# Ising spin glass: Replica-symmetric cluster expansion in finite dimensions

Anil Khurana
*Department of Physics and Astronomy, 200 South College, University of Tennessee, Knoxville, Tennessee 37996-1501*
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The replica-symmetric cluster expansion for the Ising spin glass is reviewed and explained. The expansion allows computation of various quantities and correlations directly in finite dimensions, without recourse to the mean-field theory or the saddle point. The expansion for the free energy is recovered with the use of external fields that couple to the replica-symmetric Edwards-Anderson order parameter. The expansion scheme is applied to the Edwards-Anderson susceptibility. The expansion diverges in the mean-field [infinite dimensionality] limit. The divergent expansion, when naively summed, yields the negative replicon mass of the mean-field theory. This so-called Almeida-Thouless instability of the replica-symmetric mean-field theory is argued to be absent in finite dimensions. The argument, similar to the one already reported for the entropy, depends on the low-temperature behavior of terms in the expansion at finite dimensions. [S0163-1829(96)00634-0]

## I. OVERVIEW OF STUDIES AT FINITE *d*

The accepted starting point for understanding the spin-glass phase in finite dimensions is the Edwards-Anderson model for Ising spins.[1] The long-range Ruderman-Kittel-Kasuya-Yoshida interaction between spins in the spin-glass alloy is represented in the model by a random interaction between nearest-neighbor spins placed on the sites of a $d$-dimensional hypercubic lattice. The probability distribution for the random interaction is symmetric when studying the paramagnet–spin-glass transition or the spin-glass phase.

It has been indisputably established for several years now that the Ising spin-glass model has a phase transition at finite $d$, and even at $d=3$. Evidence from one of the primary methods for studying an equilibrium phase transition, the high-temperature series expansion, was reviewed in an earlier comment.[2] The evidence from numerical simulations is reviewed in Binder and Young.[3]

The most extensive study of the spin-glass phase in the finite-dimensional Ising spin-glass model is the one pioneered by De Dominicis and Kondor.[4] Their approach is similar to that of Wilson-Fisher 4-$\epsilon$ expansion for the ferromagnetic transition: De Dominicis and Kondor start from Parisi's mean-field solution[5] and study the role of finite dimensionality by a loop expansion. This loop expansion is difficult to carry out beyond the first few orders, because the spectrum of Gaussian fluctuations about Parisi's mean-field solution, which determines the bare propagators, is infinitely degenerate and extremely singular. The main result from the loop expansion is that the Ising spin-glass phase in finite dimensions has broken replica symmetry of the form discussed by Parisi in the mean-field theory. The precise from of Parisi's order parameter $q(x)$, however, may change below some critical dimension.[6] A $1/d$ expansion for the coefficients of the Landau-Ginzburg free-energy functional for the spin-glass phase, carried out as in Ref. 7, corroborates the results of the loop expansion.

There have also been phenomenological studies of the spin-glass phase.[8–10] These find that the Ising spin glass in finite dimensions has at most a finite number of pure states, so that a description of it in the replica formalism might not require the replica-symmetry breaking discovered in the mean-field theory. One of the fundamental quantities in these studies, however, is not the order parameter but the energy cost of creating a defect of linear size $L$ in the spin-glass ground state. The energy cost varies as $L^{-y}$; the exponent $y$ is estimated to be less than $(d-1)/2$, and numerical simulations give a value of around 0.25 for it in three dimensions.[11,12]

By contrast, a Monte Carlo study of the order-parameter distribution $P(q)$, in $d=4$, finds that the weight for small $q$ accumulates to a nonzero value as the system size increases,[13] in agreement with the behavior expected in Parisi's mean-field theory. The order-parameter distribution is a measure; it is the probability for obtaining a given value of pairwise overlap $q$. It is given by the derivative of $x(q)$, the probability of obtaining a value up to $q$ for the overlap.

