Duality and universality for the Chern-Simons bosons

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By mapping the relativistic version of the Chern-Simons-Landau-Ginzburg theory in $2+1$ dimensions to the three-dimensional (3D) lattice Villain x-y model coupled with the Chern-Simons gauge field, we investigate phase transitions of Chern-Simons bosons in the limit of strong coupling. We construct algebraically exact duality and flux attachment transformations of the lattice theories, corresponding to analogous transformations in the continuum limit. These transformations are used to convert the model with arbitrary fractional Chern-Simons coefficient α to a model with α either zero or 1. We show that, unless the corrections of cubic and higher power in momenta, irrelevant near corresponding fixed points, render the phase transition first order, the phase transition in the original model is either in the 3D x-y or a ''fermionic'' universality class. [S0163-1829(96)06428-4]

INTRODUCTION

As one changes the external magnetic field, the Hall conductance of a system of two-dimensional electrons jumps from one quantized value to another, while the longitudinal conductance displays a peak. The zero temperature localization length diverges in transition points, indicating the second-order quantum phase transitions. Remarkably, experiments[1–3] show that these phase transitions are universal, i.e., the critical exponents governing the divergence of the localization length are the same for integer and fractional transitions. It has also been found[4] that the critical resistances for integer and fractional phase transitions are universal as well. Finally, a brilliant recent experiment by Shahar *et al.*[5] demonstrates a precise duality mapping of nonlinear current-voltage characteristics between the insulating and fractional quantum Hall sides of the transition, where carriers are respectively electrons and fractionally charged quasiparticles. One of the most challenging problems in the quantum Hall effect is to understand the universality of the transitions.

Apparently, there is a conceptual difference between the physics of the integer and the fractional quantum Hall regimes. In the former case the quasiparticles are fermions; their localization transition is believed to be a one-particle problem. The physics is understood quasiclassically as quantum-tunneling-assisted percolation,[6–8] while the nonlinear σ model with a topological term[9] provides the field-theoretical formalism to study this problem. On the other hand, excitations above the fractional quantum Hall effect ground state, given with extreme precision by inherently many-body Laughlin's wave functions,[10] have an infinite-range statistical interaction that seems to invalidate the usual one-particle localization theory. Field theoretically, this interaction can be described in terms of the Chern-Simons gauge field coupled to either bosons[11,12] or fermions.[13–16]

Using the bosonic description, called the Chern-Simons-Landau-Ginzburg (CSLG) theory[12] of the quantum Hall effect, Kivelson, Lee, and Zhang (KLZ) proposed[17] a global phase diagram of the quantum Hall effect. The original problem of interacting electrons in a large magnetic field is mapped onto the problem of bosons in zero or weak uniform magnetic field, interacting via the Chern-Simons gauge field. Information about the filling factor is contained only in the coefficient of the Chern-Simons term. The electromagnetic response functions in the CSLG theory can be expressed solely in terms of response functions of the bosonic field and, therefore, the different quantum Hall transitions are in the same universality class if the critical exponents of the boson superfluid to insulator phase transition are independent of the Chern-Simons coefficient.

On the basis of the CSLG theory in the random-phase approximation (RPA), KLZ argued that the critical exponents of Hall transitions are universal; they predicted values for critical conductances that have been recently confirmed experimentally.[4] However, since the Chern-Simons term is a marginal operator by naïve power counting, it could in principle change the critical exponents. If this happens, the transition will no longer be universal. Therefore, it is highly desirable to check the idea of universality in some specific field-theoretical calculation. Unfortunately, it is very hard to treat the full interacting problem in the presence of disorder. For this reason the analysis is often restricted to toy models that nevertheless share some essential features with real quantum Hall systems.

The most spectacular feature of the quantum Hall system is the symmetry of the phase diagram, and there are several field-theoretical models with a similar symmetry. A broad class of such models has been introduced by Shapere and Wilczek,[18] who generalized the construction of Cardy and Rabinovici[19,20] and constructed four- and two-dimensional self-dual models consisting of mutually dual discrete Abelian gauge lattice models coupled via the θ term. A similarly constructed $2+1$-dimensional model with the Chern-Simons term was analyzed by Rey and Zee.[21] Lütken and Ross[22,23] discussed the symmetry of fixed points of a self-dual two-dimensional clock model, and argued that the corresponding group $SL(2,Z)$ describes the symmetry of all fixed points of the quantum Hall system. This statement is analogous to the law of corresponding states;[17] unfortunately, so far no derivation of the field-theoretical realization of this symmetry group for the quantum Hall system is known.

The question of whether phase transitions in the system of

Chern-Simons bosons can even in principle be universal is so general that the answer for virtually any such model would be interesting. Particularly, one can study the phase transition driven by the amplitude of an external periodic potential at commensurate filling instead of that driven by disorder. It is known[24] that the system of bosons at fillings commensurate with external periodic potential can be mapped onto the x-y model. In the quantum Hall problem one arrives[25] at an x-y model minimally coupled with the Chern-Simons term in zero average magnetic field. Within this model one can naturally address the question of whether the order-disorder transition is altered by coupling with the Chern-Simons term and, if so, whether there is an additional universality class, or the phase transition continuously depends on the Chern-Simons coupling. Even though this is an artificial model, its successful analysis will undoubtedly shed some light on the universality of the real quantum Hall phase transitions.

Wen and Wu[25] used the representation of the the x-y model in terms of the relativistic complex-valued scalar field with quartic interaction, especially convenient for perturbational expansion. They also used the $1/N$ expansion, known to work for ordinary bosons, to regularize the perturbational series in three dimensions. Within the second loop approximation Wen and Wu found that although the phase transition remains of the second order, the critical exponents of the x-y model are continuously changed by the Chern-Simons coupling.

Soon after that, we pointed out[26] that the $1/N$ expansion artificially suppresses the gauge fluctuations in this model, and the result of Wen and Wu is not likely to hold in the physical limit of $N=1$. Our alternative calculation[26] was performed in the vicinity of a nominal tricritical point of the theory, where the quartic interaction is absent and one has to keep the six-field coupling. The effective theory in the vicinity of the tricritical point is renormalizable in three dimensions (3D), and allows a systematic loop expansion, exactly as the theory with four-field interaction is renormalizable in four dimensions where the quartic coupling is dimensionless. The presence of the Chern-Simons coupling, also dimensionless in 3D, does not introduce any classically divergent quantities, but it already has a dramatic effect at the second loop level, changing the transition from the second to the first order. This is similar to the statement[27] that the phase transition in clean type-I superconductors becomes weakly first order due to fluctuations of the electromagnetic field. In our case, since the theory without the quartic interaction is massive, it remains massive in some finite region of values of the quartic coupling. Thus, within the perturbative region of the relativistic complex scalar field model, the transition is of first order even for some nonzero values of the quartic coupling.

This conclusion obtained in the vicinity of the Gaussian fixed point of the scalar sector was not in formal contradiction with the result of Wen and Wu obtained near the strongly coupled fixed point (formally accessed with the $1/N$ expansion,) but it reopened the question about the universality of phase transitions of scalar fields coupled with the Chern-Simons field. The results of both calculations indicate that the Chern-Simons interaction is a *relevant* perturbation to the scalar theory with strong short-range repulsion, and that some kind of nonperturbative analysis is necessary.

The goal of this paper is to address the question of universality of phase transitions in relativistic Chern-Simons bosons without the limitations of the perturbation theory. We access the strong-coupling limit of the Chern-Simons bosons using the Villain form of the x-y model minimally coupled with the Chern-Simons gauge field. For this model we formulate the exact duality and flux attachment *transformations* with usual properties in the continuum limit, and build the Haldane-Halperin[28,29] hierarchy of models related by these nonlocal transformations. We show that in the absence of an external magnetic field the universality class of phase transition in the strong-coupling limit is determined by the Chern-Simons coupling $\alpha=p/q$. When either p or q is even, the phase transition is in the universality class of the order-disorder transition in the usual three-dimensional x-y model. When both p and q are odd, the system in the vicinity of a critical point (if any) is equivalent to the x-y model with the Chern-Simons coupling $\alpha=1$, corresponding to the Fermi statistics. We believe that our result is exact, and that it applies at least for fractions with small enough denominators q. With increasing denominators the terms formally irrelevant near the x-y critical point grow, effectively limiting the number of fractions with expected universality.

