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Optical properties of two interacting electrons in quantum rings: Optical absorption and inelastic light scattering

L. Wendler, V. M. Fomin,* A. V. Chaplik,† and A. O. Govorov†
Fachbereich Physik, Martin-Luther-Universität Halle, Friedemann-Bach-Platz 6, D-06108 Halle, Germany

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Two interacting electrons confined in a quantum ring form a Wigner molecule due to the interplay between the Coulomb repulsion and the radial confinement. The states with fixed total spin show a strong correlation between the quantum numbers of the relative angular vibrations and the rotation of the two electrons. On the basis of the energy band structure, which results in the occurrence of a persistent current in the ring, the optical absorption and the differential cross section of resonant inelastic light scattering is calculated. The selection rules are derived and discussed in dependence on the polarization of the incident light. [S0163-1829(96)07531-5]

I. INTRODUCTION

In recent years there has been great progress in the ability to confine electrons in semiconductor nanostructures. Modern epitaxial techniques, coupled with advances in lithography, have made possible the fabrication of a wide variety of structures in which carriers can be quantum-mechanically confined. Various techniques of surface preparation have been developed, in order to impose *lateral* constrictions on the quasi-two-dimensional electron gas. The ultimate limits are *quantum dots*[1–5] (QD's) and *quantum rings* (QR's),[6–8] where the electrons are confined in all three dimensions. QD's and QR's can be regarded as artificial atoms. The experimental conditions allow one to tune the number of electrons up to the limit of single electron charging of QD's.[9] The resulting few-electron problems have been studied theoretically for QD's (Refs. 10–18) and QR's.[19–22] The work on QR's has been heated in recent times because in the presence of a magnetic flux *persistent currents* were measured in an ensemble of about 10^7 copper rings,[23] in a single gold ring,[24] and in a single GaAs ring.[8] The theoretical explanation of these experiments has addressed two basic questions: (i) the choice of the statistical ensemble[25–30] and (ii) the role of the electron-electron interaction.[31–43]

Besides the many-particle combined impurity and Coulomb perturbation theory,[31,34] hopping models,[33–37] theory of coherent propagation of interacting particles in a random potential[38] and Luttinger liquid theory,[32,39] it has been shown,[21,22,40–43] that in the *ballistic regime* a rigorous quantum-mechanical theory of few strongly interacting electrons in QR's can be developed. It results in the picture of a rotating Wigner molecule; i.e., the interplay of the Coulomb repulsion between the electrons and the confining potential forms a relatively rigid rotator with internal azimuthal excitations and confined radial motions. On the basis of this theory the electronic states, thermodynamical properties, and persistent currents were calculated. It was shown that the finite width of the ring plays an important role for the influence of the electron-electron interaction.

In this paper we apply this theory to calculate the optical absorption and resonant inelastic light scattering of a statistical ensemble of finite-width QR's with two interacting electrons with spin in the presence of an Aharonov-Bohm magnetic flux, i.e., in the presence of a persistent current.

II. THE MODEL

A. General

The model that we use to describe *optical absorption* is as follows. We consider two electrons ($\mu=1,2$) with the effective conduction-band-edge mass m_e in the plane $z=0$, confined in a QR by the potential $V_C(\mathbf{x}_\mu)$ in the x-y plane. This potential is taken as zero in the region $R-W/2<r_\mu<R+W/2$ and infinite otherwise. It is assumed that a *static* magnetic field $\mathbf{B}_0=(0,0,B_0)$, which is restricted to the area of a circle with radius $R_{\mathrm{AB}}<R-W/2$, threads the hole of the ring with the magnetic flux $\Phi=B_0\pi R_{\mathrm{AB}}^2$ [Aharonov-Bohm (AB) geometry]. This system is considered in the presence of an external electromagnetic field, described by the vector potential $\mathbf{A}_1(\mathbf{x},t)$ and the scalar potential $\phi_1(\mathbf{x},t)$, which are related to the time-dependent electric field $\mathbf{E}_1(\mathbf{x},t)=-\boldsymbol{\nabla}\phi_1(\mathbf{x},t)-(\partial/\partial t)\mathbf{A}_1(\mathbf{x},t)$ and the time-dependent magnetic field $\mathbf{B}_1(\mathbf{x},t)=\boldsymbol{\nabla}\times\mathbf{A}_1(\mathbf{x},t)$. The resulting Hamiltonian is given by