Another approach to understanding the effect of finite dimensionality is the study of the spin glass on a Bethe lattice of finite connectivity.[14] These studies also find that replica symmetry breaking persists when the phase is studied in an expansion in the reciprocal of the connectivity parameter.[15]

Finally, there now exists a cluster expansion for the Ising spin glass. The expansion is most conveniently generated using the replica method. The cluster expansion for the free energy in the replica-symmetric phase showed that the low-temperature behavior is different at finite $d$ from that at $d\rightarrow\infty$, the mean-field limit. In particular, the negative value of the zero-temperature entropy,[16] an unphysical feature of the replica-symmetric mean-field theory that prompted attempts to break the replica symmetry,[17] could be recovered[18] from the expansions when the limit $d\rightarrow\infty$ was taken before the limit $T\rightarrow 0$.

The formalism for studying the replica-symmetric phase of the Ising spin glass directly in finite dimensions, using the cluster expansion noted above, is developed here. The expansion for the free energy in the replica-symmetric phase, which was discussed in Ref. 18, is rederived in Sec. II below using an external field that couples to the replica-symmetric order parameter. The Edwards-Anderson susceptibility is

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discussed in Sec. III. The outlook for various methods used to study the spin glass is presented in Sec. IV.

## II. FREE ENERGY

The Ising spin glass is defined by the Hamiltonian

$$H=-\sum_{(ij)} J_{ij}s_i s_j, \tag{2.1}$$

where $s_i=\pm 1$ are Ising spins on the sites of a $d$-dimensional lattice and $J_{ij}$ is the exchange interaction between nearest-neighbor spins. The replica method allows the thermodynamics of a quenched random system to be derived as a certain limit of the statistical mechanics of an effective homogeneous one. The effective Hamiltonian is obtained from the average of the $n$th power of the partition function of the quenched random system. For a Gaussian distribution of exchange interactions:

$$[Z^n]_{\mathrm{av}}=\operatorname*{Tr}_{s_i^\alpha} \exp(-\beta H_n), \tag{2.2}$$

$$-\beta H_n=\frac{1}{2}\beta^2\sum_{(ij)}\left(\sum_{\alpha=1}^{n} s_i^\alpha s_j^\alpha\right)^2. \tag{2.3}$$

When studying the replica-symmetric phase one may regard the right-hand side of (2.3) as an effective Hamiltonian for $n$ Ising spins and not worry about the fact that the limit $n\to 0$ must be taken at the end, just as is done in the replica formalism for Anderson localization.[19] For the statistical problem defined by (2.2) and (2.3) a starting configuration about which a cluster expansion may be built is then the one that maximizes $-\beta H_n$. The $2^n$ states in which each replica is independently ferromagnetically aligned throughout the system do maximize $-\beta H_n$. These states may be written as $(p,\ n-p)$, that is, the spins point down in the first $p$ replicas at each site and they point up in the remaining $n-p$ replicas.

The above choice for the initial configuration, although it seems obvious, raises a simple question: What does one do with the variable $p$? Since the thermodynamics must not depend on it, one must sum over all possible values of $p$, $0\leq p\leq n$, using binomial weights for the number of configurations for a given $p$.[20]

The initial configuration has two further peculiar properties. First, it does not contribute to the free energy because in it $H_n$ is $O(n^2)$. Second, the Edwards-Anderson (EA) order parameter vanishes: Following De Dominicis and Young,[21] the EA order parameter $q$ may be defined as

$$q=\lim_{N\to\infty}\frac{1}{N}\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}^{n}\sum_{i=1}^{N}\langle s_i^\alpha s_i^\beta\rangle.$$

In the state $(p,\ n-p)$ it is given by

$$q=\lim_{n\to 0}\frac{1}{n(n-1)}\left\langle\sum_{p=0}^{n}(n-2p)^2-n\right\rangle=0 \tag{2.4}$$

The average here means a sum over $p$ with binomial weights and a Boltzmann factor that is the exponential of $\beta^2$ times $n^2$, the overlap between spins at neighboring sites.

