Magnetic-electric two-dimensional Euclidean group

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The Galilean covariance and the symmetry group for a charged particle in planar motion is investigated. The well-known magnetic translation symmetry is extended into a magnetic two-dimensional Euclidean symmetry. We establish that the symmetry holds more generally in the presence of homogeneous magnetic and electric fields. The symmetry can have useful applications in solid-state physics. [S0163-1829(96)07532-7]

I. INTRODUCTION

It is well known[1] that Galilean symmetry can be realized in the Schrödinger equation only through projective (ray) representations. Bargmann, in his monumental work[2] on unitary ray representations of continuous groups, considered Lie groups which are relevant to quantum mechanics. He treated Abelian groups, the homogeneous and inhomogeneous pseudo-orthogonal groups, and the $(3+1)$-Galilean group. He proved that, besides the Abelian, the only groups that have genuine, nontrivial, projective representations are the inhomogeneous two-dimensional Euclidean group $E(2)$, the inhomogeneous (1,1)-Lorentz group, and the $(3+1)$-Galilean group. However, he was not interested in their realizations except for the projective representation of the $(3+1)$-Galilean that he spelled out explicitly.[1,2] Later, Zak[3] realized the translation symmetry on a plane perpendicular to a homogeneous magnetic field B (magnetic translation group). This corresponds to the projective representation of $E(2)$ that Bargmann had foreseen. Zak had one central extension parameter (one infinitesimal exponent in Bargmann's language), the product qB, where q is the charge of the particle, and opened the way for applications to the quantum theory of solids.[3–10]

In the present paper, we consider the projective representations of $(2+1)$-Galilean group in the Schrödinger equation for a charged particle. In the absence of electromagnetic fields we have the usual ray representation of the Galilean symmetry group. The Galilean mass of the particle serves as an extension parameter. In the presence of a homogeneous magnetic field the Galilean group is no longer a symmetry group. We are left with a magnetic translation symmetry[3] which, by appropriately defining the "rotation" operator, we extend into a magnetic $E(2)$ $[ME(2)]$ symmetry group. Furthermore, the boosts automatically induce an electric field. We can use it to extend the $ME(2)$ symmetry group into a magnetic-electric $E(2)$ $[MEE(2)]$ symmetry group. The generators of "translation" and "rotation" that we give for $MEE(2)$ are conserved and gauge invariant.

In Sec. II we briefly review the projective representation of the $(2+1)$-Galilean group for a free particle in the Schrödinger equation. In Sec. III, we introduce a homogeneous magnetic field and establish the magnetic $E(2)$ symmetry. In Sec. IV, we generalize the $ME(2)$ symmetry group into a $MEE(2)$ symmetry group. In Sec. V, we exhibit some applications and conclude, in Sec. VI, with a short discussion of the results.

II. (2+1)-GALILEAN GROUP

The elements of the $(2+1)$-Galilean group are given by the matrices

$$\mathcal{G}(\phi,\mathbf{u},\mathbf{a},\eta)=\begin{pmatrix}\cos\phi & \sin\phi & u_1 & a_1\\ -\sin\phi & \cos\phi & u_2 & a_2\\ 0 & 0 & 1 & \eta\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{1}$$

where $(\phi,\mathbf{u},\mathbf{a},\eta)$ are the group parameters, the angle ϕ of rotation on the plane, the velocity $\mathbf{u}=(u_1,u_2)$ of a pure Galilean transformation, and the translations in space, $\mathbf{a}=(a_1,a_2)$, and time, η. The corresponding generators are denoted by L, $\mathbf{b}$, $\boldsymbol{\tau}$, and H. They satisfy the following Lie algebra:

$$[L,\tau_k]=i\epsilon_{kl}\tau_l, \tag{2}$$

$$[L,b_k]=i\epsilon_{kl}b_l, \tag{3}$$

$$[b_k,H]=i\tau_k, \tag{4}$$

where $k,l=1,2$, and ϵ_{kl} is the antisymmetric symbol in two dimensions $(\epsilon_{12}=-\epsilon_{21}=1,\epsilon_{11}=\epsilon_{22}=0)$. All other commutators vanish.