$$H=\sum_{\mu=1}^{2}\left\{\frac{1}{2m_e}[\mathbf{p}_\mu+e\mathbf{A}_0(\mathbf{x}_\mu)+e\mathbf{A}_1(\mathbf{x}_\mu,t)]^2+V_C(\mathbf{x}_\mu)+\frac{g^*}{2}\mu_B\mathbf{B}_0(\mathbf{x}_\mu)\cdot\boldsymbol{\sigma}_\mu+\frac{g^*}{2}\mu_B\mathbf{B}_1(\mathbf{x}_\mu,t)\cdot\boldsymbol{\sigma}_\mu-e\phi_1(\mathbf{x}_\mu,t)\right\}+\frac{e^2}{4\pi\varepsilon_0\varepsilon_s|\mathbf{x}_1-\mathbf{x}_2|},\tag{1}$$

where $\mathbf{x}_\mu=(x_\mu,y_\mu,z_\mu)$ is the position vector and $\mathbf{p}_\mu=-i\hbar\boldsymbol{\nabla}_{\mathbf{x}_\mu}$ is the momentum operator of the μth electron with charge $-e$, $\mu_B=e\hbar/(2m_0)$ denotes Bohr's magneton with m_0 the free-electron mass, g^* is the effective spin-splitting factor, and $\boldsymbol{\sigma}_\mu=(\sigma_{x\mu},\sigma_{y\mu},\sigma_{z\mu})$ stands for the Pauli spin vector operator, where the components of $\boldsymbol{\sigma}_\mu$ are the Pauli matrices. Further, ε_0 is the permittivity in vacuum and ε_s is the static dielectric constant of the host semiconductor. We use the static dielectric constant because the frequency of the incident far-infrared (FIR) light is assumed to be low in comparison to the phonon frequencies of the host semiconductor. In Eq. (1) the *interparticle Coulomb* interaction potential $V_{ee}^{s}(\mathbf{x}_1,\mathbf{x}_2)$ is described by the *direct* part and the *image* part is neglected for simplicity. Further, in the Hamiltonian given in Eq. (1), we ignore the small relativistic effect of the spin-orbit coupling. According to the possible gauge transformation of the vector potential $\mathbf{A}(\mathbf{x},t)\rightarrow\mathbf{A}'(\mathbf{x},t)=\mathbf{A}(\mathbf{x},t)+\boldsymbol{\nabla}\Lambda(\mathbf{x},t)$ and of the scalar potential $\phi(\mathbf{x},t)\rightarrow\phi'(\mathbf{x},t)=\phi(\mathbf{x},t)-(\partial/\partial t)\Lambda(\mathbf{x},t)$, where $\Lambda(\mathbf{x},t)$ is a scalar function, we use different gauges for the static and dynamic potentials. For the static vector potential $\mathbf{A}_0(\mathbf{x})$, with $\mathbf{B}_0(\mathbf{x})=\boldsymbol{\nabla}\times\mathbf{A}_0(\mathbf{x})$, we use the Coulomb gauge $\mathrm{div}\mathbf{A}_0(\mathbf{x})=0$ in the form of the symmetric (or circular) gauge $\mathbf{A}_0(\mathbf{x})=\frac{1}{2}\mathbf{B}_0\times\mathbf{x}$. For the vector potential of the electromagnetic field we use the Coulomb gauge $\mathrm{div}\mathbf{A}_1(\mathbf{x},t)=0$ and the gauge $\phi_1(\mathbf{x},t)=0$. We introduce polar coordinates in the x-y plane: $\mathbf{x}_{\parallel\mu}=(r_\mu,\varphi_\mu)$, relative angular coordinate $\gamma=\varphi_1-\varphi_2$ and angular coordinate $\Theta=(\varphi_1+\varphi_2)/2$, describing the motion of the two-electron system as a whole. In this case the static vector potential has only a φ component $A_{0\varphi}(r_\mu)=B_0R_{\mathrm{AB}}^2/(2r_\mu)=\Phi/(2\pi r_\mu)$ if $r_\mu>R_{\mathrm{AB}}$ and $A_{0\varphi}(r_\mu)=B_0r_\mu/2$ if $r_\mu\leqslant R_{\mathrm{AB}}$. Treating $\mathbf{A}_1(\mathbf{x},t)$ as a perturbation, the Hamiltonian of Eq. (1) has the form

