PHYSICAL REVIEW B VOLUME 54, NUMBER 11 15 SEPTEMBER 1996-I

Green's function of a two-dimensional interacting electron gas in a perpendicular magnetic field

Xiaoguang Wu
National Laboratory for Superlattices and Microstructures, Institute of Semiconductors, Chinese Academy of Sciences, Beijing, P.O. Box 912, 100083, China

(Received 5 February 1996)

It is rigorously proved that the Green's function of a uniform two-dimensional *interacting* electron gas in a perpendicular magnetic field is diagonal with respect to single-particle states in the Landau gauge. The implication of this theorem is briefly discussed. [S0163-1829(96)02635-5]

A system of two-dimensional (2D) interacting electrons in the presence of a perpendicular magnetic field displays many fascinating properties, for example, the integer and fractional quantum Hall effects.[1] It is believed that those peculiar properties result from the interaction between electrons, and the interplay between the interaction and disorder.[1,2]

Although great advances have been made in understanding the system,[1,2] like most other interacting systems, few system properties could be calculated exactly, because the dimensionality of the system is higher than 1.[2,3] Thus exact relations, e.g., sum rules, are always desired. These exact relations not only provide criteria for making approximations, but also may give insight into the underlying physics.

In this paper, I present rigorous proof of the following theorem: the Green's function of a uniform 2D *interacting* electron gas (EG) in a perpendicular magnetic field is diagonal with respect to the single-particle states in the Landau gauge. This theorem should contribute significantly to our understanding of the physics of an interacting 2DEG, because the theorem is applicable to systems containing interaction and disorder, two key ingredients, with arbitrary strengths. The usefulness of the Green's function need not be emphasized. It is well known that the Green's function characterizes the propagation of a system with one particle more or less, and provides the energy and lifetime of excitations of the system.[3,4] Furthermore, many system properties can be evaluated with the aid of the Green's function.[3,4]

In the following, I first prove the theorem, then discuss its implications. For clarity, the theorem will be proved for a 2DEG containing electron-electron and electron-phonon interactions. If the electrons also interact with randomly distributed impurities (disorder), one can easily verify that the theorem still holds for the impurity-configuration-averaged Green's function. The electron spin is ignored, as it does not affect the validity of the theorem, and can be included straightforwardly.

Let us start by considering the model Hamiltonian[3–5]

$$H=\frac{1}{2m}\int d\mathbf{r}\ \psi^\dagger(\mathbf{r})\left[\mathbf{p}+\frac{e\mathbf{A}(\mathbf{r})}{c}\right]^2\psi(\mathbf{r})+\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\,\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$+\sum_{\mathbf{k}}\hbar\omega_k a^\dagger_{\mathbf{k}}a_{\mathbf{k}}+\int d\mathbf{r}\psi^\dagger(\mathbf{r})\left[\sum_{\mathbf{k}}\left(V_k e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}+V_k^* e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_{\mathbf{k}}\right)\right]\psi(\mathbf{r}), \tag{1}$$

where $V(\mathbf{r})$ is the electron-electron interaction potential, $\psi(\mathbf{r})$ is the electron field operator $[\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')+\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}')]$, $a^\dagger_{\mathbf{k}}$ is the phonon creation operator $[a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}-a^\dagger_{\mathbf{k}'},a_{\mathbf{k}}=\delta_{\mathbf{k},\mathbf{k}'}]$, ω_k is the phonon frequency, and V_k is the electron-phonon coupling matrix element.[3]

The Cartesian coordinate of the system is chosen such that $\mathbf{r}=(x,y)$ is in the xy plane. The vector potential $\mathbf{A}$ produces a magnetic field in the z direction. In the Landau gauge, $\mathbf{A}=(0,Bx,0)$; the single-particle states (wave functions) are given by $\varphi_{nk}(\mathbf{r})=e^{iky}\alpha^{1/2}\varphi_n(\alpha x+k/\alpha)$, with $\varphi_n(x)=(\sqrt{\pi}2^n n!)^{-1/2}e^{-x^2/2}H_n(x)$; $\alpha=(m\omega_c/\hbar)^{1/2}$; $H_n(x)$ is the Hermite polynomial, $\omega_c=eB/mc$; and n is the Landau-level index.[5,6]