In the Hilbert space of the wave functions for a free particle moving on the plane, these generators are realized by the following operators:

$$\boldsymbol{\tau}=\mathbf{p}=-i\nabla, \tag{5}$$

$$\mathbf{b}=m\mathbf{x}-t\mathbf{p}, \tag{6}$$

$$L=\epsilon_{kl}x_kp_l, \tag{7}$$

and

$$H=\frac{\mathbf{p}^2}{2m}, \tag{8}$$

where $\mathbf{x}$ is the position operator and $\mathbf{p}$ the corresponding canonical momentum of the particle, L its angular momentum, and H the Hamiltonian. The constant m is the particle mass and t the time coordinate. The Lie algebra of these generators is a projective extension of the Galilean Lie algebra, with one extension parameter the mass. The commutation relations between translations and boosts do not vanish now, but gain a c-number commutator proportional to the mass,

$$[\tau_k, b_l] = -im\delta_{kl}, \tag{9}$$

characteristic of projective representations. All other commutation relations remain unchanged. In this way the Hamiltonian and the Schrödinger equation retain Galilean symmetry. To be precise, if

$$i\partial_t \Psi(\mathbf{x},t) = -\frac{\nabla^2}{2m}\Psi(\mathbf{x},t) \tag{10}$$

is the equation of Schrödinger in an initial frame, after a boost $\mathbf{u}$ of the particle, $\mathbf{x}\rightarrow\mathbf{x}'=\mathbf{x}+\mathbf{u}t$, the wave function $\Psi(\mathbf{x},t)$ transforms into $\Psi_{\mathbf{u}}(\mathbf{x}',t)$, with an additional phase,

$$\begin{aligned}\Psi_{\mathbf{u}}(\mathbf{x}',t) &= e^{i(m\mathbf{u}\cdot\mathbf{x}+m\mathbf{u}^2 t/2)}\Psi(\mathbf{x},t)\\ &= e^{i(m\mathbf{x}'+it\nabla')\cdot\mathbf{u}}\Psi(\mathbf{x}',t),\end{aligned} \tag{11}$$

and satisfies the Schrödinger equation in the boosted frame, $i\partial_t\Psi_{\mathbf{u}}(\mathbf{x}',t) = -(\nabla'^2/2m)\Psi_{\mathbf{u}}(\mathbf{x}',t)$.

III. PROJECTIVE REPRESENTATION SYMMETRY FOR A CHARGED PARTICLE IN (2+1) DIMENSIONS IN THE PRESENCE OF A HOMOGENEOUS MAGNETIC FIELD

Let us now consider the presence of a homogeneous magnetic field. Our $E(2)$ plane can be thought of as either imbedded into three-dimensional space, in the presence of a homogeneous magnetic field $\mathbf{B}$ perpendicular to it, $\mathbf{x}=(x,y,0)$, and $\mathbf{B}=(0,0,B)$, or as a genuine two-dimensional space. In terms of the two-dimensional vector potential $\mathbf{A}=(A_x,A_y)$, the magnetic field B will be then a (rotation) scalar, $B=\partial_x A_y - \partial_y A_x$.

The Hamiltonian H is replaced by H_B,

$$H_B = \frac{(\mathbf{p}-q\mathbf{A})^2}{2m}. \tag{12}$$

Of course, the Galilean group is no longer symmetry of the Hamiltonian. However, since the magnetic field is homogeneous, our system should exhibit a translational symmetry. This is established by the well-known representation of the magnetic translation group,[3–6] with the translation operators given by[6]

$$\boldsymbol{\tau}_B = \mathbf{p} - q\mathbf{A} + q\mathbf{B}\times\mathbf{x}. \tag{13}$$