$$H=H_0+H_{\mathrm{int}}, \tag{2}$$

with the unperturbed (equilibrium) Hamiltonian

$$H_0=\sum_{\mu=1}^{2}\left\{\left[-\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial r_\mu^2}+\frac{1}{r_\mu}\frac{\partial}{\partial r_\mu}\right)+V_C(r_\mu)\right]\right\}-\frac{\hbar^2}{2m_e}\left\{\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)\left[\frac{\partial^2}{\partial\gamma^2}+\frac{1}{4}\left(\frac{\partial}{\partial\Theta}-2i\frac{\Phi}{\Phi_0}\right)^2\right]\right.$$
$$\left.+\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\frac{\partial}{\partial\gamma}\left(\frac{\partial}{\partial\Theta}-2i\frac{\Phi}{\Phi_0}\right)\right\}+\frac{e^2}{4\pi\varepsilon_0\varepsilon_s\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}}, \tag{3}$$

where $\Phi_0=h/e$ is the flux quantum. The Hamiltonian that represents the interaction of the incident light with the electrons in the ring, the perturbation, consists of three parts,

$$H_{\mathrm{int}}=H_1+H_2+H_3, \tag{4}$$

where H_1 is that of the first-order perturbation

$$H_1=\frac{e}{2m_e}\sum_{\mu=1}^{2}[\mathbf{p}_\mu\cdot\mathbf{A}_1(\mathbf{x}_\mu,t)+\mathbf{A}_1(\mathbf{x}_\mu,t)\cdot\mathbf{p}_\mu]. \tag{5}$$

With $[\mathbf{p}_\mu,\mathbf{A}_1(\mathbf{x}_\mu,t)]=-i\hbar\boldsymbol{\nabla}_{\mathbf{x}_\mu}\cdot\mathbf{A}_1(\mathbf{x}_\mu,t)$, Eq. (5) takes the form

$$H_1=\frac{e}{m_e}\sum_{\mu=1}^{2}\mathbf{A}_1(\mathbf{x}_\mu,t)\cdot\mathbf{p}_\mu. \tag{6}$$

The second part of H_{int},

$$H_2=\frac{g^*}{2}\mu_B\sum_{\mu=1}^{2}\mathbf{B}_1(\mathbf{x}_\mu,t)\cdot\boldsymbol{\sigma}_\mu \tag{7}$$

represents the interaction of the spin magnetic moment of the electrons with the oscillating magnetic field of the wave, and H_3 is the second-order perturbation

$$H_3=\frac{e^2}{2m_e}\sum_{\mu=1}^{2}\mathbf{A}_1^2(\mathbf{x}_\mu,t). \tag{8}$$

To describe the *Raman scattering* by the two interacting electrons in the QR we assume a two-step scattering mechanism:[44] in the first step, an electron is optically excited from the valence band to a virtual state associated with confined states of the ring potential in the conduction band above the Fermi surface and, in the second step, an electron in the conduction band below the Fermi surface, with either the same or the opposite spin as that of the ''excited'' electron recombines with the remaining hole in the valence band. The net effect of the two interband transitions is to generate either a spin-flip or a non-spin-flip excitation within the two-electron system of the ring. The excitation of spin-flip processes in inelastic light scattering is made possible by the substantial spin-orbit coupling in the valence-band states of III-V compound semiconductors[45] such as GaAs. Thus, to calculate the Raman scattering the starting point has to be before the effective-mass approximation with the unperturbed Hamiltonian

$$\mathcal{H}_0=\sum_{\mu=1}^{2}\mathcal{H}_0^{(\mu)}+\frac{e^2}{4\pi\varepsilon_0\varepsilon_s|\mathbf{x}_1-\mathbf{x}_2|}, \tag{9}$$

where

$$\mathcal{H}_0^{(\mu)}=\frac{1}{2m_0}[\mathbf{p}_\mu+e\mathbf{A}_0(\mathbf{x}_\mu)]^2+V(\mathbf{x}_\mu)$$
$$+\frac{\hbar}{4m_0c^2}[\boldsymbol{\sigma}_\mu\times\boldsymbol{\nabla}V(\mathbf{x}_\mu)]\cdot[\mathbf{p}_\mu+e\mathbf{A}_0(\mathbf{x}_\mu)]$$
$$+\frac{g}{2}\mu_B\mathbf{B}_0(\mathbf{x}_\mu)\cdot\boldsymbol{\sigma}_\mu \tag{10}$$

is the Hamiltonian of the μth mobile electron in the semiconductor. Preserving complete generality, $\mathcal{H}_0^{(\mu)}$ includes in $V(\mathbf{x}_\mu)$ the periodic crystal potential as well as the lateral confining potential and further, $\mathcal{H}_0^{(\mu)}$ includes the spin-orbit coupling. In Eq. (10), g is the free-electron g factor. Treating the $\mathbf{A}_1\cdot\mathbf{p}$ term of the perturbation in second order and the