Unlike previous works[30–35] analyzing lattice Chern-Simons models, we derive algebraically exact flux attachment and duality transformations. Thus we avoid the problem of relevance or irrelevance of the higher order in momenta terms appearing in different lattice definitions of the Chern-Simons coupling. Instead, we construct an exact transformation to the theory with zero α, equivalent to the usual 3D x-y model with finite lattice temperature. For this theory the algebra of local critical operators is known, and the formal irrelevance of extra terms does not require additional proof.

The paper is organized as follows. We derive the exact duality transformation for the Villain form of the x-y model in Sec. I, and the exact flux attachment transformation in Sec. II. These transformations have a correct naive continuum limit, and work for lattice theories with gauge coupling of the most general form. In Sec. III we construct the sequence of such transformations leading to the Villain x-y model with a Chern-Simons coupling of either zero or one, and utilize known properties of the usual x-y model to build the phase diagram of the original model.

I. GENERALIZED DUALITY

Particles in a superfluid repel each other at short distances. At very small temperatures most of the particles are in the same quantum state, characterized by the condensate wave function $\Psi(x,t)=\psi \exp i\theta$, where both the amplitude ψ and the phase θ are real-valued functions. Slow phase rotations have an extremely small energy, implying the existence of a linear sound mode.

Another important kind of excitation in superfluids are vortices in two dimensions, or vortex lines in three dimensions. They are characterized by a nontrivial phase of the condensate wave function gained along any surrounding contour. Far enough from the center of the vortex the gradient of the phase is small, and the superfluid density ψ^2 is close to its nonperturbed value. Although the reduced density in the core region near the center of the vortex costs some energy, the total energy of a single vortex is mostly determined by the twisted phase in the area outside the core, and any two vortices have a long-range logarithmic interaction.

Quite differently, vortices in superconductors have only short-range interactions (just like *particles* in a superfluid): the extra phase is easily screened by the vector potential. Instead, the conserved particle currents in normal phase interact logarithmically through their magnetic fields just like *vortices* in a superfluid. This analogy can be formulated more precisely as a *duality* between type-II superconductors in the London limit, where the penetration length of the magnetic field is large compared to the coherence length and the strong-coupling limit of the Ginzburg-Landau model. In this limit of strong coupling one can completely neglect the density fluctuations, considering only the phases θ_n as degrees of freedom. The core energy of the vortices can be regularized by redefining the theory at the lattice. The resulting x-y model is a collection of classical spins $s_n=(\cos\theta_n,\sin\theta_n)$ with nearest-neighbor interaction of the form $\mathbf{s}_i\mathbf{s}_j=\cos(\theta_i-\theta_j)$. It turns out that a reverse transformation from the x-y model to the strongly coupled superfluid is also possible.[36] This two-way analogy has been confirmed by comparing the results of numeric computations in the lattice x-y model with corresponding analytical results for the ϕ^4 theory.

The partition function of the x-y model,

$$Z_{x\text{-}y}=\int\frac{d\theta_n}{2\pi}\exp\sum\frac{\cos(\theta_{n+\mu}-\theta_n)}{T},\tag{1}$$

is periodic with respect to the phases θ_n and, at least in the strong-coupling limit where the lattice temperature[37] T is small, is mainly determined by the vicinity of maxima of the expression in the exponent. The shape of these maxima can be simulated using the Villain form of the x-y model,

$$\exp\frac{\cos(\theta_{n+\mu}-\theta_n)}{T}\rightarrow\sum_m\exp-\frac{(\theta_{n+\hat{\mu}}-\theta_n-2\pi m_{n\hat{\mu}})^2}{2T}$$

originally proposed by Berezinskii,[38] and later independently and in greater detail investigated by Villain.[39]

Peskin[40] proved that duality transforms the Villain x-y model into a so-called frozen superconductor, or the zero-temperature limit of the Villain x-y model coupled to the Maxwell field. This rigorous analysis has been extended by Kleinert,[36] who analyzed the lattice superconductor in a wide region of parameters using the Villain approximation, and achieved quantitative agreement[41] with numerical simulations. These model computations eventually led[42–44] to an understanding of the complete phase diagram of superconductors in the dual representation. Later Lee and Fisher[45,46] and Lee and Zhang[47] applied this duality transformation to the anyon superconductivity and the quantum Hall effect.

We are interested in a very similar class of models, namely, the strong-coupling limit of relativistic Chern-Simons bosons. The idea is to follow the successful example of a superconductor and examine nonperturbative properties of this model at the lattice using the Villain form of the x-y model coupled to the Chern-Simons gauge field. There are several[48,31,32,35] different gauge-invariant definitions of the lattice Chern-Simons coupling, and, in order to keep the analysis as universal as possible, we consider the three-dimensional Villain x-y model

$$Z[A]=\prod_{n\mu}\int_{-\pi}^{\pi}\frac{d\theta_n}{2\pi}\times\sum_{m=-\infty}^{\infty}\exp-\frac{(\triangle_\mu\theta_n-A_{n\mu}-2\pi m_{n\mu})^2}{2T},\tag{2}$$

$$Z=\int\exp\left(-\frac{1}{2}A_{n\mu}\mathcal{K}^{\mu\nu}_{nn'}A_{n'\nu}\right)Z[A]\prod_{n,\rho}dA_{n\rho},\tag{3}$$

at the cubic lattice, minimally coupled to the gauge field $A_{n\mu}$ with some generic, possibly nonlocal kernel $\mathcal{K}^{\mu\nu}_{nn'}$, or $\mathcal{K}^{\mu\nu}(\mathbf{k})=\mathcal{K}(\mathbf{k})$ in the momentum representation. By the *generic Chern-Simons coupling* we shall imply that in the continuum limit $\mathcal{K}$ must be an antisymmetric matrix linear in momenta.

The summation over links μ in (2,3) is performed only in the positive direction $\hat{\mu}$; the phases θ_n are defined in the nodes, while both the integer-valued field $m_{n\mu}$ and the gauge field $A_{n\mu}$ are defined at the links of three-dimensional cubic lattice. We use the notations $\triangle_\mu f_n=f_{n+\hat{\mu}}-f_n$ and $\tilde{\triangle}_\mu f_n=f_n-f_{n-\hat{\mu}}$ for the forward and backward lattice differences, respectively, and imply the summation over repeated indices.

It will be convenient to specify the Fourier expansion of vector fields in the form

$$A_{n\mu}=\frac{1}{\sqrt{N}}\int\int\int_{-\pi}^{\pi}\frac{d^3k}{(2\pi)^3}A_\mu(\mathbf{k})e^{ik_\mu/2}e^{i\mathbf{k}\cdot\mathbf{r}_n}\tag{4}$$

(no summation in μ) to emphasize their location at the links of the lattice. With this definition the gradient $\triangle_\mu f_n=\tilde{\triangle}_\mu f_{n+\hat{\mu}}$ or $P_\mu f(\mathbf{k})$ in the momentum representation is a vector, while the divergence $\tilde{\triangle}_\mu A_{n\mu}=\triangle_\mu A_{n-\hat{\mu}\mu}$ or $P_\mu A_\mu(\mathbf{k})$ is a scalar field, as one would expect from the geometrical interpretation of the lattice fields. We shall use the vector $P_\mu=2\sin(k_\mu/2)$ to define different functions of lattice momentum. For example, the kernel of the local in the gauge-field Chern-Simons term used in Refs. 48, 31, and 32 can be written as $\mathcal{K}_{\mu\nu}=\varepsilon^{\mu\nu\rho}P_\rho Q_0/2\pi\alpha$, with $Q_0=\cos\Sigma_\mu k_\mu/2$, and in Sec. II we shall define the Chern-Simons term with the kernel $\mathcal{K}_{\mu\nu}=\varepsilon^{\mu\nu\rho}P_\rho/2\pi\alpha Q_0$, providing for the local coupling between the gauge-invariant scalar currents on the lattice.