Because H is time independent, the Green's function $G(\mathbf{r},\mathbf{r}',t)$ can be defined as (without the loss of generality, only the case of $t\geqslant 0$ needs to be considered) (Refs. 3 and 4)

$$G(\mathbf{r},\mathbf{r}',t)=-i\mathrm{Tr}[\rho\psi(\mathbf{r},t)\psi^\dagger(\mathbf{r}',0)], \tag{2}$$

where $\rho=e^{\beta(\Omega-K)}$ is the density matrix; $\psi(\mathbf{r},t)=e^{iKt/\hbar}\psi(\mathbf{r})e^{-iKt/\hbar}$; $K=H-\mu N$; $N=\int d\mathbf{r}\ \psi^\dagger(\mathbf{r})\psi(\mathbf{r})$, the total number of particles; and μ is the chemical potential.[4]

Let us consider a unitary transformation

$$U_1=\exp\left[-\frac{i}{\hbar}\mathbf{s}\cdot\int d\mathbf{r}\ \psi^\dagger(\mathbf{r})\mathbf{p}\psi(\mathbf{r})\right]\exp\left[-i\mathbf{s}\cdot\sum_{\mathbf{k}}\mathbf{k}a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right]. \tag{3}$$

0163-1829/96/54(11)/7640(3)/$10.00

One can verify that $U_1^{\dagger}=U_1^{-1}$, $U_1\psi(\mathbf{r})U_1^{-1}=\psi(\mathbf{r}+\mathbf{s})$, and $U_1 a_{\mathbf{k}} U_1^{-1}=a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{s}}$. Applying this transformation to H, one finds that $H_1=U_1HU_1^{-1}$ differs from H only in the vector potential, i.e., in H_1, $\mathbf{A}=[0,B(x-s_x),0]$.

Let us examine another unitary transformation

$$U_2=\exp\left[-\int d\mathbf{r}\psi^{\dagger}(\mathbf{r})[i\alpha^2 s_x y]\psi(\mathbf{r})\right], \tag{4}$$

with s_x the x component of $\mathbf{s}$. One can verify that $U_2^{\dagger}=U_2^{-1}$, $U_2\psi(\mathbf{r})U_2^{-1}=e^{i\alpha^2 s_x y}\psi(\mathbf{r})$, and $U_2 a_{\mathbf{k}} U_2^{-1}=a_{\mathbf{k}}$. Applying this transformation to H_1, one finds that $U_2H_1U_2^{-1}=H$ exactly. Note that $U_1NU_1^{-1}=U_2NU_2^{-1}=N$.

Inserting $1=U^{-1}U$ (U denotes U_1 and U_2, respectively) into the trace defining the Green's function, since the value of the trace does not change, one obtains

$$G(\mathbf{r},\mathbf{r}',t)=-i\mathrm{Tr}[e^{\beta(\Omega-K)}e^{iKt/\hbar}\psi(\mathbf{r}+\mathbf{s})e^{-iKt/\hbar}\psi^{\dagger}(\mathbf{r}'+\mathbf{s})]e^{i\alpha^2 s_x(y-y')}. \tag{5}$$

Choosing $\mathbf{s}=-(\mathbf{r}+\mathbf{r}')/2$, and denoting $\mathbf{z}=\mathbf{r}-\mathbf{r}'$, one obtains

$$G(\mathbf{r},\mathbf{r}',t)=G(\mathbf{z}/2,-\mathbf{z}/2,t)e^{-i\alpha^2(x+x')(y-y')/2}. \tag{6}$$

Next, let us consider unitary transformation

$$U_3=\exp\left[-\int d\mathbf{r}\psi^{\dagger}(\mathbf{r})[-i\alpha^2 xy/2]\psi(\mathbf{r})\right]. \tag{7}$$

One can verify that $U_3^{\dagger}=U_3^{-1}$, $U_3\psi(\mathbf{r})U_3^{-1}=e^{-i\alpha^2 xy/2}\psi(\mathbf{r})$, and $U_3 a_{\mathbf{k}} U_3^{-1}=a_{\mathbf{k}}$. Note that $U_3NU_3^{-1}=N$. Applying this transformation to H, one finds that $H_S=U_3HU_3^{-1}$ differs from H only in the vector potential. In H_S, the vector potential is $\mathbf{A}=B(-y,x,0)/2$, the symmetrical gauge.

