Static and dynamic processes in a two-dimensional Josephson junction

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A two-dimensional planar Josephson junction of an arbitrary cross section is considered. Electrodes are assumed to have a cross section identical with that of the junction. This configuration, which is correct from the physical point of view, allows the boundary conditions for the two-dimensional sine-Gordon equation to be determined uniquely through a distribution of a surface electrode current. The resulting problem, with this surface current distribution induced by the bias and/or external magnetic field, is trivial since it is a linear one. Square and rectangular cases are considered. The numerically derived results concerning the static and dynamic regimes differ significantly from the one-dimensional model because of a nonuniform transversal phase distribution. Moreover, a degeneration of static modes in the absence of an external magnetic field also appears. [S0163-1829(96)08533-5]

I. INTRODUCTION

Considerable effort has already been devoted to the solution of the problem of a two-dimensional (2D) flat Josephson junction in a static or dynamic regime. The problem has a long history and the majority of authors accept as correct the natural generalization of the sine-Gordon equation (SGE) with a 2D Laplace operator. Essential differences exist, however, because of the applied boundary conditions.

Let us confine ourselves to a rectangular, flat Josephson junction biased by a time-independent (i.e., dc) current I_b and embedded in a homogeneous external magnetic field H_e. Usually the boundary conditions are imposed as a requirement that the normal derivative of the order parameter on the boundary edge be equal to a linear combination of I_b and H_e. As a rule, this derivative is chosen as to be a constant along each side of the rectangle and applied to the linearized version of the sine-Gordon equation.[1] This approach is rather typical,[2–4] and in fact it means that assumed boundary conditions are given in the form of a piecewise constant function of the point on an edge of a junction. It is a natural extrapolation of boundary conditions imposed in the 1D case, e.g., in Ref. 5. If such a choice has a deep justification in the 1D case, there is no basis for the assumption that the considered expression is constant along each side of a 2D junction. The inflowing current depends on the electrode configuration and thus indirectly on the local magnetic field produced by the biasing current whose distribution is in fact unknown. Moreover, one can suppose that the density of an injected current has singularities at the sharp corners of a junction.[4]

There are a few exceptions to this general assertion. Since the problem is important from the applied point of view, in Ref. 3 it was investigated by making use of a water tank model or proposing some boundary conditions on the basis of the 1D configuration.[6] Another way was chosen in Refs. 7 and 8 where the 2D junction is considered there as a tunneling ‘‘window’’ in a uniform strip line configuration. A completely different approach was proposed in Ref. 9. The extremalization of some functional, in this case the maximum of the total current, was chosen for the determination of an internal current distribution.

One feels, however, the lack of a model which would be correct from both the mathematical and physical points of view. This is the motivation of the present paper.

II. BASIC CONCEPTS

We propose a model which, while being correct with respect to the physical aspects and relatively simple from the mathematical point of view, remains sufficiently general so that the conclusions seem to be applicable to other configurations. We assume the junction to be an infinitesimally thin region cutting perpendicularly through an infinitely long (along the z direction) and uniform superconducting cylinder, which represents the junction electrodes on both sides of the junction plane. The cross section of the electrodes and thus the junction shape can be arbitrary, e.g., circular, rectangular, or of any other form.

The whole structure is therefore homogeneous in the z direction, with the exception of the thin junction, as shown in Fig. 1.

Since the junction constitutes only a relatively small perturbation in the structure, the boundary conditions are determined by the surface current distribution in an infinitely long, perfectly superconducting slab. This also follows from energetic considerations. Hence the solution of the problem can be decomposed into two steps. In the first, the current on the surface of an infinite superconducting slab is determined (including also the effect of an external magnetic field). This distribution imposes boundary conditions for the junction whose nonlinear SGE is solved in the second step. Thus the role of electrodes is taken into account, although we are aware that the geometry is chosen in some particular manner. Nevertheless, the problem is now well posed from the physical point of view, since it excludes an arbitrariness in the bias current distribution.