$\mathbf{A}_1^2$ term in first order, it was shown[46,47] that the Raman scattering can be described by the Hamiltonian of Eq. (2), where H_0 is the same as that of Eq. (3) but with the effective interaction Hamiltonian

$$H_{\text{int}}=\frac{e^2}{2m_e}\mathcal{M}(\mathbf{q})A(\omega_L)A(\omega_S), \tag{11}$$

where

$$\mathcal{M}(\mathbf{q})=\sum_{\alpha,\beta}\gamma_{\alpha\beta}\,c_\alpha^\dagger c_\beta, \tag{12}$$

with

$$\begin{aligned}\gamma_{\alpha\beta}=(\mathbf{e}_L\cdot\mathbf{e}_S^*)\xi_{\alpha\beta}+\frac{1}{m_0}\sum_{\beta'}{}'\Bigg[&\frac{\langle\alpha|\mathbf{e}_S^*\cdot(\boldsymbol{\pi}+e\mathbf{A}_0)e^{-i\mathbf{k}_S\cdot\mathbf{x}}|\beta'\rangle\langle\beta'|\mathbf{e}_L\cdot(\boldsymbol{\pi}+e\mathbf{A}_0)e^{i\mathbf{k}_L\cdot\mathbf{x}}|\beta\rangle}{E_\beta-E_{\beta'}+\hbar\omega_L}\\&+\frac{\langle\alpha|\mathbf{e}_L\cdot(\boldsymbol{\pi}+e\mathbf{A}_0)e^{i\mathbf{k}_L\cdot\mathbf{x}}|\beta'\rangle\langle\beta'|\mathbf{e}_S^*\cdot(\boldsymbol{\pi}+e\mathbf{A}_0)e^{-i\mathbf{k}_S\cdot\mathbf{x}}|\beta\rangle}{E_\beta-E_{\beta'}-\hbar\omega_L}\Bigg],\end{aligned} \tag{13}$$

$$\xi_{\alpha\beta}=\langle\alpha|e^{i\mathbf{q}\cdot\mathbf{x}}|\beta\rangle \tag{14}$$

and

$$A(\omega_{L,S})=\left(\frac{\hbar}{2V\varepsilon_0\varepsilon(\omega_{L,S})\omega_{L,S}}\right)^{1/2} \tag{15}$$

are the amplitudes of the vector potential of the incident wave of frequency ω_L and of the scattered wave of frequency ω_S. Further, $\boldsymbol{\pi}=\mathbf{p}+(\hbar/4m_0c^2)[\boldsymbol{\sigma}\times\boldsymbol{\nabla}V(\mathbf{x})]$, $\mathbf{q}=\mathbf{k}_L-\mathbf{k}_S$, V is the scattering volume, and $\varepsilon(\omega_{L,S})$ is the dielectric function at the respective frequency of the medium embedding the ring. In these equations, the states $|\alpha\rangle$ are the single-particle eigenstates: $\mathcal{H}_0^{(\mu)}|\alpha\rangle=E_\alpha|\alpha\rangle$ of the semiconductor (Bloch states), where the symbol α labels the band index n, orbital and spin quantum numbers, and $c_\alpha^\dagger$ and c_α are the single-particle creation and destruction operators. The Hamiltonian H_{int} of Eq. (11) is responsible for the scattering of light due to $|\alpha\rangle\rightarrow|\beta\rangle$ electronic transitions and $|\beta'\rangle$ are intermediate states. In order to investigate the inelastic light scattering, it becomes necessary to invoke a suitable model of the band structure formation. With this scope we will use below, in Sec. IV, a $\mathbf{k}\cdot\mathbf{p}$ method and the Kane two-band model adequate for the quantum rings on GaAs. The restrictions imposed on the sum over the intermediate states $\Sigma'_{\beta'}$ depend on the type of scattering process; thus, for resonant scattering, when both the initial $|\alpha\rangle$ and the final state $|\beta\rangle$ belong to the conduction band, the sum is extended to the valence-band states that are connected to the conduction-band states by the optical transitions associated with the incident and scattered photons.