Let us treat the action $Z[A]$ as the Villain x-y model in the presence of some external gauge field A, and follow the derivation[40] of the duality transformation for this model. While the integration in phases θ_n is performed over restricted intervals, the gauge transformation

$$\theta_n\rightarrow\theta_n+2\pi N_n$$
$$m_{n\mu}\rightarrow m_{n\mu}+N_{n+\hat{\mu}}-N_n\tag{5}$$

does not change the integrand of (2). One can extend the integration region in θ_n to the infinite interval by simultaneously constraining the field $m_{n\mu}$ to avoid overcounting. This can be done, for example, by applying the usual gauge condition

$$0=\sum_{\mu}\,(m_{n+\hat{\mu}\mu}-m_{n\mu}). \tag{6}$$

Now introduce an auxiliary field $b_{n\mu}$ by writing

$$Z[A]=\prod_{n}\int_{-\infty}^{\infty}\frac{d\theta_n}{2\pi}\prod_{\mu}\left(\frac{T}{2\pi}\right)^{1/2}\int db_{n\mu}\times\sum_{m_{n\mu}}\exp\left[-\frac{T}{2}b_{n\mu}^2+ib_{n\mu}(\triangle_\mu\theta_n-A_{n\mu}-2\pi m_{n\mu})\right]. \tag{7}$$

The integration by θ_n can be done after regrouping the terms in the exponent,

$$\int_{-\infty}^{\infty}\frac{d\theta_n}{2\pi}e^{-i\theta_n\tilde{\triangle}_\mu b_{n\mu}}=\delta(\tilde{\triangle}_\mu b_{n\mu}), \tag{8}$$

the resulting constraint being merely the lattice version of the equation $\nabla\cdot\mathbf{b}=0$ indicating that the field $\mathbf{b}$ is a pure curl,

$$b_{n\mu}=\varepsilon^{\mu\nu\rho}\tilde{\triangle}_\nu a_{n-\hat{\rho}\rho}. \tag{9}$$

So far the only difference from the duality transformation[40] for the pure x-y model is the presence of the product $-ib_{n\mu}A_{n\mu}$ in the exponent of (7). This extra term in the action is responsible for important additional symmetry. Indeed, the original gauge field in the full partition function (3) can be integrated away,

$$Z=\prod_{n\mu}\left(\frac{T}{2\pi}\right)^{1/2}\int_{\theta_n,b_{n\mu}}\sum_{m_{n\mu}}e^{ib_{n\mu}(\triangle_\mu\theta_n-2\pi m_{n\mu})}\times\exp\left[-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}b_\mu(-\mathbf{k})(T\delta_{\mu\nu}+\mathcal{K}_{\mu\nu}^{-1})b_\mu(\mathbf{k})\right], \tag{10}$$

and, since the field b is transverse, it is clear that the partition function Z depends only on the transverse combination

$$T\delta_{\mu\nu}^t+\mathcal{K}_{\mu\nu}^{-1}, \tag{11}$$

indicating the exact equivalence of any two models with the parameters satisfying the relationship

$$T_1\delta_{\mu\nu}^t+\mathcal{K}_{1\mu\nu}^{-1}=T_2\delta_{\mu\nu}^t+\mathcal{K}_{2\mu\nu}^{-1}, \tag{12}$$

where $\delta_{\mu\nu}^t=\delta_{\mu\nu}-\hat{P}_\mu\hat{P}_\nu$ denotes the transverse part of the Kronecker symbol, and $\hat{P}_\nu\equiv P_\nu/|P|$ is the unit vector. This reparametrization changes both the gauge and scalar couplings; it is an exact symmetry of the partition function of the Villain x-y model or any correlators that do not involve the gauge field directly. These are the only physical correlators as long as we treat the Chern-Simons field as an auxiliary field needed to define the fractional statistics[49] for the particles. Since the lattice temperature T measures the strength of the local repulsion, relationship (12) implies that a part of this repulsion can be mediated by the gauge field if its propagator has a nonzero symmetric part.

One may further verify that the system averages taken in the presence of any number of vortex-antivortex pairs $\exp i(\theta_n-\theta_{n'})$, as well as the monopoles dual to them, do not depend on a particular selection of parametrization. Moreover, although at the tree level the properties of quasiparticles appear to be changed, any correlators involving the gauge-invariant current retain their values. This can be checked by performing the same transformation in the presence of an additional external gauge field A_0: the reparametrization does not affect the minimal gauge coupling with this external field.

Having established the reparametrization symmetry (12) of the complete model (2) and (3), let us resume the transformations of the partition function (7). As usual, the field $a_{n\rho}$ introduced in (9) is defined up to a gauge transformation

$$a_{n\mu}\to a_{n\mu}+(\lambda_{n+\mu}-\lambda_n) \tag{13}$$

with arbitrary λ_n; for simplicity we imply the Coulomb gauge $\tilde{\triangle}_\mu a_{n\mu}=0$ for all our gauge fields. The integration over θ_n results in

$$Z[A]=\int_a\sum_m e^{-Tb_{n\mu}^2/2-2\pi ib_{n\mu}m_{n\mu}-iA_{n\mu}b_{n\mu}}, \tag{14}$$

or, after regrouping the second term in the exponent,

$$Z[A]=\int_a\sum_m e^{-Tb_{n\mu}^2/2+2\pi ia_{n\mu}M_{n\mu}-iA_{n\mu}b_{n\mu}}, \tag{15}$$

where the integer-valued vorticity is defined as

$$M_{n\mu}=\varepsilon^{\mu\nu\rho}(m_{n+\hat{\mu}+\hat{\nu}\rho}-m_{n+\hat{\mu}\rho}). \tag{16}$$

Since the vorticity is locally conserved,

$$\sum_\mu\,(M_{n\mu}-M_{n-\mu\mu})=0, \tag{17}$$

we may exchange the summation over $m_{n\mu}$ for a sum over $M_{n\mu}$ subject to condition (17). This integer-valued constraint can in turn be removed by additional phases θ_n,

$$\delta_{0,\tilde{\triangle}_\mu M_{n\mu}}=\int_{-\pi}^{\pi}\frac{d\theta_n}{2\pi}e^{i\theta_n\tilde{\triangle}_\mu M_{n\mu}}. \tag{18}$$

To finish the transformation of the dual model toward the Villain form analogous to the original model (2) and (3), Peskin[40] introduced a convergence factor

$$1=\lim_{t\to 0}\exp\left[-\frac{t}{2}M_{n\mu}^2\right] \tag{19}$$

into Eq. (15) and transformed the resulting sum

$$\lim_{t\to 0}\sum_M e^{-tM_{n\mu}^2/2+2\pi ia_{n\mu}M_{n\mu}+i\theta_n\tilde{\triangle}_\mu M_{n\mu}} \tag{20}$$

with the Poisson summation formula

$$\sum_{M=-\infty}^{\infty}e^{i\varphi M-tM^2/2}=\left(\frac{2\pi}{t}\right)^{1/2}\sum_{m=-\infty}^{\infty}e^{-(\varphi-2\pi m)^2/2t}. \tag{21}$$

After rescaling the gauge fields $2\pi a_{n\mu}=\tilde{A}_{n\mu}$, $2\pi b_{n\mu}=\tilde{B}_{n\mu}$, the dual action can be finally written as

$$Z[A]=\lim_{t\to 0}\prod_n \int_{-\pi}^{\pi}\frac{d\theta_n}{2\pi}\prod_\mu \int d\widetilde{A}_{n\mu}$$
$$\times\sum_m \exp\left[-\frac{1}{2t}(\triangle_\mu\theta_n-2\pi m_{n\mu}-\widetilde{A}_{n\mu})^2 -\frac{i}{2\pi}A_{n\mu}\widetilde{B}_{n\mu}-\frac{T}{8\pi^2}\widetilde{B}_\mu^2\right]. \qquad (22)$$

Again, we can integrate away the original gauge field $A_{n\mu}$ to obtain the final form of the dual action,

$$Z=\lim_{t\to 0}\int_{\widetilde{A},\theta}\sum_m \exp\left[-\frac{1}{2t}(\triangle_\mu\theta_n-2\pi m_{n\mu}-\widetilde{A}_{n\mu})^2 +\frac{1}{8\pi^2}\widetilde{A}_{-k}\mathbf{P}\times(T+\mathcal{K}^{-1})\times\mathbf{P}\widetilde{A}_k\right], \qquad (23)$$

where the appropriate summation in each term of the exponent is implied. In accord with our previous conclusion, the action depends only on the transverse part of the combination $(\mathcal{K}^{-1}+T\delta^t)_{\mu\nu}$. In addition, the dual model has a form of (2) and (3) with zero lattice temperature $t=0$ and the dual gauge kernel[50]

$$\widetilde{\mathcal{K}}_0=\frac{TP^2-\mathbf{P}\times\mathcal{K}^{-1}\times\mathbf{P}}{(2\pi)^2}=\frac{P^2}{(2\pi)^2}(T+\mathcal{K}^{-1}), \qquad (24)$$

where only the transverse part enters because of the gauge fixing for the field $\widetilde{A}$. The dual model has the same form as the original one and, since the additional gauge field was introduced as an auxiliary field, this model has the same reparametrization freedom (12). Therefore, the most general form of the dual model with the lattice temperature $\widetilde{T}$ and the gauge kernel $\widetilde{\mathcal{K}}$ satisfies the equation

$$\widetilde{T}+\widetilde{\mathcal{K}}^{-1}=0+\widetilde{\mathcal{K}}_0^{-1}=\frac{(2\pi)^2}{P^2}(\mathcal{K}^{-1}+T)^{-1}, \qquad (25)$$

or, more symmetrically,

$$(T+\mathcal{K}^{-1})(\widetilde{T}+\widetilde{\mathcal{K}}^{-1})=\frac{(2\pi)^2}{P^2}. \qquad (26)$$

An alternative direct derivation of the finite-temperature dual action, as well as the vortex-monopole mapping explicitly relating the scalar sectors of the two models is provided in Appendix A. This algebraically exact mapping proves that dual models of the form (2) and (3) related by the generalized duality (26) are equivalent, being just two different representations of the same model.