The most important feature is, however, that the first step leads to a linear problem and hence it can be easily and effectively solved. The determination of the surface current distribution follows from the requirement that the normal

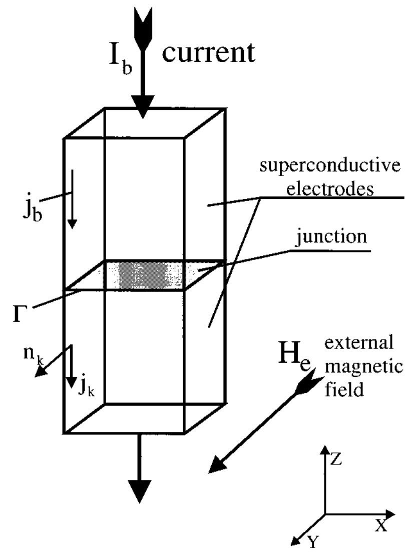

FIG. 1. Geometrical configuration of the junction (uniform in the z direction).

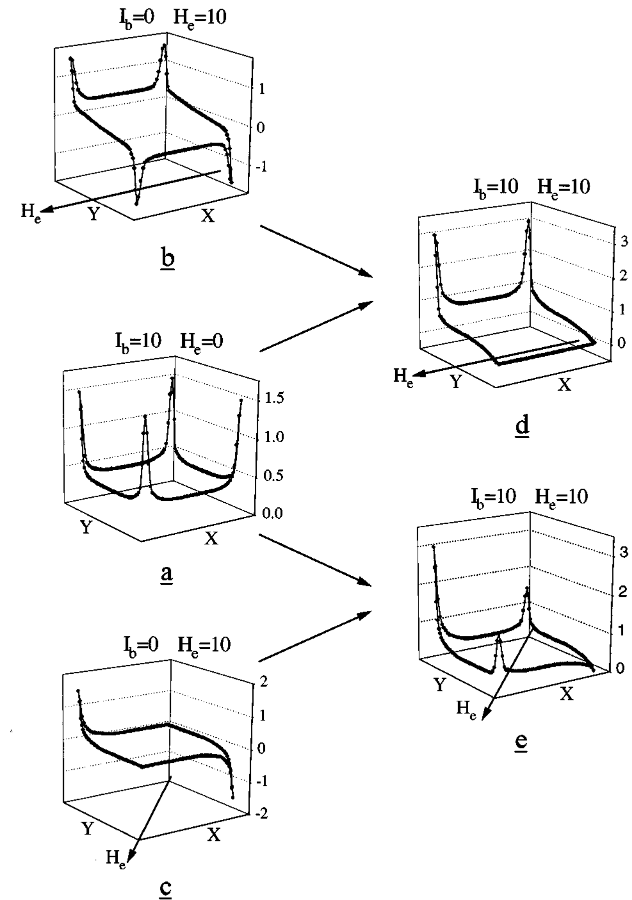

FIG. 2. Distribution of the surface current density on the surface of superconducting electrodes (uniform in the z direction): (a) $H_e = 0$, (b) and (c) $I_b = 0$ (i.e., screening current only), magnetic field parallel to the junction side and to the diagonal, respectively, (d) and (e) $I_b \neq 0$, $H_e \neq 0$, magnetic field parallel to the junction side and to the diagonal, respectively.

derivative of the total magnetic field (i.e., including the external one, if it exists) vanishes on the surface of the slab. This first, linear problem can be decomposed in turn into two additive subproblems:

$$\text{(i)}\ \ H_e = 0,\quad I_b \neq 0 \quad \text{and (ii)}\ \ H_e \neq 0,\quad I_b = 0, \tag{1}$$

each one accompanied by the condition that the normal derivative of the magnetic field vanish on the superconducting slab surface.

Denoting by j_b the surface density of the current on the slab surface and by $\mathbf{h}_b$ the magnetic field produced by this current, both problems reduce to the determination of either j_b' or j_b'' such that

$$\text{(i)}\ \oint j_b'(P)\,d\Gamma = I_b, \quad \mathbf{h}_b' \cdot \mathbf{n}|_\Gamma = \mathbf{0}, \tag{2}$$

$$\text{(ii)}\ \oint j_b''(P)\,d\Gamma = 0, \quad (\mathbf{h}_b'' + \mathbf{H}_e) \cdot \mathbf{n}|_\Gamma = \mathbf{0}, \tag{3}$$

where Γ represents a flat contour of the slab cross section and $\mathbf{n}$ is a unit vector normal to the slab. As a consequence both subproblems lead to the Laplace equation with specified boundary conditions and numerous methods of its solution can be found in any textbook.