B. Ground state

In this section we consider the eigenvalue problem of H_0; in all the other sections we discuss the perturbations H_{int}. It is obvious that the z component of the angular momentum operator of the two electrons, $L_z=-i\hbar\partial/\partial\Theta$, is a constant of motion, i.e., $[L_z,H_0]=0$ is valid, from which the rotational invariance of the problem follows. The corresponding wave function $\Psi(\mathbf{x}_1,\sigma_1;\mathbf{x}_2,\sigma_2)$ depends on the spatial coordinates $\{\mathbf{x}_\mu\}$ and the spin coordinates $\{\sigma_\mu\}$. This two-electron wave function must satisfy the following conditions: (i) *antisymmetry condition*:

$$\mathcal{P}_{12}\Psi(\mathbf{x}_1,\sigma_1;\mathbf{x}_2,\sigma_2)=-\Psi(\mathbf{x}_1,\sigma_1;\mathbf{x}_2,\sigma_2); \tag{16}$$

(ii) *single-valuedness boundary conditions* (SVBC's):

$$\begin{aligned}\Psi(r_1,r_2;\varphi_1+2\pi,\varphi_2+2\pi;z_1,z_2)&=\Psi(r_1,r_2;\varphi_1,\varphi_2+2\pi;z_1,z_2)\\&=\Psi(r_1,r_2;\varphi_1+2\pi,\varphi_2;z_1,z_2)\\&=\Psi(r_1,r_2;\varphi_1,\varphi_2;z_1,z_2),\end{aligned} \tag{17}$$

where $\mathcal{P}_{12}$ is the permutation operator. Because the Hamiltonian of Eq. (3) does not depend on the spin operator, the wave function separates: $\Psi(\mathbf{x}_1,\sigma_1;\mathbf{x}_2,\sigma_2)=\Psi(\mathbf{x}_1,\mathbf{x}_2)\chi(\sigma_1,\sigma_2)$ in the orbital part $\Psi(\mathbf{x}_1,\mathbf{x}_2)$ and the spin part $\chi(\sigma_1,\sigma_2)$. In dependence on the symmetry of the spin function under particle permutation, we have $\mathcal{P}_{12}\Psi(\mathbf{x}_1,\mathbf{x}_2)=\eta\Psi(\mathbf{x}_2,\mathbf{x}_1)$, where $\eta=\pm1$ is the parity of the orbital wave function. In a pure AB geometry the spin direction is arbitrary. But in practice, at least an infinitesimal magnetic field penetrates the electron orbits to stabilize the z direction for the spin quantization. Thus, in dependence on the two possible spin quantum numbers $m_{s\mu}=\pm\frac{1}{2}$ of both electrons, we have the following

(i) *para state*: $m_{s1}=-m_{s2}\equiv m_s(=\frac{1}{2}$ or $-\frac{1}{2})$ and $\eta=1$,

$$\Psi_{\text{para}}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{\sqrt{2}}[\Psi(\mathbf{x}_1,\mathbf{x}_2)+\Psi(\mathbf{x}_2,\mathbf{x}_1)], \tag{18}$$

and

$$\chi_{\text{para}}(\sigma_1,\sigma_2)=\frac{1}{\sqrt{2}}[\chi_{m_s,-m_s}(\sigma_1,\sigma_2)-\chi_{-m_s,m_s}(\sigma_1,\sigma_2)], \tag{19}$$

which is an eigenstate of both the z component S_z and the square $\mathbf{S}^2$ of the total spin operator $\mathbf{S}=(\hbar/2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$, belonging to the eigenvalues $S_z=\hbar M_S$ with $M_S=0$ and $S(S+1)=0$: *singlet state* $|S=0, M_S=0\rangle$; and