To provide an example of the duality in the presence of the Chern-Simons coupling, let us consider the simplest linear in momenta form of the gauge kernel

$$\mathcal{K}^{\mu\nu}(k)=\frac{1}{2\pi\alpha}\varepsilon^{\mu\nu\rho}P_\rho. \qquad (27)$$

While this definition is very well convergent toward the continuum limit, and generally looks like a plausible definition of the Chern-Simons kernel, it is actually nonlocal in a coordinate representation. The conventional duality transformation results in the zero-lattice-temperature model (23) with the gauge kernel

$$\widetilde{\mathcal{K}}_0^{\mu\nu}(k)=-\frac{\alpha}{2\pi}\varepsilon^{\mu\nu\rho}P_\rho+\frac{T}{(2\pi)^2}[P^2\delta^{\mu\nu}-P^\mu P^\nu], \qquad (28)$$

analogous to the frozen superconductor.[40] The two terms here are a nonlocal Chern-Simons term of the same form as (27), and the usual lattice Maxwell action. The zero lattice temperature implies a zero tree level propagator for the matter field, in analogy with the dual theory[12] of nonrelativistic continuum bosons coupled to a purely statistical Chern-Simons gauge field. However, already in the RPA approximation, the dynamics of the dual gauge field $\widetilde{A}$, associated with the nonzero dimensionful charge $e^2=(2\pi)^2/T$, renders this propagator finite, therefore introducing some nonzero lattice temperature for the dual matter field.

Let us try to understand this result using our generalized duality transformation (26). Specifically, we want to find a form of the dual model with a purely statistical gauge field, so that all dynamics would be associated with the scalar field.

To simplify the subsequent algebra, it is convenient to introduce special notation for typical matrices appearing in the expressions. The matrices $\delta^t_{\mu\nu}=\delta_{\mu\nu}-\hat{P}_\mu\hat{P}_\nu$ and $[\hat{I}]_{\mu\nu}=\varepsilon^{\mu\nu\rho}\hat{P}_\rho$ commute with each other, while $[\hat{I}^2]_{\mu\nu}=-\delta^t_{\mu\nu}$. Algebraically, the symmetric matrix $\delta^t_{\mu\nu}$ is equivalent to unity, while $\hat{I}$ plays a role of $i=\exp(i\pi/2)$. Since we always assume a transverse gauge, it is possible not to write $\delta^t_{\mu\nu}$ at all, and treat $\hat{I}$ as a commuting number. In these notations the dual gauge kernel (28) can be written as

$$\widetilde{K}_0=-\hat{I}\frac{\alpha P}{2\pi}+\frac{TP^2}{(2\pi)^2},$$

so that the corresponding inverse matrix in the transverse gauge is just

$$\frac{1}{\widetilde{K}}+\widetilde{T}\overset{(25)}{=}\frac{1}{\widetilde{K}_0}=\frac{(2\pi)^2}{-2\pi\hat{I}\alpha P+TP^2} \approx\frac{2\pi\hat{I}}{\alpha P}\left(1-\frac{\hat{I}TP}{2\pi\alpha}\cdots\right)=\frac{2\pi\hat{I}}{\alpha P}+\frac{T}{\alpha^2}+\cdots. \qquad (29)$$

Naively, the first term of the expansion may be associated with the inverse of the modified dual Chern-Simons kernel, while the second constant term plays a role of the dual temperature. We notice that while the *Chern-Simons term is the same as the one in the representation with* $T=0$, *the* F^2 *term has been traded for the nonzero lattice temperature* $\widetilde{T}=T/\alpha^2$ *in the scalar sector.*

In reality, we cannot just discard the extra terms denoted by ellipsis in Eq. (29), but we certainly have the freedom to choose the value $\widetilde{T}=T/\alpha^2$ for the dual temperature as long as the general duality condition (26) is satisfied. With this particular choice the exact dual gauge kernel becomes

$$\widetilde{\mathcal{K}} = -\frac{\hat{I}\alpha P}{2\pi}\frac{1}{\hat{I}x+(1+\hat{I}x)^{-1}}, \tag{30}$$

where we denoted $x = TP/2\pi\alpha$. The exact kernel $\widetilde{\mathcal{K}}$ differs from the truncated dual kernel

$$-\frac{\alpha}{2\pi}\varepsilon^{\mu\nu\rho}P_\rho \equiv -\frac{\alpha}{2\pi}[\hat{I}]_{\mu\nu}P$$

only by corrections of the higher order $\widetilde{A}\mathcal{O}(P^3)\widetilde{A}$, so that in the naive continuum limit the dual gauge field $\widetilde{A}$ does not have any F^2 term or associated dynamics.

It is tempting to propose that, as long as these corrections are small, as terms of higher order in momenta they should be irrelevant, and that the scaling properties of both the truncated and exact forms of the dual theory should be equivalent, with their corresponding critical indices equal. Then, using the previously established mapping between theories that are dual as specified by Eq. (26), one could conclude that the two models with gauge kernels of the form (27) and the corresponding parameters (T,α) and $(T/\alpha^2,-1/\alpha)$ should also be equivalent at large scales. The first statement is, however, not so obvious because the Chern-Simons theories are intrinsically nonlocal, and the critical dimensions of different operators may substantially differ from their classical values. We shall prove the legitimacy of this procedure in Sec. III.

Along with the dimensionless parameter α we carried out a transformation of the classically irrelevant dimensionful lattice temperature T. Clearly, since the values of the lattice momentum P are limited from the above, the small values of the ratio $T/|\alpha| = \widetilde{T}/|\widetilde{\alpha}|$ ensure the smallness of x, so that the effect of higher-order irrelevant terms is negligible, and the bare value of the lattice temperature T should be meaningful. Remarkably, in the presence of the Chern-Simons term the duality transformation does not lead to a reversion[44,51] of the temperature axis characteristic of the duality between the superconductor and the superfluid. In our case only the Chern-Simons coupling constant is inverted, while the scalar sector may remain in the strong-coupling regime.

The physical properties of excitations in the generalized x-y model are revealed by their coupling with external fields in 2+1 dimensions. Luckily, the introduction of the additional external gauge field eA_0 does not influence the exact procedure of the duality transformation; now it results in the partition function (22) up to the substitution $A_{n\mu} \to A_{n\mu} + eA_{0n\mu}$. Integration over the original gauge field A and a proper shift of the dual gauge field $\widetilde{A}$ results in the frozen model (23) with additional minimal coupling to the external field

$$-\frac{e}{\alpha}\frac{1}{1+\hat{I}x}A_0, \quad x = \frac{TP}{2\pi\alpha}$$

and the constant term

$$\frac{e^2}{4\pi\alpha}A_0(-k)\frac{\hat{I}P}{1+\hat{I}x}A_0(k) \tag{31}$$

in the exponent. These two terms remain intact in reparametrized models with nonzero lattice temperatures.

It is clear that the model describes particles with the charge $\widetilde{e} = -e/\alpha$, while the momentum-dependent term $1+\hat{I}x$ in the denominator of the coupling can be regarded as a form factor indicating the finite size of excitations and the presence of the Magnus force. Term (31), dependent only on the external gauge field, shows that the duality transformation produced the condensate with the charge density $-e^2\widetilde{\triangle}\times A_0/2\pi\alpha = -e^2B_0/2\pi\alpha$ determined by the external magnetic field B_0. More specifically, if the original theory had the average charge density ρ, the total charge density of excitations in the dual representation is given by

$$\widetilde{\rho} = \rho - \frac{e^2B_0}{2\pi\alpha}. \tag{32}$$

One flux quantum of the magnetic field binds the charge e/α, or exactly one quasiparticle in the dual representation. Since the duality translates the original quasiparticles with charge e into vortices with unit vorticity, this condensate can be also interpreted as the average vorticity $1/\alpha$ per quasiparticle in the dual model.