Actually, the symmetry group is a wider one. Indeed, we can immediately construct a conserved angular momentum operator

$$L_\Phi = [\mathbf{x}\times(\mathbf{p}-q\mathbf{A})]\cdot\frac{\mathbf{B}}{B} + \frac{q}{2}B\mathbf{x}^2. \tag{14}$$

The first term $[\mathbf{x}\times(\mathbf{p}-q\mathbf{A})]\cdot\mathbf{B}/B$ is the kinetic angular momentum of the particle, and the second term $qB\mathbf{x}^2/2$ expresses the field angular momentum. The magnetic field $\mathbf{B}$ and the Coulomb field $\mathbf{E}$ of the point charge q produce, through the Poynting vector $\mathbf{E}\times\mathbf{B}$, a field angular momentum density which is a function of the position of the charge, $\mathbf{x}$. Integrating appropriately in all space, we find exactly this term. The second term also corrects for the additional phase that the wave function acquires upon rotation by sweeping magnetic flux.

The generator L_Φ together with the magnetic translation operators $\boldsymbol{\tau}_B$ and the Hamiltonian H_B close the following Lie algebra:

$$[\tau_{Bk}, \tau_{Bl}] = -iqB\epsilon_{kl}, \tag{15}$$

$$[\tau_{Bk}, H_B] = 0, \tag{16}$$

$$[L_\Phi, H_B] = 0, \tag{17}$$

$$[L_\Phi, \tau_{Bk}] = i\epsilon_{kl}\tau_{Bl}. \tag{18}$$

The Lie commutators (15) and (16) are the well-known commutators of the magnetic translation symmetry group. Including the generator L_Φ in the algebra, we established that the symmetry group for a charged particle in homogeneous magnetic field is the $ME(2)$ which contains the magnetic translation group as an invariant subgroup. The $ME(2)$ group is the semidirect product of magnetic translations and rotations, $ME(2)=MT(2)\otimes_s SO(2)$.

We remark that our angular momentum L_Φ, which generates a symmetry of the Hamiltonian, is, as it should be, gauge invariant. It coincides with the usual $L=-i\partial_\phi$ angular momentum in the symmetric gauge $\mathbf{A}=(-By/2,Bx/2)$, and saves us from the awkward statement that the angular momentum is conserved only in the symmetric gauge. Physical quantities should be gauge independent. Commutator (18) simply expresses the fact that translators are vectors. With the usual definition of angular momentum, $xp_y - yp_x$, even the vector character of translators was lost in the arbitrary gauge.

Let us now consider the boost operators in the presence of a magnetic field. Although the Galilean group is no longer symmetry of the Hamiltonian, the Galilean covariance will be useful.

The boost operators must change the velocity $\mathbf{v}$ of the particle into

$$e^{-i\mathbf{b}\cdot\mathbf{u}}\mathbf{v}e^{i\mathbf{b}\cdot\mathbf{u}} = \mathbf{v}+\mathbf{u}, \tag{19}$$

or, in commutator form,

$$[b_k, v_l] = i\delta_{kl}. \tag{20}$$

From the boost generators for the free-particle Galilean group, Eq. (6), the generators for the magnetic translations, Eq. (16), and condition (20), we easily guess that the boost generators $\mathbf{b}_B$ in the presence of magnetic field $\mathbf{B}$ must be given by

$$\mathbf{b}_B = m\mathbf{x} - t(\mathbf{p} - q\mathbf{A} + q\mathbf{B}\times\mathbf{x}). \tag{21}$$

The boost operators become now noncommutative:

$$[b_{Bk}, b_{Bl}] = -iqBt^2\epsilon_{kl}. \tag{22}$$

This commutation relation is characteristic of the ray representations of the (2+1)-dimensional Galilean group. Other authors[11–13] recently showed interest in the projective representations of the (2+1)-dimensional Galilean group from the group-theoretical point of view; Eq. (22) offers a physical realization of such projective representations. In the limit $B \to 0$, $\mathbf{b}_B \to \mathbf{b}$, and the boosts commute.