Inserting $1=U_3^{-1}U_3$ into the trace that defines $G(\mathbf{z}/2,-\mathbf{z}/2,t)$ above, one obtains $G(\mathbf{z}/2,-\mathbf{z}/2,t)=G^{(S)}(\mathbf{z}/2,-\mathbf{z}/2,t)$, where the superscript emphasizes that $G^{(S)}$ is calculated with the Hamiltonian H_S. Thus one arrives at[7]

$$G(\mathbf{r},\mathbf{r}',t)=G^{(S)}(\mathbf{z}/2,-\mathbf{z}/2,t)e^{-i\alpha^2(x+x')(y-y')/2}. \tag{8}$$

In a symmetrical gauge, the Hamiltonian H_S is invariant under a unitary transformation of a rotation about the z axis.[8] This is because, in the model Hamiltonian H_S, the electron-electron and electron-phonon interaction terms are invariant under the unitary transformation, while the kinetic-energy term (in the symmetrical gauge) is also invariant. Consequently, $G^{(S)}(\mathbf{r},-\mathbf{r},t)$ is a function of $|\mathbf{r}|$ only, independent of the direction of $\mathbf{r}$. Note that the possibility of spontaneous symmetry breaking is not considered here.

Now, let us calculate $G_{nk,n'k'}(t)=\int d\mathbf{r}\int d\mathbf{r}'\varphi_{nk}^{*}(\mathbf{r})G(\mathbf{r},\mathbf{r}',t)\varphi_{n'k'}(\mathbf{r}')$. After a change of integration variables, $\mathbf{z}=\mathbf{r}-\mathbf{r}'=(u,v)$ and $\mathbf{R}=(\mathbf{r}+\mathbf{r}')/2=(X,Y)$, one has

$$G_{nk,n'k'}(t)=\int d\mathbf{z}\, G^{(S)}\left(\frac{\mathbf{z}}{2},-\frac{\mathbf{z}}{2},t\right)\int d\mathbf{R}\varphi_{nk}^{*}\left(\mathbf{R}+\frac{\mathbf{z}}{2}\right)\varphi_{n'k'}\left(\mathbf{R}-\frac{\mathbf{z}}{2}\right)e^{-i\alpha^2 Xv}. \tag{9}$$

Note that the domains of $\mathbf{r}$ and $\mathbf{r}'$ integrals are the entire xy plane, and so are the domains of $\mathbf{z}$ and $\mathbf{R}$ integrals. The $\mathbf{R}$ integral can be done exactly as,[9]

$$\begin{aligned} I&=\int d\mathbf{R}\varphi_{nk}^{*}(\mathbf{R}+\mathbf{z}/2)\varphi_{n'k'}(\mathbf{R}-\mathbf{z}/2)e^{-i\alpha^2 Xv}\\ &=\delta(k-k')F_{n,n'}H_{n,n'}(\alpha^2z^2/2)e^{i(n'-n)\phi}, \end{aligned} \tag{10}$$

where ϕ is introduced via $u=z\cos\phi$, $v=z\sin\phi$, $F_{n,n'}=(-1)^{(n'-n+|n'-n|)/2}$, and $H_{n,n+l}(x)=(n!/(n+l)!)^{1/2}e^{-x/2}x^{l/2}L_n^l(x)$. Thus one obtains

$$G_{nk,n'k'}(t)=\int d\mathbf{z}\, G^{(S)}\left(\frac{\mathbf{z}}{2},-\frac{\mathbf{z}}{2},t\right)\delta(k-k')F_{n,n'}H_{n,n'}\left(\frac{\alpha^2z^2}{2}\right)e^{i(n'-n)\phi}. \tag{11}$$

Since $G^{(S)}(\mathbf{r},-\mathbf{r},t)$ is independent of the direction of $\mathbf{r}$, the ϕ integral immediately leads to

$$G_{nk,n'k'}(t)=G_{nk}(t)\delta(k-k')\delta_{n,n'}. \tag{12}$$

Thus I have rigorously proved that the nondiagonal part of the Green's function is exactly zero, even in the presence of electron-electron and electron-phonon interactions.[10] It should be emphasized that it is the symmetry properties of the system that lead to the validity of the theorem.