Certainly, the external magnetic field does not simplify the lattice model at all: even the one-particle Hofstadter spectrum in the magnetic field is very complicated. However, if both the magnetic field and particle density are small at the scale of the lattice, only the filling factors of electrons $\nu = 2\pi\rho/e^2B$ and quasiparticles $\widetilde{\nu} = 2\pi\widetilde{\rho}/\widetilde{e}^{\,2}B$ are important. In this case Eq. (32) becomes

$$\widetilde{\nu} = \alpha(\alpha\nu - 1). \tag{33}$$

II. PERIODICITY IN CHERN-SIMONS COUPLING

Nonrelativistic continuum models display periodicity in the Chern-Simons coupling because geometrically this coupling is just the linking number between the trajectories of quasiparticles. As long as this linking number remains an integer, equal increments in the Chern-Simons coefficient result in the total shift of the phase of the partition function by integer multiples of 2π. This is always so when quasiparticles avoid each other—for example, if they are fermions or if they have strong hardcore repulsion. Although the x-y model *is* already in the limit of pointlike strong repulsion as viewed from the corresponding continuum model, the current lines can intersect or even join each other, and special effort is needed to define a Chern-Simons action that is periodic in its coupling. In this section we find such a definition and use it to derive the exact transformation analogous to the flux attachment transformation in the continuum, valid for an arbitrary form of the lattice Chern-Simons action.

The current of quasiparticles in the x-y model is defined as the field canonically conjugated with the gradients of phases $\triangle_\mu\theta_n$. Since the partition function is periodic in these phases, they are cyclic variables and the current is integer valued. As one would expect, the current operator is not diagonal in terms of the phases θ_n; it is more convenient to deal with currents in the special current representation. In the Villain approximation the explicit form of this representation can be obtained with the inverse of the Poisson summation formula (21). The integration over phases θ_n ensures the conservation of the current $M_{n\mu}$, and the partition function (2) and (3) rewritten in terms of this current becomes

$$Z=\sum_{M} \delta_{0,\tilde{\triangle}_{\mu}M_{n\mu}}\int_{dA} e^{-TM_{n\mu}^{2}/2-iM_{n\mu}A_{n\mu}-A_{n\mu}\mathcal{K}_{nn'}^{\mu\nu}A_{n'\nu}/2}. \tag{34}$$

It is important to emphasize that the transformation to the integer current representation (34) did not change the gauge field A in any way, so that the gauge kernel $\mathcal{K}_{nn'}^{\mu\nu}$ in Eqs. (3) and (34) is exactly the same.

The linking number between lattice current flow lines does not have a natural geometrical meaning at intersection points and therefore cannot be uniquely defined to fit all intuitive requirements. There is, however, no difficulty in defining the linking number for any two conserved fields $M_{n\mu}$ and $N_{n\mu}$ determined on mutually dual lattices since they never intersect. Formally, this coupling can be introduced via the solution a^0 of equations

$$\varepsilon^{\mu\nu\rho}\triangle_{\mu}a_{n\nu}^{0}=N_{n-\hat{\rho}\rho}, \quad \tilde{\triangle}_{\nu}a_{n\nu}^{0}=0 \tag{35}$$

as $\mathcal{N}=\Sigma M_{n\mu}a_{n\mu}^{0}$. Since the field $N_{n\mu}$ is defined on the links of the dual lattice (or at the plaquettes of the primary lattice,) the auxiliary field $a_{n\nu}^{0}$ is defined on the links of the primary lattice, and the summation is well defined. One can prove that this definition is indeed the integer-valued linking number by rewriting the conserved current $M_{n\mu}$ as a superposition of some number n_{ℓ} of directed loops L^i, $i=1,\ldots,n_{\ell}$ carrying unit current each. This splits the expression for $\mathcal{N}$ into a number of sums over independent closed loops, and by the Stokes theorem each of them is exactly equal to the (integer) flux of current N through the surface delimited by the corresponding loop; obviously the total $\mathcal{N}$ is the integer-valued linking number.

Now let us define the regularized integer *self-linking* number of the integer-valued current M in exactly the same way but with additional identification $M_{n\mu}\equiv N_{n\mu}$. This identifies neighboring parallel links of the original and dual lattices mutually displaced in the direction (1,1,1) by half a period. This displacement, uniquely determining the integer-valued regularized linking number, is revealed in Fourier representation, where the auxiliary equations (35) take the form

$$\varepsilon^{\mu\nu\rho}(e^{ik_{\mu}}-1)e^{ik_{\nu}/2}a_{\nu}^{0}(\mathbf{k})=e^{-ik_{\rho}/2}M_{\rho}(\mathbf{k}),$$

$$(1-e^{-ik_{\nu}})a_{\nu}^{0}(\mathbf{k})=0,$$

(no summation in ρ) or, somewhat more symmetrically,

$$\varepsilon^{\mu\nu\rho}P_{\mu}a^{0}{}_{\nu}=ie^{-i(k_x+k_y+k_z)/2}M_{\rho}, \quad P_{\nu}a_{\nu}^{0}(\mathbf{k})=0. \tag{36}$$

Solving these equations with respect to auxiliary field a^0, we obtain the regularized self-linking number in Fourier representation

$$\mathcal{N}=\mathrm{Tr}a^{0}{}_{\mu}(-\mathbf{k})M_{\mu}(\mathbf{k})=i\ \mathrm{Tr}\varepsilon^{\mu\nu\rho}M_{\mu}(-\mathbf{k})\frac{Q_0P_{\nu}}{P^2}M_{\rho}(\mathbf{k}),$$

where the original exponent $\exp i/2\Sigma_{\mu}k_{\mu}$ in (36) was replaced with $Q_0\equiv Q_0(\mathbf{k})=\cos\Sigma_{\mu}k_{\mu}/2$, allowing for the symmetry of the total sum. The exponent of the linking number $\mathcal{N}$ can be further transformed to form a gauge coupling

$$\exp-i\pi\alpha\mathcal{N}\rightarrow\int_{da}\exp\sum_{k}\ -iM_{\mu}(-k)a_{\mu}(k)$$
$$+a(-k)\frac{\hat{I}P}{4\pi\alpha Q_0}a(k), \tag{37}$$

with an additional transverse fluctuating field a introduced as the Hubbard-Stratonovich field. Clearly, the gauge coupling has exactly the form of that in (34) with the gauge kernel

$$\mathcal{K}_{\mu\nu}(\mathbf{k})=\varepsilon^{\mu\nu\rho}\frac{P_{\rho}}{2\pi\alpha Q_0}. \tag{38}$$

The momentum expansion of this kernel starts with a linear antisymmetric term and has no quadratic in the momenta part; therefore, it corresponds to the Chern-Simons term in the long-distance limit. The apparent singularity of this gauge kernel at the plane $k_x+k_y+k_z=\pi$ does not lead to any divergences, but merely suppresses fluctuations of the field a in the vicinity of this plane. One can totally avoid introducing a somewhat unpleasant division by zero by considering periodic lattices with an odd number of lattice nodes in each direction, so that at any finite system size the singularity of the kernel is never reached. This also eliminates the problem of a zero functional denominator arising from integration over the transverse part of the gauge field a in (37).

Consider model (34) with the specific form of the gauge kernel (38). This partition function is convergent at all positive lattice temperatures T even though the second term can be zero for certain configurations. One can use the Poisson summation formula to rewrite the model in the form (2) and (3) depending only on the phases θ. By construction, this model is a periodic function of α with the period $\triangle\alpha=2$, so that any two models with the couplings related by

$$\alpha\rightarrow\alpha+2m, \quad T\rightarrow T \tag{39}$$

are absolutely equivalent to each other. Definitely, this periodicity is precisely the flux attachment symmetry as it was formulated for nonrelativistic systems. It is important to mention that none of the previously considered definitions[48,31–34] of the lattice Chern-Simons term provide for this property; this is why we needed to construct yet another form of the Chern-Simons coupling. The flux attachment transformation changes the properties of the gauge field, but our main interest is to understand the effect of the fractional statistics on the dynamics of the scalar field; we use the *auxiliary* Chern-Simons field only to get rid of the nonlocal interaction.

Even though one can construct other forms of gauge coupling leading to the same symmetry of the partition function, these forms are very special; generic gauge couplings do not reveal the exact flux attachment *symmetry*. Moreover, it is easy to check that this property is not even preserved by the duality transformation (26). Therefore, we need to define the flux attachment *transformation* to display the same periodicity (39) at least in the continuum limit for some broad class of possible lattice Chern-Simons terms.