The new boosts are vectors

$$[L_\Phi, b_{Bk}] = i\epsilon_{kl} b_{Bl}, \tag{23}$$

and transform the Schrödinger equation covariantly. If $\Psi(\mathbf{x},t)$ satisfies the Schrödinger equation

$$i\partial_t \Psi(\mathbf{x},t) = \frac{(\mathbf{p}-q\mathbf{A})^2}{2m}\Psi(\mathbf{x},t), \tag{24}$$

the boosted wave function

$$\Psi_{\mathbf{u}}(\mathbf{x},t) = e^{i\mathbf{b}_B\cdot\mathbf{u}}\Psi(\mathbf{x},t) \tag{25}$$

satisfies

$$i\partial_t \Psi_{\mathbf{u}}(\mathbf{x},t) = \left[\frac{(\mathbf{p}-q\mathbf{A})^2}{2m} + q\mathbf{x}\cdot(\mathbf{u}\times\mathbf{B})\right]\Psi_{\mathbf{u}}(\mathbf{x},t)$$
$$= \left[\frac{(\mathbf{p}-q\mathbf{A})^2}{2m} - q\mathbf{x}\cdot\mathbf{E}\right]\Psi_{\mathbf{u}}(\mathbf{x},t). \tag{26}$$

The physical meaning of this equation is apparent. The finite boost $e^{i\mathbf{b}_B\cdot\mathbf{u}}$ has induced the electric field $\mathbf{E}=\mathbf{B}\times\mathbf{u}$, in agreement with Lorentz transformations to first order in velocity $\mathbf{u}$.[14] This fact opens additional possibilities.

IV. GENERALIZING THE $E(2)$ SYMMETRY IN THE PRESENCE OF A HOMOGENEOUS MAGNETIC AND ELECTRIC FIELD

Starting with the problem of a charged particle in homogeneous magnetic field $(\mathbf{B}\neq 0, \mathbf{E}=0)$, we are led to a problem of a charged particle in homogeneous magnetic and electric field $(\mathbf{B}\neq 0, \mathbf{E}=\mathbf{B}\times\mathbf{u})$.

The dynamical system should again exhibit translation symmetry.[15,16] Indeed, we can define the generators of the $MEE(2)$ symmetry group:

$$\boldsymbol{\tau}_{BE} = \mathbf{p} - q\mathbf{A} - m\frac{\mathbf{E}\times\mathbf{B}}{B^2} + q\mathbf{B}\times\mathbf{x} - q\mathbf{E}t, \tag{27}$$

$$L_{\Phi E} = [\mathbf{x}\times(\mathbf{p}-q\mathbf{A}-q\mathbf{E}t)]\cdot\frac{\mathbf{B}}{B} + [m\mathbf{x}-t(\mathbf{p}-q\mathbf{A})]\cdot\frac{\mathbf{E}}{B} + \frac{q}{2}B\left(\mathbf{x}^2+\frac{E^2}{B^2}t^2\right). \tag{28}$$

Together with the Hamiltonian

$$H_{BE} = \frac{(\mathbf{p}-q\mathbf{A})^2}{2m} - q\mathbf{x}\cdot\mathbf{E}, \tag{29}$$

these operators form the following Lie algebra:

$$[\tau_{BEk}, \tau_{BEl}] = -iqB\epsilon_{kl}, \tag{30}$$

$$[L_{\Phi E}, \tau_{BEk}] = i\epsilon_{kl}\tau_{BEl}, \tag{31}$$

$$[\tau_{BEk}, H_{BE}] = iqE_k, \tag{32}$$

$$[L_{\Phi E}, H_{BE}] = i\frac{E_k}{B}\tau_{BEk}, \tag{33}$$

where E_k are the Cartesian components of the electric field. The Lie algebra of Eqs. (30) and (31) is a projective extension of the $E(2)$ group, the $MEE(2)$ group. It is a symmetry group for our Hamiltonian, H_{BE}. Indeed, using expressions (27) and (28) and the commutators (32) and (33), we find that $\boldsymbol{\tau}_{BE}$ and $L_{\Phi E}$ are conserved in time,

$$\frac{d\boldsymbol{\tau}_{BE}}{dt} = \partial_t\boldsymbol{\tau}_{BE} + \frac{1}{i}[\boldsymbol{\tau}_{BE}, H_{BE}] = 0, \tag{34}$$

$$\frac{dL_{\Phi E}}{dt} = \partial_t L_{\Phi E} + \frac{1}{i}[L_{\Phi E}, H_{BE}] = 0. \tag{35}$$