The first of the peculiarities noted above says merely that the expansion is not about the true spin-glass ground state, because the ground-state energy is nonzero. The second, however, is more tricky. Since ordered phases are described by the values of their order parameters, might not one require that the initial configuration at least has the value expected for $q$ at $T=0$, which is $q=1$? Is something wrong about the choice $(p,\ n-p)$ for the initial state?

The reason $q=0$ in the state $(p,\ n-p)$ is none other than the one that causes the magnetization of a ferromagnet to be zero if states with preferred magnetizations both up and down are kept in the Gibbs average while carrying out a low-temperature expansion. It is for this reason—the analogy with the ferromagnet—that in discussions of the cluster expansion for the spin glass reported to date,[18,20] the starting configuration was chosen to be $(n)$, that is, one with spins in all replicas pointing up. This state also does not contribute to the free energy, but in it $q=1$. The cluster expansion about this state yields, in the limit $d\to\infty$, a low-temperature expansion for the replica-symmetric Sherrington-Kirkpatrick (SK) solution.[18]

But in the low-temperature expansion for the ferromagnet the Gibbs average may be unrestricted if one adds a uniform field. Indeed, low-temperature expansions for the ferromagnetic phase are recovered if the uniform field is set equal to zero *after* taking the thermodynamic limit. Similarly, to generate the low-temperature expansion for the replica-symmetric Ising spin-glass phase, it is useful to consider adding to the right-hand side of (2.3) a term

$$\frac{\beta^2 h^2}{2}\sum_i\sum_{\alpha\neq\beta} s_i^\alpha s_i^\beta, \tag{2.5}$$

which has the same form as the EA order parameter. Such a term would arise if a Gaussian random external field of variance $h^2$ acted on the system and one was interested in the average response to this field. In the presence of (2.5), $q$ in the state $(p,\ n-p)$ is

$$\int [dy]\tanh^2(\beta h\sqrt{N})\to 1,\quad N\to\infty. \tag{2.6}$$

In (2.6) and in the following $\int[dy]$ stands for an integral with Gaussian weight, of variance unity unless otherwise specified.[22]

Having introduced the notion of an external field that couples to the replica-symmetric order parameter, it is easy to verify that all the results reported to date for the cluster expansion for the replica-symmetric phase,[18] which were obtained using $(n)$ as the starting configuration (i.e., spins in all replicas point up), can also be obtained starting from the state $(p,\ n-p)$ in the presence of (2.5) and taking the limit $h\to 0$ after the thermodynamic limit. As an example, consider the contribution to the free energy from states with one flipped site, that is, when the spin configuration at one site in the lattice is of the form $(q,\ p-q;\ r,\ n-p-r)$, the number of down replicas being $q+r$ and of up replicas being $n-q-r$. The contribution to the free energy reads[22]

$$-\beta F_1 = N\left\{ \int [dy][dx_1][dx_2]\ln[4\cosh(\beta\tilde{x}_1)\cosh(\beta\tilde{x}_2)\cosh(\beta\tilde{y}) + 4\sinh(\beta\tilde{x}_1)\sinh(\beta\tilde{x}_2)\sinh(\beta\tilde{y})] - \int [dx]\ln[2\cosh(\beta\tilde{x})]\right\}, \quad (2.7)$$

where

$$\tilde{x}_1 = \sqrt{(N-1)}hx_1, \quad \tilde{x}_2 = \sqrt{h}x_2, \quad \tilde{y} = \sqrt{2d}y,$$

$$\text{and } \tilde{x} = \sqrt{N}hx.$$

In the thermodynamic limit the external field $h$ is chosen to be $O(1/\sqrt{N})$. The $x_2$ integral gives unity in that limit; the $x_1$ and $y$ integrals factorize, so that the $x_1$ integral cancels the second terms on the right-hand side of (2.7). This gives the free energy per spin