A sum of the results for both subcases (i) and (ii) solves the problem, since we deal with a linear problem and

$$\oint [j_b'(P) + j_b''(P)]\,d\Gamma = I_b, \tag{4}$$

while

$$(\mathbf{h}_b' + \mathbf{h}_b'' + \mathbf{H}_e) \cdot \mathbf{n}|_\Gamma = 0. \tag{5}$$

In the next, this time a nonlinear step, the tangential component of the magnetic field will be used as a boundary condition for the phase order parameter φ through the equation

$$\varphi,_n = (\mathbf{h}_b' + \mathbf{h}_b'' + \mathbf{H}_e)_t, \tag{6}$$

where $\varphi,_n$ denotes a normal derivative and $(\cdot)_t$ a tangential component.

The problem is complicated somewhat when the contour Γ forms an irregular curve with edges. For some particular contour shapes a complete solution can be obtained by using the methods of conformal transformation.

One can easily show that if the edge forms an angle $\alpha = 2\pi/s$, where s is an integer, the tangential magnetic field is given by

$$H_t \sim m r^{m-1} \cos(m\varphi)|_\Gamma \quad \text{with } m = \frac{ks}{2(s-1)}, \tag{7}$$

where k is a second integer and r is a distance from the edge. For a rectangular junction the field has a singularity at the edge since $H_t \sim r^{-1/3}$, as shown in the Appendix. Field energy and integrated current are of course finite.

III. METHODS AND RESULTS

Instead of solving the Laplace equation, we adopted the following procedure, both for regular and irregular contours.

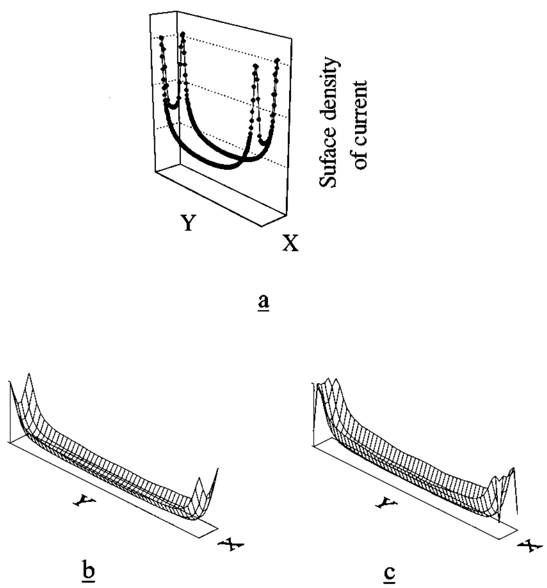

FIG. 3. Long rectangular junction: (a) surface current distribution on supeconducting rectangular electrodes, and (b), (c) the two lowest current modes corresponding to different bias currents.

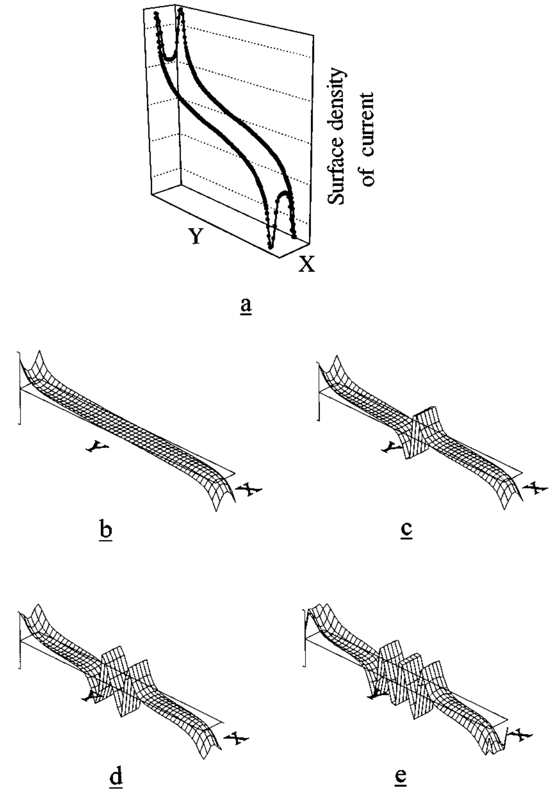

FIG. 4. Long rectangular junction: (a) screening surface current distribution on superconducting rectangular electrodes, and (b)–(d) the four lowest magnetic modes ($H_e \| x$).