Our generators $\boldsymbol{\tau}_{BE}$ differ slightly from the corresponding electric magnetic translators[15] $\boldsymbol{\pi}$ given by Ashby and Miller. In their gauge, we have $\boldsymbol{\tau}_{BE}=\boldsymbol{\pi}-m(\mathbf{E}\times\mathbf{B})/B^2$; $\boldsymbol{\tau}_{BE}$ is, of course, gauge invariant. Both $\boldsymbol{\tau}_{BE}$ and $\boldsymbol{\pi}$ are conserved, but $\boldsymbol{\tau}_{BE}$ is needed for the correct closure of the Euclidean algebra, Eqs. (30) and (31). Our translators are vectors. The recent very interesting work of Zak[16] is concerned with homogeneous constant and/or time-periodic electric fields. In our language, such a case is in the electric-type electromagnetic field sector $(\mathbf{E}^2-\mathbf{B}^2>0)$. In the present work, we remained in the magnetic sector $(\mathbf{B}^2-\mathbf{E}^2>0)$.

Finally, we remark that the boosts can also be defined for a given B and $\mathbf{E}$ as the operators which tangentially add velocity $\mathbf{v}\to\mathbf{v}+\mathbf{u}$ to the particle, leaving $\mathbf{E}$ unchanged as $\mathbf{u}\to 0$,

$$\mathbf{b}_{BE} = e^{i\mathbf{b}_B\cdot\mathbf{u}}\nabla_{\mathbf{u}} e^{-i\mathbf{b}_B\cdot\mathbf{u}}\big|_{\mathbf{u}=[(\mathbf{E}\times\mathbf{B})/B^2]}$$
$$= m\mathbf{x} + \frac{q}{2}\mathbf{E}t^2 - t(\mathbf{p}-q\mathbf{A}+q\mathbf{B}\times\mathbf{x}). \tag{36}$$

The term $q\mathbf{E}t^2/2$ expresses the ''fall'' of the particle inside the electric field. The commutator between the boosts remains unchanged, Eq. (22), but their commutator with the ''rotation'' operator is modified:

$$[L_{\Phi E}, b_{BEk}] = i\epsilon_{kl}\left(b_{BEl} + \frac{q}{2}E_l t^2\right). \tag{37}$$

V. APPLICATIONS

The above symmetries may find useful applications. For example, using the boost operators $\mathbf{b}_B$, out of the Green's function for a charged particle in homogeneous magnetic field[17]

$$G_B(\mathbf{x},\mathbf{x}';t,t') = \frac{qB\Theta(t-t')e^{-iq\mathbf{B}\cdot(\mathbf{x}\times\mathbf{x}')/2}}{4\pi i\sin[qB(t-t')/2m]} \times \exp\left\{i\frac{qB}{4}(\mathbf{x}-\mathbf{x}')^2\cot\left[\frac{qB(t-t')}{2m}\right]\right\}, \tag{38}$$

where $\Theta(x)$ is the Heaviside unit step function, we obtain the Green's function