$$-\beta f_1 = \int [dy]\ln[2\cosh(\beta\tilde{y})], \quad (2.8)$$

which has the same form as the SK free energy when $q_{\mathrm{SK}} = 1$.[18,20] It would belabor the point to show how the contributions listed in Ref. 18 arise when the initial configuration is $(p, n-p)$, rather than $(n)$. Suffice it to say that the contribution of a flipped cluster of $r$ sites when the background state is $(p, n-p)$ and a random field of the form (2.5) acts on every site, in some ways resembles the contribution of a flipped cluster of $(r+1)$ sites in the background state $(n)$, but that this contribution reduces in the thermodynamic limit to the one obtained for a cluster of $r$ flipped sites in the background state $(n)$. Comparing Eq. (2.7) above with Eq. (5) in Ref. 18 might serve as a useful illustration of this general observation.

## III. EDWARDS-ANDERSON SUSCEPTIBILITY

The Almeida-Thouless (AT) instability[23] ranks foremost among the concepts and notions of spin-glass theory because of which the theory has come to be regarded as a sort of paradigm for ''complexity.''[24] The AT instability, as it is currently understood, means that theoretical descriptions of a spin-glass-like phase that regard its phase space as a simple one similar to that of a ferromagnet, rather than as one with many almost degenerate valleys separated by barriers that diverge in the thermodynamic limit, will be plagued by unstable behavior of physical quantities.[5,24]

The AT instability first arose in a mean-field theory for the Ising spin glass using the replica method, but it has since been found to appear in all mean-field treatments of that phase: in the familiar Sherrington-Kirkpatrick (SK) model, in which each of $N$ spins interacts with all the rest,[16] it appears as a negative eigenvalue in the spectrum of fluctuations about the saddle point;[23] in the Landau-Ginzburg-Wilson functional for the spin-glass phase, it appears as a negative mass for a mode (called the replicon) when the coefficients in the functional are approximated by their mean-field values[17,25] or expanded in $1/d$ about the mean-field values ($d$ is the space dimensionality);[7] in mean-field studies of dynamical relaxation to a *unique* equilibrium configuration, it appears as a negative value of the kinetic coefficient,[26] and is thought to mark the onset of irreversible behavior.[27] To date Parisi's ansatz for breaking the replica symmetry[5]—or considerations based on what that ansatz implies for dynamical relaxation[28] or for the structure of spin-glass phase space[5]—is the only known mechanism for healing the instability.[4,29]

The equilibrium quantity underlying the AT instability is the Edwards-Anderson susceptibility.[17,25] The EA correlation between two sites $i$ and $j$ is the average, over the distribution of exchange bonds, of the square of the correlation between the spins on those sites:

$$\chi_{ij}^{\mathrm{EA}} = \langle(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle)^2\rangle_J, \quad (3.1)$$

so that for a system of $N$ Ising spins, $\chi^{\mathrm{EA}}$ may be defined as

$$\chi^{\mathrm{EA}} = \frac{1}{N}\sum_{i,j=1}^{N} \chi_{ij}^{\mathrm{EA}}. \quad (3.2)$$

In the mean-field approximation, the left-hand side of (3.2) factors into products of single-site correlations and is usually written as[30]

$$\chi^{\mathrm{EA}} \xrightarrow{\text{Mean-field}} \frac{\chi_{ii}^{\mathrm{SK}}}{1-\beta^2\chi_{ii}^{\mathrm{SK}}}, \quad (3.3)$$

$$\chi_{ii}^{\mathrm{SK}}(q_{\mathrm{SK}}) = \int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2)[\operatorname{sech}(y\beta\sqrt{q_{\mathrm{SK}}})]^4, \quad (3.4)$$

where $\beta$, the inverse temperature, is measured in units of the square root of the variance of the bond distribution. In Eq. (3.4) $q_{\mathrm{SK}}$ is the replica-symmetric Edwards-Anderson order parameter in the mean-field approximation; it is given by the self-consistent SK equation.