N equidistant points were chosen along the contour with initial currents j_k, such that $\Sigma_{k=1}^{N} j_k = I_b$. At each point l $(1 \leqslant l \leqslant N)$, there was then calculated the component of the magnetic field normal to the contour (including an external field but excluding the contribution of j_l) and the currents were corrected to minimize the sum of the absolute values of normal components, i.e., the quantity $q = \Sigma_{k=1}^{N} |\mathbf{h}_k \cdot \mathbf{n}_k|$, where $\mathbf{n}_k$ is the normal vector at point k. This procedure allows the determination of the surface current distribution relatively quickly, without the troublesome process of solving a partial differential equation. Edge points were always excluded from the calculations. The error, arising from the fact that an infinite current density at the edge is not included, does not influence the current distribution but only the total current value and it is relatively small. Indeed, as the current close to the edges or a rectangular junction has the form $j(x) = j_1[\Delta x/(2x)]^{1/3}$, where Δx is the distance between two adjacent points and j_1 is the surface current density at the point closest to the edge, the total error (per one edge) is $\Delta j = 2\int_0^{\Delta x/2} j(x)dx - j_1 \Delta x = j_1 \Delta x/2$, and thus it decreases with increasing mesh density.

As the first step, the initial current distribution was chosen to be uniform for case (i) and antisymmetric for case (ii). As seen in Fig. 2, the final current distribution is far from uniform in both cases, justifying the need of this analysis. Moreover, in the case of a rectangular and relatively long junction (side ratio 10:1) the current density distribution along the longer side of the rectangle, although relatively small, gives as the result that total currents flowing over longer and shorter sides are of the same range. This means of course that the practically realized 2D junctions (with side ratio less than 5:1) can hardly be considered as 1D ones, since the lateral current cannot be assumed to be negligibly small.

The distribution of the surface current density, without the presence of external magnetic field, has of course fourfold symmetry [Fig. 2(a)]. In contrast, an application of the external magnetic field produces an antisymmetric pattern of the screening current, as can be seen in Fig. 2(b) for different orientations of the magnetic field with respect to the square junction. Since, at this level, the problem is linear, the pattern in the simultaneous presence of bias current and magnetic field can be obtained as the sum of the previous, independent patterns. This is shown in Figs. 2(d) and 2(e) and it can be written symbolically as Fig. 2(d) = Fig. 2(a) + Fig. 2(b) or Fig. 2(e) = Fig. 2(a) + Fig. 2(c). These patterns will serve as boundary conditions according to Eq. (6).

Calculations of the surface current distribution on a lateral surface of a rectangular slab of ratio 10:1 show that the current density along a shorter side of the rectangle is slightly higher than along a longer side. The relevant patterns are presented in Figs. 3(a) and 4(a) without and with an external magnetic field (applied perpendicularly to a longer side of a junction), respectively. However, the partial currents, integrated along longer and shorter sides, have comparable values and this means that the 1D model needs to be applied with caution.

Having determined boundary conditions, we solved the 2D attenuated sine-Gordon equation

$$\varphi_{,xx} + \varphi_{,yy} - \varphi_{,tt} - \sigma\varphi_{,t} = j_c \sin\varphi \qquad (8)$$

using the scheme[10,11]

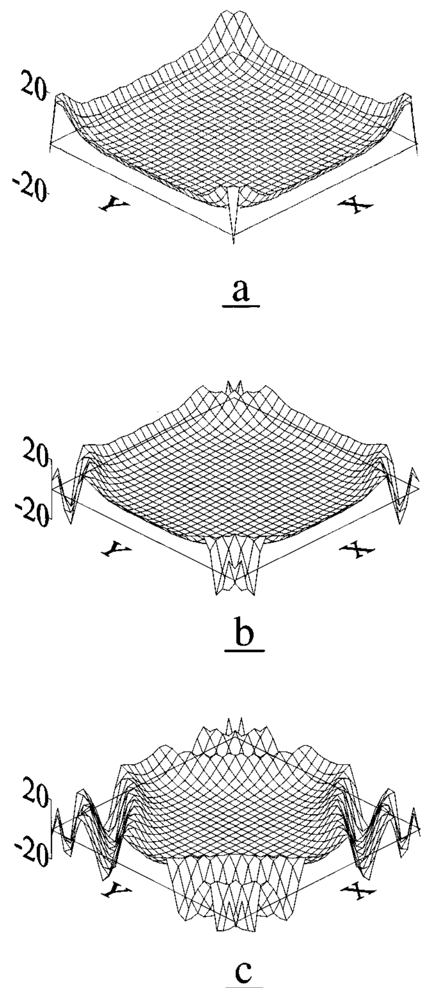

FIG. 5. The three lowest static current modes in a square junction ($H_e=0$).