$$G_{BE}(\mathbf{x},\mathbf{x}';t,t')=\frac{qB\Theta(t-t')e^{-iq\mathbf{B}\cdot(\mathbf{x}\times\mathbf{x}')/2}}{4\pi i\sin[qB(t-t')/2m]}e^{im(\mathbf{E}\times\mathbf{B})\cdot(\mathbf{x}-\mathbf{x}')/B^2}\exp\left\{-i(t-t')\left[\frac{m}{2}\frac{E^2}{B^2}-q\mathbf{E}\cdot\frac{(\mathbf{x}+\mathbf{x}')}{2}\right]\right\}$$
$$\times\exp\left\{i\frac{qB}{4}\left[\mathbf{x}-\mathbf{x}'-(t-t')\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right]^2\cot\left[\frac{qB(t-t')}{2m}\right]\right\}, \tag{39}$$

for a charged paticle in the presence of both magnetic and perpendicular electric fields. Expression (39) is derived from (38), through the transformation

$$G_{BE}(\mathbf{x},\mathbf{x}';t,t')=e^{-iq(\mathbf{u}\times\mathbf{B})\cdot(\mathbf{x}t-\mathbf{x}'t')/2}e^{-im\mathbf{u}^2(t-t')/2}\times e^{im\mathbf{u}\cdot(\mathbf{x}-\mathbf{x}')}G_B(\mathbf{x}-\mathbf{u}t,\mathbf{x}'-\mathbf{u}t';t,t'), \tag{40}$$

as a consequence of the transformation of the wavefunctions, Eq. (25). The expressions of Eqs. (38) and (39) have been calculated for simplicity in the symmetric gauge. The corresponding expressions for any gauge are easily derived through the gauge transformation for the Green's function.

Of special interest is the time-independent partial wave Green's function $G_{BE}(E,\kappa)$ characterized by the energy $\mathbf{E}$ diagonalized simultaneously with one component of the translation operator $\boldsymbol{\tau}_{BE}$. This Green's function may be appropriate for Hall-effect applications. The electric field $\mathbf{E}$ is perpendicular to $\mathbf{B}$, and the time-independent conserved component of $\boldsymbol{\tau}_{BE}$, which we chose to diagonalize together with the Hamiltonian, is perpendicular to both $\mathbf{B}$ and $\mathbf{E}$. We denote by κ the eigenvalue of this component of the translation operator. Conveniently in the Landau gauge, $\mathbf{A}=(-By,0)$, we have

$$G_{BE}(\mathrm{E},\kappa)=\frac{m}{i\sqrt{\pi qB}}\Gamma(\tfrac{1}{2}+\alpha)[\Theta(y-y')U(\alpha,w)U(\alpha,-w')+\Theta(y'-y)U(\alpha,w')U(\alpha,-w)], \tag{41}$$

where the electric field was taken along the y axis, $\mathbf{E}=(0,E_y)$ and $U(\alpha,w)$ are the parabolic cylinder functions,[18] $\alpha=-m[\mathrm{E}-\kappa E_y/B-m/2(E_y/B)^2]/qB$, $w=\sqrt{2qB}(y+\kappa/qB)$, and $w'=\sqrt{2qB}(y'+\kappa/qB)$.

The $ME(2)$ or $MEE(2)$ symmetries may also find applications in solid-state physics. Let us add a periodic potential $V(\mathbf{x})=V(\mathbf{x}+\mathbf{a})$ where $\mathbf{a}$ is a lattice vector, to the Hamiltonian for a charged particle in homogeneous magnetic and/or electric field. Then, out of the previously studied continuous symmetry, there remains a discrete symmetry typical for crystals. This certainly contains the overlap symmetry between the $MEE(2)$ and the discrete translation symmetry of the crystal.[15,16] Since $MEE(2)$, besides translations, contains rotations the symmetry is a wider one, it is the overlap of $MEE(2)$ with the space-group symmetry of the crystal. This extended discrete symmetry, in the case of $ME(2)$, may be applied to simplify the construction of the band structure of solids in the presence of an external homogeneous magnetic field.