In order to do this, let us introduce the trivial phase $2\pi m\mathcal{N}$ proportional to the integer-valued self-linking number $\mathcal{N}$ *in addition* to the already present in Eq. (34) gauge coupling with the gauge kernel $\mathcal{K}$. Clearly, this trivial phase does not change the partition function, and yet, after rewriting the introduced term as the additional gauge coupling (37) and integrating away one of the gauge fields, we obtain the same model but with the gauge kernel

$$\mathcal{K}' = \left(\mathcal{K}^{-1} - \frac{4\pi m \hat{I} Q_0}{P} \right)^{-1}. \tag{40}$$

The resulting model is rigorously equivalent to the original model with the gauge kernel $\mathcal{K}$, and it is convergent at any nonzero lattice temperature T.

The long-range properties of the flux-attached gauge kernel (40) depend on the properties of the original kernel $\mathcal{K}$. If $\mathcal{K}$ has the Chern-Simons form, i.e., if its momentum expansion starts with the linear in momenta term $\mathcal{K}_{\mu\nu} \sim \varepsilon^{\mu\nu\rho} k_\rho / 2\pi\alpha$, the expansion of the gauge kernel $\mathcal{K}'$ has the same form with the coefficient $\alpha' = \alpha + 2m$. In the particularly simple case when the matrix structure of the gauge kernel $\mathcal{K}_{\mu\nu}$ is the combination of $\delta^t_{\mu\nu}$ and $[\hat{I}]_{\mu\nu}$, we may further simplify the flux attachment transformation by defining the form factor Q through $\mathcal{K} = \hat{I} P Q^{-1} / 2\pi\alpha$. Then the appropriately defined form factor Q' for the kernel $\mathcal{K}'$ is determined simply as the linear combination

$$Q' = \frac{\alpha Q + 2 m Q_0}{\alpha + 2m}. \tag{41}$$

Earlier we adjusted the lattice temperature to obtain the purely statistical Chern-Simons field without any dynamics in the naive continuum limit, the appropriate lattice gauge kernel having no quadratic terms in the small momentum expansion. This is equivalent to the expansion of the form factor $Q(k) = 1 + \mathcal{O}(k^2)$, $k \to 0$; obviously this property is preserved by Eq. (41).

Before discussing the implications of this flux attachment transformation on the universality of the phase transitions in Chern-Simons models, let us reflect on the fact that the derived expression (40) is an exact symmetry of any lattice model with the integer-valued current, moreover, it is valid not only for Villain models, but also for the usual x-y models minimally coupled with with some fluctuating gauge field. If the original model has no long-range interaction between the quasiparticles, it is in the universality class of the x-y model, while the transformed model has a Chern-Simons term with the gauge kernel (38) and the coupling $\alpha = 2m$. The duality transformation results in the model of the same form with the lattice temperature $\widetilde{T} = T/4m^2$ and Chern-Simons coupling $\widetilde{\alpha} = -1/2m$, the small momentum expansion being valid as long as $TP \ll 4\pi m$. The continuum limit of this model is that of scalar particles coupled with the even-denominator Chern-Simons field; yet this theory is rigorously equivalent to the original x-y model.

A completely different situation arises if we try to add such additional Chern-Simons interaction to a lattice superconductor described by the same model (2) and (3) but with *quadratic* in momenta gauge kernel $\mathcal{K}$. Qualitatively, the interaction introduced with the CS coupling is working independently of the original one. In the coordinate representation this additional interaction between current lines decays as $1/r$ at large distances; in order to have any effect it should not be screened by the original interaction. However, current lines in the normal state already have the magnetic interaction *logarithmic* at large distances, and the additional coupling cannot have any effect on the long-distance physics: it will not be visible at all at some finite scale.

Although this statement appears to be rather evident, in some cases the form of interaction at large distances may not be obvious. For example, Schultka and Manousakis[52] considered the 3D x-y model with the Chern-Simons coupling introduced for *vortices* instead of the real charges. Numerically, the phase transition in this model is in the x-y universality class for all values of the Chern-Simons coupling, and this was interpreted as explicit proof of the universality of phase transitions in the Chern-Simons theories. We do not agree with this interpretation: clearly, the model dual to the x-y model is the frozen lattice superconductor, and the CS coupling between the vortices of the original model is just the extra coupling between the currents of the dual theory. But the currents in the normal state of the superconductor already have long-range magnetic interaction; the additional coupling is screened, and the large distance behavior is effectively merely that of the original x-y model. This statement can be obtained more rigorously from the small-momentum expansion of the combined gauge kernel derived in Appendix B.

III. LAW OF CORRESPONDING STATES

For the Villain x-y model (2) and (3) with fractional statistics α introduced by coupling to an auxiliary gauge field with the kernel obeying $\mathcal{K}^{\mu\nu} = \varepsilon^{\mu\nu\rho} k_\rho / 2\pi\alpha + \mathcal{O}(k^3)$ at small momenta, we found that the exact duality transformation [(26), $\widetilde{T} = T/\alpha^2$] and flux attachment transformation [(40), $T' = T$] have very natural and simple forms,

$$\widetilde{\alpha} = -1/\alpha, \quad \widetilde{T} = T/\alpha^2,$$

$$\alpha' = \alpha + 2m, \quad T' = T,$$

in terms of the Chern-Simons coefficient α. Although α fully describes the gauge coupling only in the limit of small momenta, we would like to check whether these relationships are meaningful by themselves at least for some values of the lattice temperature T and the Chern-Simons coupling α.

In the continuum, the duality and flux attachment transformations can be used to ''unwind'' the Chern-Simons coupling with the fractional parameter $\alpha_0 = p_0/q_0$. Shifting α_0 by an appropriate even integer number, we can always satisfy inequalities $-1 < \alpha_0' = \alpha_0 + 2m_0 \leqslant 1$. If $\alpha_0' = 1$, or $\alpha_0' = 0$, we cannot proceed any further with our transformations and should stop. Otherwise the duality transformation yields $\alpha_1 = -1/\alpha_0' = p_1/q_1$ with the integer denominator $0 < q_1 < q_0$. Repeating such transformations, in a finite number of steps $n < q_0$ we arrive at either $\alpha_n' = 1$ or $\alpha_n' = 0$ with the final lattice temperature $T_n = q_0^2 T_0$ given simply by the denominator q_0 of the original Chern-Simons coupling. Since both the shift and duality transformation preserve the parity of the sum of the numerator and denominator of α, it is clear that any fraction with this sum odd (i.e., either nu-

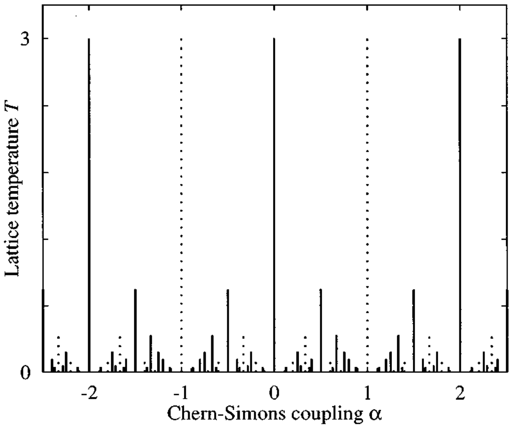

FIG. 1. The phase diagram of the Chern-Simons Villain x-y model. Vertical lines represent ordered phases of different symmetry, corresponding to different levels of hierarchy. The phases with the sum of numerator and denominator *odd* can be mapped to usual Villain x-y model; they are shown with solid lines. The phases with ''fermionic'' Chern-Simons coupling are only shown schematically with dotted lines, since we know only the ratios of corresponding transition temperatures. Unlike the case of usual superconductor, the duality transformation does not invert the direction of the lattice temperature axis, and the high-temperature phases are always disordered.

merator or denominator is an even number) will lead to $\alpha_n=0$, and the original fraction with both p and q odd will result in $\alpha_n=1$.

In the former case the final theory is just the pure x-y model without any additional coupling. Up to a finite renormalization correction, the phase-transition temperature in this model is just that in the usual Villain x-y model,[41] $T_n \sim T_{\rm VXY}=3.03$. We immediately obtain values

$$T_0\left(\alpha=\frac{p_0}{q_0}\right)\sim\frac{T_{\rm VXY}}{q_0^2} \tag{42}$$

of the phase-transition temperatures in the Villain Chern-Simons models with either the numerator or denominator given by even numbers, as plotted in Fig. 1 with solid lines. Similarly, once the phase-transition temperature in a single x-y model with ''fermionic'' Chern-Simons coupling $\alpha=1$ is known, one can use the same relationship to obtain the approximate values of the phase-transition temperatures for all ''fermionic'' fractions with both the numerator and denominator odd.