At low temperatures, when $q_{\mathrm{SK}} \to 1$, the integral in (3.4) is of order $1/\beta$, so that the denominator on the left-hand side of (3.3) is large, $O(\beta)$, and negative. This negative value of $\chi^{\mathrm{EA}}$—and therefore of the replicon eigenvalue or mass,[23,17,25] which are inversely related to $\chi^{\mathrm{EA}}$—lies at the root of the AT instability.

A cluster expansion for $\chi^{\mathrm{EA}}$ is discussed below. Analysis of terms in the cluster expansions of $\chi^{\mathrm{EA}}$ shows that at any *finite* $d$, $\chi^{\mathrm{EA}}$ will be given by a sum of positive terms which are all small at low temperatures,[31] but that the form (3.3) will be recovered *only* in the limit $d \to \infty$. This means that the feature that makes the mean-field $\chi^{\mathrm{EA}}$ negative, namely, that each term is larger than the previous one by a factor of $\beta$, will be absent at finite $d$. A negative value for $\chi^{\mathrm{EA}}$ in the replica-symmetric spin-glass phase at finite $d$ therefore seems highly unlikely.

This behavior of $\chi^{\mathrm{EA}}$ at finite $d$ was determined from analysis of terms with up to four flipped sites in its cluster expansion. These terms, and especially their behavior at $d \to \infty$, are discussed next for a Gaussian distribution of bonds.

To understand the $d \to \infty$ limit of the cluster expansion, it is useful to recall a low-temperature expansion for the SK equation of state for the EA order parameter, $q_{\rm SK}$. The result of iterating that equation may be written as $q_{\rm SK} = 1 - q_1 T - q_2 T^2 - q_3 T^3 + \cdots$, where $q_1$, $q_2$, etc. are functions of $T$ but their expansions in powers of $T$ have nonzero constant terms, so that the iteration series gives a sort of low-$T$ expansion for $q_{\rm SK}$. In the cluster expansion, only flipped sites contribute to the order parameter, and the $d \to \infty$ limit of the contributions from one-, two-, and three-flipped sites gives $q_1$, $q_2$, and $q_3$, respectively.

Consider next the site-diagonal terms in $\chi^{\rm EA}$, which involve averages of the second and fourth powers of the local magnetization. Here again only flipped sites contribute. The one-flipped site contribution gives at $d \to \infty$ $\chi_{ii}^{\rm SK}$ for $q_{\rm SK} = 1$. The two- and three-flipped sites contributions in that limit reduce to, respectively, the next two terms in the low-$T$ expansion obtained when the low-$T$ expansion for $q_{\rm SK}$ is substituted in $\chi_{ii}^{\rm SK}(q_{\rm SK})$.

Two-site correlations $\chi_{ij}^{\rm EA}$ contribute to $\chi^{\rm EA}$ only when both $i$ and $j$ belong to a cluster of 7 flipped sites. The first such contribution arises from states with two flipped sites, 1 and 2, say, and reads

$$\int [dy_{12}][dy_1][dy_2] \frac{t_{12}^2 (1-t_1^2)^2 (1-t_2^2)^2}{[1+t_1 t_2 t_{12}]^4}, \qquad (3.5)$$