Our symmetry group generators are related to fundamental physical quantities: the generator $\boldsymbol{\tau}_B$ to the crystal momentum $\mathbf{k}$, and L_Φ to the rotational elements of the point group of the crystal. Their conservation, we hope, will have useful applications. The conservation of $\boldsymbol{\tau}_B$ leads immediately to the well-known acceleration theorem in the presence of a magnetic field. In our scheme the proof is simple and straightforward. Consider first the case without an external magnetic field. Let $T(\mathbf{a})$ be the translation operator along a lattice vector $\mathbf{a}$, $T(\mathbf{a})=e^{i\mathbf{p}\cdot\mathbf{a}}$. As we can easily verify for an eigenstate of $T(\mathbf{a})$, a Bloch state, the operator which corresponds to the crystal momentum $\mathbf{k}$, is

$$\mathbf{k}=-i\ln[T(\mathbf{a}_l)]\mathbf{a}_l^*, \tag{42}$$

where $\mathbf{a}_l,\mathbf{a}_l^*$ are the primitive vectors of the direct and reciprocal lattices,

$$\mathbf{a}_l^*=\epsilon_{lk}\frac{\mathbf{a}_k\times\mathbf{B}}{\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{B})}. \tag{43}$$

The operator $\mathbf{k}$ commutes with the Hamiltonian $\mathbf{p}^2/2m+V(\mathbf{x})$, and gives rise to a conserved quantity, the crystal momentum or quasimomentum. In the presence of a homogeneous electric field $\mathbf{E}$, a term $-q\mathbf{E}\cdot\mathbf{x}$ is added to the Hamiltonian, and $T(\mathbf{a})$ no longer commutes with it,

$$\left[T(\mathbf{a}),\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-q\mathbf{E}\cdot\mathbf{x}\right]=-q\mathbf{E}\cdot\mathbf{a}T(\mathbf{a}). \tag{44}$$

The Heisenberg equation of motion then gives:

$$\frac{d}{dt}T(\mathbf{a})=iq\mathbf{E}\cdot\mathbf{a}T(\mathbf{a}), \tag{45}$$

and integrating, we obtain

$$T(\mathbf{a})=e^{i(q\mathbf{E}\cdot\mathbf{a}t+\mathbf{p}\cdot\mathbf{a})}. \tag{46}$$

Taking the logarithm of $T(\mathbf{a})$ and using the definition of the crystal momentum, Eq. (42), we obtain

$$\mathbf{k}(t)=q\mathbf{E}t+\mathbf{k}(0). \tag{47}$$

Thus we have the acceleration theorem. The time derivative of the crystal momentum operator equals the electric force applied to the particle,

$$\frac{d}{dt}\mathbf{k}=q\mathbf{E}. \tag{48}$$

We now proceed to the proof of the acceleration theorem in the presence of a homogeneous magnetic field. The Hamiltonian is $H_B+V(\mathbf{x})$, and we define the appropriate magnetic quasimomentum operator by

$$\mathbf{k}_B=-i\ln[T_B(\mathbf{a}_l)]\mathbf{a}_l^*, \tag{49}$$

where

$$T_B(\mathbf{a}_l)=\exp(i\boldsymbol{\tau}_B\cdot\mathbf{a}_l) \tag{50}$$

is a primitive magnetic translation. The operator $\mathbf{k}_B$ corresponds to the magnetic wave vector,[19] and is conserved:

$$\frac{d}{dt}\mathbf{k}_B=\frac{1}{i}[\mathbf{k}_B,H_B+V(\mathbf{x})]=0. \tag{51}$$

If, in addition to **B**, there is an external electric field, the Hamiltonian becomes $H_{BE}+V(\mathbf{x})$, and the magnetic translation operator $T_B(\mathbf{a})$ is again nonconserved

$$\frac{d}{dt}T_B(\mathbf{a})=\frac{1}{i}[T_B(\mathbf{a}),H_{BE}+V(\mathbf{x})]=iq\mathbf{E}\cdot\mathbf{a}T_B(\mathbf{a}). \tag{52}$$