Let us rectify the outlined procedure by using the exact duality (26) and the flux attachment (40) transformations at every step, starting with the local theory with the gauge kernel (38) and the initial parameters α_0 and T_0. The partition function with this gauge kernel is periodic with respect to the coupling α_0, and we can safely assume that $0<|\alpha_0|<1$. The dual model has the lattice temperature $T_1=T_0/\alpha_0^2>T_0$ and the gauge kernel

$$\mathcal{K}_1\equiv\widetilde{\mathcal{K}}_0=\frac{\hat{I}P}{2\pi\alpha_1}\,\frac{1}{\hat{I}x_0+(Q_0+\hat{I}x_0)^{-1}}, \tag{43}$$

where $x_0=PT_0/2\pi\alpha_0$, $|x_0|\leqslant T_0/\pi|\alpha_0|$ and the the dual coupling $\alpha_1=-1/\alpha_0$. It is convenient to rewrite the dual gauge kernel as

$$\mathcal{K}_1=\frac{\hat{I}P}{2\pi\alpha_1 Q_1}, \tag{44}$$

by introducing the next level form factor

$$Q_1=\hat{I}x_0+\frac{1}{Q_0+\hat{I}x_0}. \tag{45}$$

A similarly introduced form factor of the CS coupling resulting from the flux attachment transformation (40) can be expressed merely as

$$Q_1'=\frac{\alpha_1 Q_1+2m_1 Q_0}{\alpha_1+2m_1}, \tag{46}$$

where $\alpha_1+2m_1=\alpha_1'$ is the resulting gauge coupling; obviously, both transformations preserve the small momentum expansion of the form factor $Q(k)=1+\mathcal{O}(k^2)$. At the nth step α_n becomes an integer, and the universality class of the resulting theory is determined by the parity of α_n.

If this number is an even integer, the final shift similar to (46) results in the gauge kernel *singular* at small momenta, and the long-distance gauge interaction is suppressed. The remaining short-range interaction between the currents perturbs the x-y critical point, and one can prove that it is irrelevant in this point by simple power counting. As formally irrelevant, this additional interaction may result either in some finite correction to the transition temperature, or it has to change the symmetry of the phase transition completely. In the former case, since the models at every level of hierarchy are exactly equivalent to the original Chern-Simons model with the fractional coupling α_0, we conclude that the phase transition in the original model is in the same x-y universality class.

Similar arguments apply when the final α_n is an odd number corresponding to the Fermi statistics of the quasiparticles. Although we do not know the universality class of the phase transition for Chern-Simons bosons with odd-integer-valued coupling, we can claim that it should be the same for all original models in this class unless the irrelevant terms drive it away from the critical point.

So far we have concentrated our attention on the more pleasant possibility that the irrelevant terms are too weak to change the symmetry of the critical point, and assumed that the phase transition does not change its universality class or become the first order. Let us try to analyze the irrelevant terms more carefully to see what fractions are likely to follow the universality scenario.

We saw that the bare lattice temperatures of the transitions decrease rapidly with the hierarchy level; therefore the temperature-dependent convergence condition $x_k=PT_k/2\pi\alpha_k\ll 1$ is not very limiting. There is, however, the momentum contribution from the form factor $Q_0=\cos\Sigma_\mu k_\mu/2$ of the local Chern-Simons kernel (38). To investigate the extreme possible effect of this contribution, let us write the sequence of hierarchical form factors at zero lattice temperature for the fraction

$$\alpha_0=\cfrac{1}{2m_1+\cfrac{1}{2m_2+\cdots}}, \tag{47}$$

starting with the form factor Q_0. The first duality transformation in the unwinding procedure results in $Q_1=1/Q_0$, $\alpha_1=-1/\alpha_0$, and the flux attachment transformation (46) leads to the form factor

$$Q_1'=\frac{1}{Q_0}\left[1-2m_1\left(2m_2+\frac{1}{2m_3+\cdots}\right)\mathcal{P}^2\right], \tag{48}$$

where

$$\mathcal{P}^2=1-Q_0^2=\sin^2\sum_\mu k_\mu/2\approx\left(\sum_\mu k_\mu\right)^2\Big/4 \tag{49}$$

vanishes at the origin. Clearly, the coefficient in front of the quadratic in momenta part increases with increased levels of the hierarchy, and eventually it may become the main driving term of the phase transition in the system.

Making yet another duality transformation, we obtain $Q_2=1/Q_1'$; the resulting zero-temperature kernel has zero surface at finite momenta determined by the equation $\mathcal{P}^2=1/2m_1[2m_2+(1/2m_3+\cdots)]<1/2m_1$. Although the divergence does not occur at any finite lattice temperature, the system develops a soft mode much closer to the origin then the location of the original soft plane in the middle of the Brilluen zone $\mathcal{P}^2=1$. The exact location and the orientation of this soft surface depends on the details of the selected regularization procedure [in our case it was determined by the chosen form of the local kernel (38)], but its existence is probably unavoidable as long as we want to define the integer-valued linking numbers. The fluctuations in the vicinity of this mode will grow with the hierarchy level and, again, are capable of destroying the second-order phase transition.

We see that the ''unwinding'' procedure creates additional instabilities of the strength increasing with the level of hierarchy and, generally, with the denominator of the statistical coupling. Therefore, the phase transitions in the system of particles with fractional statistics $\alpha=p/q$ are expected to be universal only for small enough denominators q.

One could argue that, instead of starting with Chern-Simons bosons with fractional statistics, we could have followed the usual direction of the hierarchy sequence and started with the x-y model or Chern-Simons bosons with odd ''fermionic'' statistical coupling. In this case the sequence of exact transformations also results in a nonlocal Chern-Simons model with fractional coupling, and the truncated local model with the same Chern-Simons coefficient differs from the nonlocal one by classically irrelevant terms. Again, the two models should be in the same universality class as long as the phase transitions remain of the second order. This procedure has the advantage that one can construct the sequence of lattice models without any apparent divergences. Indeed, starting with $\alpha_0=2m_0$, the duality (45) and the flux attachment (46) transformations result in the form factor

$$Q_1'=\frac{1}{Q_0}\left(1-\cfrac{2m_1\mathcal{P}^2}{2m_1+\cfrac{1}{2m_0}}\right); \tag{50}$$

the coefficient in front of $\mathcal{P}^2$ here is less than 1, and the original convergence condition $|k|\ll 1$ is preserved. Unfortunately, this argument also fails to prove universality at high hierarchy levels: the perturbation is not small compared to the relevant scale given by the vanishing lattice temperature (42).

CONCLUSIONS

In this paper we studied a relativistic version of the Chern-Simons-Landau-Ginzburg theory of bosons in the limit of strong coupling. We used the lattice representation of this model in terms of Villain x-y model minimally coupled with the Chern-Simons gauge field to access the strong-coupling limit without the perturbation theory. This model has duality and flux attachment symmetries in the long-wavelength limit, i.e., these symmetries are accurate up to irrelevant cubic and higher-order derivative terms. There is no single lattice model that can obey both symmetries exactly, but we construct algebraically exact nonlocal duality and flux attachment *transformations* corresponding to these symmetries in the continuum limit. These nonlocal transformations were used to show that there are only two universality classes in this model: one corresponding to the pure x-y transition, and another corresponding to the ''fermionic'' x-y transition, or the transition in the x-y model with the Chern-Simons coefficient $\alpha=1$. The value of such an investigation is to establish a theoretical model in which universality in the CSLG theory can be proven beyond the bounds of perturbation theory.

Although it may be possible to construct an experimental system that would be described by such a relativistically invariant model, this model is not in the same universality class as the experimental quantum Hall systems, where the relativistic symmetry is broken by nonzero charge density, external magnetic field and the disorder potential. Theoretically, these effects can be incorporated into the lattice model as additional external gauge fields without breaking the exactness of the performed hierarchy transformations, but the extra fields lower the symmetry of the problem. Now every phase is characterized by the nontrivially transforming filling factor ν in addition to the original Chern-Simons coefficient α, and the exact mapping between different quantum Hall states is absent.

This is not surprising, since the mapping preserves information about the exact configuration of the disorder. We believe that disorder averaging with a proper identification of the relevant degrees of freedom will increase the symmetry and reveal the universality of phase transitions in this model. Qualitatively, we saw that the Chern-Simons interaction has no effect if it is screened by some other independent long-range force. A two-dimensional time-independent random scalar potential may be treated as an interaction of infinite range in the time direction; apparently this interaction is strong enough to suppress or modify the effect of the statistical coupling on localization phase transitions.