where $t_{1,2} \equiv \tanh(\beta y_{1,2})$ and $t_{12} = \tanh(\beta y_{12})$ and the integrals have Gaussian weights: of variance $2d-1$ for $y_1$ and $y_2$, and of variance unity for $y_{12}$. These variances arise because $(2d-1)$ bonds connect each of the two flipped sites to the rest of the lattice, but only one bond connects them to each other. To obtain the mean-field limit of this contribution to $\chi^{\rm EA}$, multiply it by $d$—the lattice factor, which arises from the sum on the right-hand side (rhs) of (3.2)—scale $\beta^2$ by $1/2d$, and let $d \to \infty$. The denominator is then replaced by unity, $t_{12}$ in the numerator by its linear term, and the remaining integrals on $y_1$ and $y_2$ have Gaussian weights with variance 1. In other words, it gives $\beta^2 [\chi_{ii}^{\rm SK}]^2$ for $q_{\rm SK} = 1$. Not unexpectedly, in view of the discussion above, $\chi_{12}^{\rm EA}$ and $\chi_{23}^{\rm EA}$ from states with three flipped sites (1, 2, and 3, say) at $d \to \infty$ give the first order in the low-$T$ expansion of $\beta^2 [\chi_{ii}^{\rm SK}(q_{\rm SK})]^2$. But $\chi_{13}^{\rm EA}$ for three-flipped sites reads

$$\int [dy_{12}][dy_{23}][dy_1][dy_2] \times [dy_3] \frac{t_{12}^2 t_{23}^2 (1-t_1^2)^2 (1-t_2^2)^2 (1-t_3^2)^2}{[1+t_1 t_2 t_{12} + t_2 t_3 t_{23} + t_1 t_3 t_{12} t_{23}]^4}. \qquad (3.6)$$

Here the variances for the Gaussian weights are $2d-1$ for $y_1$ and $y_3$, $2d-2$ for $y_2$ and 1 for $y_{12}$ and $y_{23}$. Using a lattice factor of $d(2d-1)$ for a chain of three connected sites, the $d \to \infty$ limit gives $\beta^4 [\chi_{ii}^{\rm SK}(1)]^3$. Similarly, the $d \to \infty$ limit of $\chi_{14}^{\rm EA}$ for a chain of four connected sites gives $\beta^6 [\chi_{ii}^{\rm SK}(1)]^4$.

The factorization of $\chi_{12}^{\rm EA}$, $\chi_{13}^{\rm EA}$, and $\chi_{14}^{\rm EA}$ noted above occurs only at $d \to \infty$; because $\chi_{ii}^{\rm SK}$ is $O(T)$ at low temperatures the factorization makes the leading temperature dependence of the contributions to be of order unity, $\beta$ and $\beta^2$.

Just as in the case of free energy,[18] calculations for the contributions of small clusters discussed above immediately lead to generalizations, and it is possible to write the contribution of any cluster to $\chi^{\rm EA}$ or to the order parameter. Furthermore, just as the temperature dependence obtained by explicit calculations of small-cluster contributions to the free energy was argued to apply to larger clusters as well,[18] the contributions of larger clusters to $\chi^{\rm EA}$ will be all small at low $T$. In particular, it is almost impossible that the integrals involved in the cluster contributions to $\chi^{\rm EA}$ will yield terms proportional to positive powers of $\beta$, whose sum may lead to a negative $\chi^{\rm EA}$ at low temperatures. And what other mechanism might give a negative sum for a series of non-negative terms? It therefore seems safe to conclude that $\chi^{\rm EA}$ in the replica-symmetric phase of the Ising spin glass is non-negative at finite $d$.

As an illustration of the calculations involved in the cluster expansion for $\chi^{\rm EA}$, consider $\langle\langle s_i s_j \rangle^2 \rangle_J$. If site $i$ has $p$ flipped replicas, $1 \le p \le n$, and site $j$ has $r_1 + r_2$ flipped replicas, $1 \le r_1 \le p$ and $1 \le r_2 \le n-p$, then

$$\langle\langle s_i s_j \rangle^2 \rangle_J \equiv \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \langle s_i^\alpha s_j^\alpha s_i^\beta s_j^\beta \rangle = \lim_{n \to 0} \frac{1}{n(n-1)} [\langle (n - 2p - 2r_2 + 2r_1)^2 \rangle - n]. \qquad (3.7)$$

The average on the rhs means sums over $p$, $r_1$, and $r_2$ with the following weights: binomial factors for the number of ways $p$, $r_1$, and $r_2$ replicas can be chosen; a ''Boltzmann'' weight, which is the exponential of $\beta^2$ times the sum of squares of overlaps between flipped sites and their nearest neighbors. The other two-site correlations needed in the calculation of $\chi^{\rm EA}$ can be similarly expressed as averages over the numbers of flipped replicas on the two sites. Details of the calculation will be published elsewhere.