Integrating Eq. (52), we obtain

$$T_B(\mathbf{a})=\exp[i(q\mathbf{E}\cdot\mathbf{a}t+\boldsymbol{\tau}_B\cdot\mathbf{a})], \tag{53}$$

and, taking the logarithm, we find the evolution of the magnetic wave vector

$$\mathbf{k}_B(t)=q\mathbf{E}t+\mathbf{k}_B(0). \tag{54}$$

This establishes the acceleration theorem in the presence of a magnetic field,

$$\frac{d}{dt}\mathbf{k}_B=q\mathbf{E}. \tag{55}$$

Using the operator $\boldsymbol{\tau}_{BE}$ instead of $\boldsymbol{\tau}_B$, one could of course arrive at the same result, through the conservation of the magnetic-electric translations, $T_{BE}(\mathbf{a})=\exp(i\boldsymbol{\tau}_{BE}\cdot\mathbf{a})$.

The previous results also hold for the more general case of simultaneous electric and magnetic periodicity. Let us assume for simplicity that we have a periodic magnetic field $\mathbf{B}_p$ in addition to the homogeneous magnetic field **B**. The vector potential is $\mathbf{A}+\mathbf{A}_p$, where $\nabla\times\mathbf{A}=\mathbf{B}$ and $\nabla\times\mathbf{A}_p=\mathbf{B}_p$. Without loss of generality, $\mathbf{A}_p$ is considered periodic. The Hamiltonian becomes

$$H_B'=\frac{(\mathbf{p}-q\mathbf{A}-q\mathbf{A}_p)^2}{2m}+V(\mathbf{x}). \tag{56}$$

We easily verify that $T_B(\mathbf{a})$ commutes with H_B' and consequently we have the conservation of the magnetic wave vector and the corresponding Bloch theorem. Indeed, according to Eq. (56), H_B' can be written as

$$H_B'=\frac{(\mathbf{p}-q\mathbf{A})^2}{2m}+q^2\frac{\mathbf{A}_p^2}{2m}+V(\mathbf{x})-q\frac{(\mathbf{p}-q\mathbf{A})\cdot\mathbf{A}_p+\mathbf{A}_p\cdot(\mathbf{p}-q\mathbf{A})}{m}. \tag{57}$$

The sum of the first three terms is of the type already considered, and therefore commutes with $T_B(\mathbf{a})$; **A** is a vector potential for the homogeneous magnetic field **B** and the sum $q^2\mathbf{A}_p^2/2m+V(\mathbf{x})$ acts as a periodic potential. The last term of Eq. (57) also commutes with $T_B(\mathbf{a})$. We have

$$[\mathbf{A}_p,T_B(\mathbf{a})]=0, \tag{58}$$

since $\mathbf{A}_p$ is periodic, and

$$[(p-qA)_k,T_B(\mathbf{a})]=0, \tag{59}$$

from the well-known[3,5] commutators

$$[(p-qA)_k,\tau_{Bl}]=0. \tag{60}$$

The Bloch theorem in the presence of magnetic periodicity has been considered recently[20] by other authors.

Finally, in the presence of an additional external electric field, we again establish the acceleration theorem Eq. (55), since

$$[T_B(\mathbf{a}),H_B'-q\mathbf{E}\cdot\mathbf{x}]=-q\mathbf{E}\cdot\mathbf{a}T_B(\mathbf{a}). \tag{61}$$

VI. CONCLUSIONS

We have investigated the Galilean covariance, the symmetry group, and the corresponding projective representations for a charged particle in planar motion in the presence of a homogeneous magnetic field. We thus offered physical realization for the projective representations of the Euclidean group. The well-known magnetic translation symmetry has been extended into a magnetic two-dimensional Euclidean symmetry, and more generally into a magnetic-electric $E(2)$ symmetry. All generators, rotations, and translations were gauge invariant and conserved. Their conservation leads to acceleration theorems.

Our scheme is useful for applications to solid-state physics. We proved that the acceleration theorem holds more generally in the presence of electric and/or magnetic periodicity. The $ME(2)$ group together with the space group of the crystal may also be useful for finding a band structure in the presence of homogeneous magnetic fields.

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