Our model also sheds some light on the important question of the existence of high-order hierarchy states in the quantum Hall effect. Since the duality and periodicity transformations can simultaneously become symmetries of the theory only in the long-distance limit, the accumulation of extra terms, irrelevant near the critical point, will eventually render a phase transition of the first order or change its symmetry.

Further research on this class of models should be concentrated on understanding models with several species of the scalar fields. A clear understanding of a model with a finite number of components is necessary to apply the replica trick, which is the only possible way to perform disorder averaging in the presence of interaction.

Another perspective direction is to understand the hierarchical picture within the rigorous approach by Chen and Itoi.[53,54] Starting from the representations of the $2+1$-dimensional Lorenz group, they established the exact relationship of the spin of particles to the coefficient of their Chern-Simons coupling, namely, that all theories of particles with spin s and Chern-Simons coupling α describe the same irreducible representation of the Lorenz group with the fractional spin $s+\alpha/2$. Although nominally the action of particles with spin s has $2s+1$ components, in $2+1$ dimensions the equations of motion for massive particles leave only one independent component, and the long-range properties of all particles with the same α but with spins s differing by an integer should be similar. This can be considered as a rigorous definition of the flux attachment transformation for hardcore relativistic particles with Chern-Simons interaction.

While finishing this work we learned that a similar model was studied by Fradkin and Kivelson.[55] They also used duality and periodicity transformations to argue that phase transitions in three-dimensional lattice-regularized models are universal, although they were less specific about the form of the Chern-Simons coupling at short distances. In addition to the behavior considered in this paper, Fradkin and Kivelson considered a model with an additional dimensionless nonlocal interaction between the integer-valued currents, corresponding to the dissipative conductivity in the quantum Hall effect. As in Refs. 17 and 23, this leads to a phase diagram with multiple self-dual fixed points. Perturbatively, such an interaction of finite strength can be generated near the phase-transition point,[26] and the accumulation of higher-order irrelevant terms considered in the present work can in principle imply the change of the symmetry of the second-order phase transition instead of the first order phase transition.

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APPENDIX A: ALTERNATIVE DERIVATION OF THE FINITE-TEMPERATURE DUAL MODEL

Instead of using the previously established symmetry property (12), we could have arrived directly at the dual action with nonzero lattice temperature by equivalent transformations of the gauge field in the exponent. Indeed, averaging Eq. (15) in the original gauge field $A_{n\mu}$ produces the kinetic term (24) for the dual gauge field a, and one can further rewrite the exponent in terms of the auxiliary transverse field $\underline{b}$ as

$$iaM-\tfrac{1}{2}a\tilde{\mathcal{K}}_0a\rightarrow iaM-ia\underline{b}-\tfrac{1}{2}\underline{b}\tilde{\mathcal{K}}_0^{-1}\underline{b}.$$

Now one can add and subtract the term $\tilde{T}\underline{b}^2/2$ with the arbitrary constant $\tilde{T}$, so that the introduction of yet another transverse field $\underline{c}$ to dispatch the part $\underline{b}(\tilde{\mathcal{K}}_0^{-1}-\tilde{T})\underline{b}/2$ of the obtained gauge kernel results in the exponent

$$iaM-ia\underline{b}+i\underline{bc}-\frac{\tilde{T}}{2}\underline{b}^2-\frac{1}{2}\,\underline{c}(\tilde{\mathcal{K}}_0^{-1}-\tilde{T})^{-1}\underline{c}.$$

This expression is linear in field a, and an easy integration yields the constraint $\underline{b}=M$; the subsequent integration in $\underline{b}$ trivially yields

$$iM\underline{c}-\frac{\tilde{T}}{2}M^2-\frac{1}{2}\,\underline{c}(\tilde{\mathcal{K}}_0^{-1}-\tilde{T})^{-1}\underline{c}.$$

With the transverse integer-valued current M, this is precisely the exponent of expression (20) with the gauge kernel $\tilde{\mathcal{K}}=(\tilde{\mathcal{K}}_0^{-1}-\tilde{T})^{-1}$ and the lattice temperature $\tilde{T}>0$ replacing the artificially introduced infinitesimal variable t. Now the summation formula (21), and a subsequent rescaling of the gauge field, lead directly to the dual model of the form (2) and (3) with the finite temperature $\tilde{T}$ and the gauge kernel $\tilde{\mathcal{K}}$ satisfying (26).

The derived mapping is a generalization of the well-known duality in three dimensions.[40] It can be understood as the duality between vortices and monopoles in three spatial dimensions, or the charge-vortex duality in $2+1$-dimensional systems. Indeed, the average in the presence of an arbitrary number of vortex-antivortex pairs introduced by the integer vortex strength L_n,

$$\left\langle \exp i\sum_n L_n\theta_n\right\rangle\equiv\left\langle \exp i\int\frac{d^3k}{(2\pi)^3}L_{-k}\theta_k\right\rangle, \qquad (A1)$$

with zero total vorticity $\Sigma_nL_n=0$, results in the constraint

$$\int\frac{d\theta_n}{2\pi}e^{iL_n\theta_n-i\theta_n\tilde{\triangle}_\mu b_{n\mu}}=\delta(\tilde{\triangle}_\mu b_{n\mu}-L_n) \qquad (A2)$$

instead of (8). Evidently, the integers L_n serve as the sources for the field b, or monopoles with appropriate quantized charges. The values of these charges are not affected by any single-valued transformation of the vector potential, therefore the statement of equivalence generally holds for any two models related by Eq. (26). Similarly, the dual transformation of the original model (2) and (3) in the presence of monopoles introduced by appropriate multivalued external vector potential results in vortices of the form (A1) located exactly in the original positions of the monopoles.

Unfortunately, any averages involving the gauge fields, particularly the averages of gauge-invariant currents, do not have simple dual representation since the fluctuating gauge fields change upon reparametrization. Therefore, only scalar sectors of arbitrary models of the form (2) and (3) related by the generalized duality transformation (26) are equivalent to each other.

APPENDIX B: CS COUPLING BETWEEN VORTICES OF THE *x-y* MODEL: DETAILED ANALYSIS

Here we analyze the model numerically considered by Schultka and Manousakis:[52] the pure *x-y* model with the additional Chern-Simons interaction between the *vortices*, introduced to study the universality of transitions in a system of particles with fractional statistics. The original model is defined simultaneously in terms of original phases θ_n and the vorticity $M_{n\mu}$. Working in the Villain approximation, let us perform the exact duality transformation on the scalar sector and express the model entirely in terms of dual variables. The duality transformation results in a model of the form (20) with an additional coupling of the vorticity M to the second gauge field A with the Chern-Simons kinetic term. Shifting the gauge field

$$A_{n\mu}\to A_{n\mu}-a_{n\mu}$$

and integrating away the field a, we obtain the gauge kernel of the combined interaction in the form

$$\mathcal{K}=\frac{\hat{I}P}{2\pi\tilde{\alpha}Q}\left(1-\frac{1}{1-\hat{I}T\tilde{\alpha}QP/2\pi}\right)=\frac{TP^2}{4\pi^2}+\mathcal{O}(P^3),$$

$$\tilde{\alpha}TPQ\ll 2\pi, \tag{B1}$$

instead of $TP^2/4\pi^2$ present without any gauge coupling; the statistical coupling of the vortices does not change the long-range properties of the model at all.

This is precisely the result of numerical study;[52] from our analysis it is clear that the additional Chern-Simons interaction between the vortices is completely screened by the Coulomb interaction already present in the dual model. Indeed, the original vortices of the *x-y* model map to charges in the superconductor, which is the correct dual model. The charges have the Coulomb interaction decaying like $1/r$ at large distances, while the additional CS interaction falls off like $1/r^2$; there always exists some range above which the usual Coulomb force dominates. It is because of the existence of this finite range that the asymptotic form is correct only in the restricted range of momenta as specified in (B1). This phenomenon can also be understood in terms of the original *x-y* model: the vortices in a superfluid are not really local objects because of their long-range phase structure; therefore the additional Chern-Simons interaction does not do anything at large enough distances.

It is important to emphasize that the previous analysis, including Eq. (B1), holds for an *x-y* model coupled to a Chern-Simons gauge field via the dual current only. If the long-range structure around the vortices is already broken by some mechanism, the addition of the Chern-Simons interaction between vortices does change the parameters or even the qualitative behavior of the model. In this case the long-range phase structure of the vortices is destroyed (in the dual representation—the charges are screened), and the additional statistical coupling is the main interaction at large distances.

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