Next let us discuss the implications of the theorem. First, let us make a comparison between the Green's function of a free 2DEG without interaction and that of an interacting one. For a free 2DEG, the Green's function is diagonal with respect to n and k,[4] and is k independent. Equations (11) and (12) show that these two properties are inherited by the interacting 2DEG. This of course does not mean that the interaction plays no role at all. As the self-energy[3,4] must also be diagonal in n and be k independent, the correction due to electron-electron interaction is applicable to a Landau level as a whole. This could be interpreted to mean that, if one treats the electron-electron interaction as a perturbation, then all k states in a Landau level are equally coupled by the interaction.

In a quantum system, the symmetry usually leads to degeneracy.[5] For a noninteracting 2DEG, all Landau levels are highly degenerate and have the same degeneracy.[5] What could one say about an interacting 2DEG? From Eq. (6), one observes that $G(\mathbf{r},\mathbf{r},t)$ is $\mathbf{r}$ independent. The same is true of a free 2DEG. For an interacting 2DEG, this is a result much stronger than stating that the electron-density distribution is spatially uniform. From the definition of $G_{nk}(t)$, one may write $G(\mathbf{r},\mathbf{r},t)$ as $\Sigma_{nk}G_{nk}(t)|\varphi_{nk}(\mathbf{r})|^2$. As $G_{nk}(t)$ is k independent, after the sum over k, one is left with $\Sigma_n G_{nk}(t)$. Since $G_{nk}(t)$ is believed to characterize the spectra of the system,[3,4] one could interpret this sum over n as a sum over all Landau levels. Thus the number of particles allowed in each Landau level is the same (the notion of a Landau level should be still appropriate for an interacting system, if the electron-electron interaction is treated as a perturbation). This could be viewed as a kind of degeneracy, though not exactly.

It is well known that, in the zero-temperature limit, the Green's function defined by Eq. (2) characterizes the propagation of a true ground state containing an additional particle.[3,4] Let us perform an imaginary experiment: at time $t=0$, one first adds a particle of state nk to the true ground state, lets it evolve according to the full Hamiltonian H for a finite time $t>0$, and then takes a particle of state $n'k'$ away from it. What is the overlap between this evolved state with the true ground state? This overlap is precisely given by the Green's function $G_{nk,n'k'}(t)$,[3,4] and this overlap is zero, unless $n=n'$ and $k=k'$, according to Eq. (12). For a noninteracting 2DEG, the result of this imaginary experiment can be easily understood, as there is no coupling between different nk states. For an interacting 2DEG, however, electrons are coupled by the interaction, and become correlated. In view of this, one could interpret Eq. (12) as the manifestation of a kind of ''stiffness'' of the true ground state, as the particle added seems to be marked somehow by the system. It is interesting to note that this stiffness exists for arbitrary electron densities. This stiffness should not be confused with the well-established notion that the quantum Hall liquid is incompressible.[1,2]

The theorem provides a very helpful guideline for calculating the Green's function approximately. In the theories treating the interaction as perturbation, the selection of Feynman diagrams must be consistent with the theorem. Practical numerical calculations should also benefit. It is interesting to point out that the so-called strong magnetic-field approximation widely used in the literature[1] is actually necessary for moderate field strengths as well. The approach employed in the procedure of proving the theorem is useful for other purposes as well. For example, one can easily verify that the electron density-density correlation function $D(\mathbf{r},\mathbf{r}',t)$ (Refs. 3 and 4) is a function of $\mathbf{r}-\mathbf{r}'$, though the system is not translational invariant.

In summary, it is rigorously proved that the Green's function of an interacting 2DEG in a perpendicular magnetic field is diagonal with respect to single-particle states in the Landau gauge. The investigation of further implications of the theorem is currently underway.

This work was partly supported by the Chinese Natural Science Foundation. Helpful discussions with B.S. Wang are appreciated.

---

[1] *The Quantum Hall Effect*, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, New York, 1987).

[2] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, New York, 1991).

[3] G.D. Mahan, *Many-Particle Physics* (Plenum, New York, 1990).

[4] A. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).

[5] E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970).

[6] M. Abramovitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[7] An expression similar to Eq. (8), but for the Green's function of noninteracting electrons, was derived by N.J. Horing [Ann. Phys. **31**, 1 (1965)].

[8] This can be verified more easily by changing into a cylindrical polar coordinate. Note that this unitary transformation of rotation about the z axis can also be written down explicitly; see, for example, Ref. 5.

[9] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1979).

[10] It should be pointed out that factor $\delta(k-k')$ can be obtained more easily, as H is translational invariant along the y direction. Note that the δ function arises because the wave function is not normalized to a unit.