k+1}p_k\dots p_1}.$$

Working in a similar manner to the above for the computation of the unwanted terms for the nested case leads to the equations,

$$\breve{\Lambda}_k^{(1)}=-\frac{b(\theta_k^{(1)}-\theta')}{a(\theta_k^{(1)}-\theta')}$$
$$\times\prod_{j=1,j\neq k}^{n_1}\frac{[a(\theta_j^{(1)}-\theta_k^{(1)})+b(\theta_j^{(1)}-\theta_k^{(1)})]}{a(\theta_j^{(1)}-\theta_k^{(1)})}$$
$$\times\prod_{i=1}^{n}[a(\theta_k^{(1)}-\theta_i)+b(\theta_k^{(1)}-\theta_i)], \tag{A5}$$

$$\Lambda_k^{(1)}=-\frac{b(\theta'-\theta_k^{(1)})}{a(\theta'-\theta_k^{(1)})}$$
$$\times\prod_{j=1,j\neq k}^{n_1}\frac{[a(\theta_k^{(1)}-\theta_j^{(1)})+b(\theta_k^{(1)}-\theta_j^{(1)})]}{a(\theta_k^{(1)}-\theta_j^{(1)})}$$
$$\times\prod_{i=1}^{n}a(\theta_k^{(1)}-\theta_i). \tag{A6}$$

The cancellation of $\Lambda_k^{(1)}$ and $\breve{\Lambda}_k^{(1)}$ leads to the equation

$$\prod_{j=1,j\neq s}^{n_1}\frac{[a(\theta_s^{(1)}-\theta_j^{(1)})+b(\theta_s^{(1)}-\theta_j^{(1)})]a(\theta_j^{(1)}-\theta_s^{(1)})}{a(\theta_s^{(1)}-\theta_j^{(1)})[a(\theta_j^{(1)}-\theta_s^{(1)})+b(\theta_j^{(1)}-\theta_s^{(1)})]}$$
$$=\prod_{i=1}^{n}\frac{a(\theta_s^{(1)}-\theta_i)+b(\theta_s^{(1)}-\theta_i)}{a(\theta_s^{(1)}-\theta_i)},\quad s=1\dots n_1. \tag{A7}$$

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