$$(1+\sigma d_t/2)\varphi_{00}^1=2(1-2d_t^2/d_x^2)\varphi_{00}^0-(1-\sigma d_t/2)\varphi_{00}^{-1}$$
$$+(\varphi_{10}^0+\varphi_{-10}^0+\varphi_{01}^0+\varphi_{0-1}^0)d_t^2/d_x^2$$
$$-j_c d_t^2 \sin[(\varphi_{10}^0+\varphi_{-10}^0+\varphi_{01}^0+\varphi_{0-1}^0)/4], \tag{9}$$

where σ is an attenuation coefficient, j_c the junction critical current, and d_t and d_x time and space steps, respectively. A shorthand notation $\varphi_{00}^0\equiv\varphi_{xy}^t$ is here adopted. We are aware that the above scheme can be simplified by a proper choice of d_t/d_x, but because of the singularity in the boundary conditions, we preferred, to be on safe side, a very small time step $d_t \ll d_x$. As the argument of the sine function, the quantity φ_{00}^0 was sometimes substituted for simplicity during calculations.

The qualitative results in principle follow those known from the analysis of 1D junctions; i.e., when the bias current is below some threshold value, the solution is represented by a static pattern, which can be degenerate in the presence of the external magnetic field. When the biasing current exceeds this threshold, a static solution does not exist at all and, after some transient interval, the dynamic state periodic in time is reached.

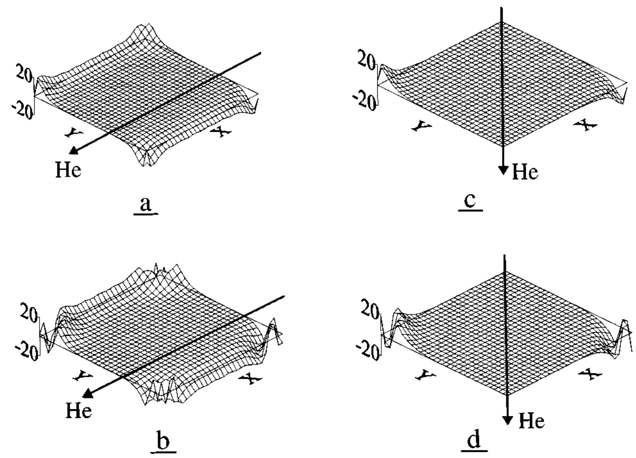

FIG. 6. The lowest static magnetic modes in a square junction ($I_b=0$) for different orientations of external magnetic field.

In the 2D junction the main features are the same, although there are some differences in the static state degeneracy since one has additional free parameters: the second space coordinate and the orientation of the external electromagnetic field. This leads to a degeneracy of static states also in the absence of an external magnetic field. Moreover, the distribution of Josephson current density rather strongly depends on the transversal coordinate, even when a long rectangular junction is considered. As a result, the behavior of a square or near-square junction is considerably different from predictions of the 1D model. The same observations relate also to dynamic states. Often a flux creep starts from the corners of the junction and it is highly dependent on the direction and magnitude of an external magnetic field.

We first considered a square junction. In the absence of a magnetic field we have found a few static modes presented in Fig. 5. With the increase of the bias current the patterns of the Josephson current develop stronger and stronger oscillations close to the corners. These patterns we call the (static) current modes. Total phase φ differences are of course 2π, 4π, and 6π, respectively.

In the presence of a magnetic field only (no bias current) applied either parallel to the side of square junction or to its diagonal, another, rather antisymmetric pattern of the screening current appears, as seen in Fig. 6. We deal with loop currents (vortices) perpendicular to the plane of the junction, since the net current through the junction vanishes. These types of patterns we denote as the (static) magnetic modes.