Finally, a word about the nonlinear susceptibility $\chi^{\rm nl}$, which is inversely related to the second distinct eigenvalue (also called the longitudinal mode) in the spectrum of fluctuations about the replica-symmetric saddle point.[17,23] Cluster contributions to $\chi^{\rm nl}$ reduce in the limit $d \to \infty$ to a low-temperature expansion of the replica-symmetric mean-field result, just as $\chi^{\rm EA}$ does.

## IV. OUTLOOK

In 1990–1991, a value of zero was reported for the zero-temperature entropy in the replica-symmetric phase using a cluster expansion for the free energy.[18] The unphysical mean-field result for the replica-symmetric phase—namely, the negative value of the zero-temperature entropy—could be recovered from the free-energy expansion if the limit $d \to \infty$ (the mean-field limit) was taken before the limit $T \to 0$. It was remarked at that time that the noncommutativity of the limits in the behavior of the entropy was in agreement with an earlier result by Fisher and Huse, who have argued that the mean-field approximation might be a singular limit of finite-dimensional spin-glass models,[8] but that nothing definitive could be said about the nature of the finite-dimensional phase until $\chi^{\mathrm{EA}}$ was examined for AT instability. The unphysical behavior of the $\chi^{\mathrm{EA}}$ expansion in the $d \to \infty$ limit discussed above strengthens both the Fisher and Huse result about singular behavior at $d \to \infty$, as well as the suggestion by them,[8] by Bray and Moore,[9] and by Bovier and Fröhlich,[10] that the spin-glass phase space in finite dimensions might be much simpler than the one with many valleys arranged in an ultrametric hierarchy, which underlies Parisi's ansatz for breaking the replica symmetry to heal the AT instability of the mean-field theory.

The non-negative value of the replica-symmetric $\chi^{\mathrm{EA}}$ discussed here contradicts the results for $\chi^{\mathrm{EA}}$ obtained using the Landau-Ginzburg-Wilson free-energy functionals.[6,7] The reason for the contradiction might be that studies of the spin-glass transition using free-energy functionals reported to date all start from the mean-field limit, wherefore it is necessary to implement Parisi's replica symmetry breaking and stabilize the ''bare'' replicon mass before the role of non-Gaussian fluctuations (that is, of finite dimensionality) can be considered, and they find that the replica-symmetry broken phase persists in finite dimensions. The contradiction cannot be resolved until the ground-state energies in various spin-glass phases have been compared. But it might also be important to resolve before any such comparison is carried out which of the two possible methods for calculating the ground-state energies for a given value of $d$ will be more reliable: the ones, such as the cluster expansion discussed here, that do not start from the $d \to \infty$ limit, or the ones that expand about that limit.

One might argue that the absence of replica-symmetry-breaking instability in the cluster expansion arises because of improper boundary conditions, as has been found to be the case in studies of the spin glass on a Bethe lattice of finite connectivity.[14] Regarding this or similar criticisms, it is useful to remember that the expansion does recover the instability in $d \to \infty$. In fact, it is the only method to date whereby both the instability at $d \to \infty$ and its possible absence in finite $d$ can be derived from the same analytic expression.

Finally, it must be admitted—and especially so in light of recent developments[32]—that the cluster expansion, as developed so far, may not be a suitable method for determining the nature of the spin-glass phase in finite dimensions. It may, however, be fruitful to develop this expansion further, for it allows evaluation of quantities of interest directly in finite dimensions. One obvious challenge, in developing the expansion further, is to use it to study phases with broken replica symmetry.

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