For a rectangular junction some lowest current and magnetic modes are shown in Figs. 3(b), 3(c), and 4(b)–4(e). The current distribution is evidently two dimensional and the dependence on the ''transversal'' coordinate cannot be neglected, since the density of the Josephson current close to a longer side is greater than in the junction center giving a non-negligible contribution to the total bias current. In other words a local current exceeds the critical current density. This means indirectly that the 2D model gives smaller critical (maximal) currents than follow from the 1D model. One can say that the 1D model is more ''optimistic.''

As has already been mentioned, when the bias current is greater than some threshold current, which depends on the junction geometry and on the magnitude and direction of

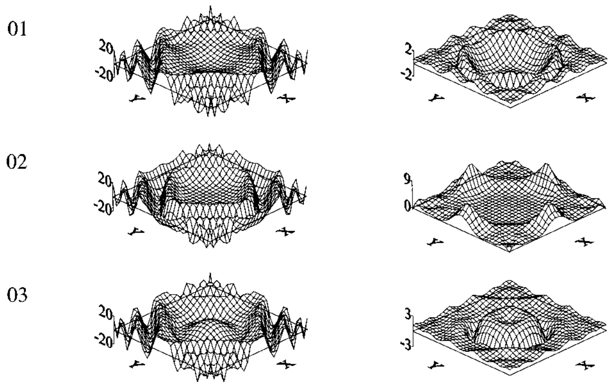

FIG. 7. Dynamics of a square junction ($I_b=100$, $H_e=0$): the density of Josephson current (left column) and Ohmic current (right column). Some intermediate stages of the time-periodic solution. Note the different vertical scales.

magnetic field, the time evolution according to Eq. (9) does not reach a static state, but after some transient period, it becomes periodic in time.

In the case of a 1D junction with a sufficiently high current, the periodic solutions exist with traveling fluxons originating at the junction ends; see, e.g., Ref. 3. This feature is retained by the 2D junction (10:1), but the process is somewhat more complex. For a small bias current the process evolves into a classical single fluxon-antifluxon solution, but with some subtle structure in the transversal direction. For slightly higher currents two objects, a fluxon and antifluxon, are generated at both ends and annihilate in the middle of the junction.

In a square junction the processes are far more complicated. Dynamic processes without magnetic field are shown in Fig. 7. There is an interaction (annihilation) of four fluxons generated at the four corners. Generation starting from the sides was never observed as far as the Josephson current is concerned. This is not true, however, in the case of the Ohmic current. Let us observe that the Josephson current is accompanied by an Ohmic current caused by the damping term in the sine-Gordon equation (8). Since the attenuation is

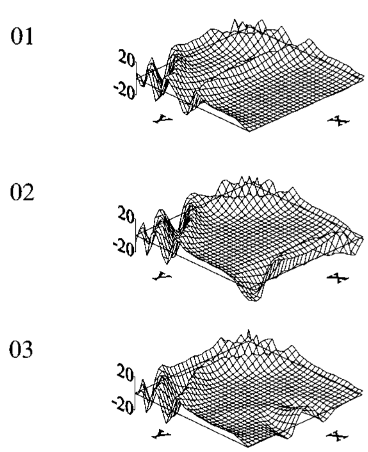

FIG. 8. Dynamics of a square junction ($I_b=50$, $H_e=50$, and $H_e \| x$): the density of the Josephson current. Some intermediate stages of the time-periodic solution.

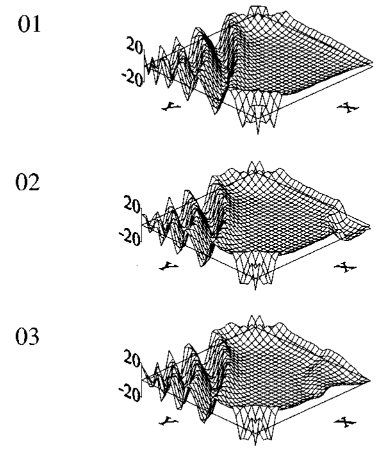

FIG. 9. Dynamics of a square junction ($I_b=50$, $H_e=50$, and $H_e \|$ diagonal: the density of the Josephson current. Some intermediate stages of the time-periodic solution.

small, the Ohmic current is also relatively small, but for graphical reasons the scales in both columns of pictures are different.

In the presence of an external magnetic field the periodic process ceases to be symmetric.

In Fig. 8 the magnetic field is parallel to a junction side; in Fig. 9 it is parallel to the diagonal. In both cases the fluxon sources are located at the corners, but on inspecting the intermediate stages in Fig. 8, one has sometimes an impression that a fluxon parallel to the side appears. In the configuration presented in Fig. 9, the fluxons always started from the corners, but one of them was dominant. The profile of the fluxons seems to be circular, which is rather surprising in a Cartesian structure.

To ensure that we deal with a periodic process, the local voltage defined by

$$V(x,y,T)=\frac{\sigma}{T}\int_{T_0}^{T}\varphi_t(x,y,t)dt$$
$$=\frac{\sigma}{T}[\varphi(x,y,T)-\varphi(x,y,T_0)] \qquad (10)$$

was determined in different points of the junction. For static solutions this voltage vanishes. For dynamical processes, after some transient time, this voltage was constant in time and also independent of the mesh point. This confirms the argument formulated in the past, that in a physically correct model the dc voltage has to be independent of spatial coordinates,[12] which favors periodic processes. This observation was used to test whether a static or a truly periodic state was reached.

IV. CONCLUSION

The idea that the current distribution on the surface of superconducting electrodes determines boundary conditions for the 2D SGE and the choice of junction geometry as a uniform structure in the direction perpendicular to the junction plane appear to be very fruitful. Since this current distribution and in consequence the boundary conditions cease to be constant along the sides of a rectangular junction, one obtains, contrary to the past propositions, static and dynamic states of the 2D Josephson junction and such a model is closer to the physical realization. For a long 2D junction the situation is similar qualitatively to that in a 1D one; however, for a square junction the picture is completely different. There appears a degeneracy of static current modes and a flux creep starting from the corners. Moreover, due to the strongly warped current pattern, the estimation of the maximal bias current allowed in the static regime (overall junction critical current) seems to be less optimistic than follows from the 1D model. It can be anticipated that these conclusions are valid also for other symmetric configurations.

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APPENDIX

We estimate a singularity of the magnetic field tangential component close to the sharp edge of a uniform superconducting slab.

The solution of the Laplace equation $\nabla^2\varphi=0$, in a 2D cylindrical coordinate system r,ψ with boundary conditions

$$\frac{\partial\varphi(r,\psi)}{\partial\psi}=0 \quad \text{for} \begin{cases}\psi=0,\\ \psi=2\pi-\alpha,\end{cases} \tag{A1}$$

where $\alpha=2\pi/s$ and s an integer, is given by $\varphi=(C_1r^m+C_2r^{-m})\cos(m\varphi)$, with

$$m=\frac{k\pi}{2\pi-\alpha}=\frac{k}{2(1-1/s)}=\frac{ks}{2(s-1)}, \tag{A2}$$

where k is the second integer. If $\mathbf{H}=\nabla\psi$, then boundary conditions indicate that the normal component of **H** vanishes on the surfaces $\psi=0$ and $\psi=2\pi-\alpha$. The tangential component is then

$$H_t=\partial\varphi/\partial r$$
$$=m(C_1r^{m-1}-C_2r^{-m-1})\cos(m\psi)|_{\psi=0,\,2\pi-\alpha}. \tag{A3}$$

The requirement $\int_0^R H_t dr|_{b.\mathrm{sur}}<\infty$ (condition of finite surface current) reduces to $m>0$. Then $C_2=0$, and by Eq. (A2) $s>1$. Thus, due to Eq. (A2) on the superconductive surface

$$\varphi\sim(-1)^k r^m, \quad H_t\sim(-1)^k m r^{m-1}. \tag{A4}$$

For $\alpha=\pi/2$, H_t is singular for $k=1$ and then

α	=	$2\pi/3$,	$\pi/2$,	$2\pi/5$,	$\pi/3$,
m	=	3/4,	2/3,	5/8,	3/5,
H_t	$\sim$	$r^{-1/4}$,	$r^{-1/3}$,	$r^{-3/8}$,	$r^{-2/5}$.

Thus the tangential component of a magnetic field close to the $\alpha=\pi/2$ corner of a superconducting electrode and hence the surface current have a singularity $H_t\sim r^{-1/3}$.

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