(ii) *ortho state*: $m_{s1}=-m_{s2}\equiv m_s(=\frac{1}{2}$ or $-\frac{1}{2})$ and $\eta=-1$,

$$\Psi_{\mathrm{ortho}}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{\sqrt{2}}[\Psi(\mathbf{x}_1,\mathbf{x}_2)-\Psi(\mathbf{x}_2,\mathbf{x}_1)] \tag{20}$$

and

$$\chi_{\mathrm{ortho}}(\sigma_1,\sigma_2)=\begin{cases}\chi_{m_s,m_s}(\sigma_1,\sigma_2)\\ \chi_{-m_s,-m_s}(\sigma_1,\sigma_2)\\ \frac{1}{\sqrt{2}}[\chi_{m_s,-m_s}(\sigma_1,\sigma_2)+\chi_{-m_s,m_s}(\sigma_1,\sigma_2)],\end{cases} \tag{21}$$

which is an eigenstate of $\mathbf{S}^2$ with $S(S+1)=2$ and of S_z with $M_S=+1$, $M_S=-1$ and $M_S=0$: *triplet state* $|S=1, M_S=0,\pm 1\rangle$. In the following calculation we consider only the orbital part $\Psi(\mathbf{x}_1,\mathbf{x}_2)$, which is a scalar, but include the spin of the two electrons via the Pauli principle.

Removing the magnetic flux from the Hamiltonian H_0 by a gauge transformation with $\Lambda(\mathbf{x}_\mu)=-B_0R^2_{\mathrm{AB}}\varphi_\mu/2$ for $r_\mu>R_{\mathrm{AB}}$, where $\nabla^2_{\mathbf{x}_\mu}\Lambda(\mathbf{x}_\mu)=0$ and $\Psi(\mathbf{x}_1,\mathbf{x}_2)\rightarrow\exp[-(ie/\hbar)\Sigma^2_{\mu=1}\Lambda(\mathbf{x}_\mu)]\Psi(\mathbf{x}_1,\mathbf{x}_2)$, the SVBC's change to *twisted boundary conditions* (TBC's):

$$\Psi(r_1,r_2;\gamma+2\pi,\Theta+\pi;z_1,z_2)=\exp\left(i2\pi\frac{\Phi}{\Phi_0}\right)\Psi(r_1,r_2;\gamma,\Theta;z_1,z_2) \tag{22}$$

and

$$\Psi(r_1,r_2;\gamma,\Theta+2\pi;z_1,z_2)=\exp\left(i4\pi\frac{\Phi}{\Phi_0}\right)\Psi(r_1,r_2;\gamma,\Theta;z_1,z_2). \tag{23}$$

The only effect of the static vector potential, which is now absent in the Hamiltonian of Eq. (3), is to enter the TBC's.

For narrow-width rings, i.e., if $W\ll R$, the radial motion is much faster than the angular motions. Thus, the radial motion is adiabatically decoupled from the angular motions with the result[40]

$$\Psi(r_1,r_2;\gamma,\Theta;z_1,z_2)=\sum_{P,K_1,K_2}\Xi^P_{K_1,K_2}(r_1,r_2;\gamma,\Theta)\times\psi_{P,K_1,K_2}(\gamma,\Theta)\;\varphi(z_1,z_2), \tag{24}$$

$$|\varphi(z_1,z_2)|^2=\delta(z_1)\,\delta(z_2) \tag{25}$$

for the orbital part of the *two-electron wave function*. Substituting Eq. (24) into the Schrödinger equation $H_0\Psi=E\Psi$ with the unperturbed Hamiltonian we obtain, according to the adiabatic approximation and under the condition that the confinement energy of the lateral potential is much larger than the typical Coulomb energy, a decoupling of the radial motions of both electrons:

$$\sum_{\mu=1}^{2}H_{\mathrm{rad}}(r_\mu)\;\Xi^P_{K_1,K_2}(r_1,r_2)=\mathcal{E}^{\mathrm{rad}}_{K_1K_2}\;\Xi^P_{K_1,K_2}(r_1,r_2), \tag{26}$$

where $H_{\mathrm{rad}}(r_\mu)$ is the radial part of H_0 and $\mathcal{E}^{\mathrm{rad}}_{K_1K_2}=\mathcal{E}_{K_1}+\mathcal{E}_{K_2}$. Herein, we have defined $\Xi^P_{K_1,K_2}(r_1,r_2)\equiv\Xi^P_{K_1,K_2}(r_1,r_2;\gamma,\Theta)$, where the slow variables γ and Θ are fixed. The radial wave function follows in the form

$$\Xi^P_{K_1,K_2}(r_1,r_2)=\begin{cases}\chi_{K_1}(r_1)\chi_{K_2}(r_2) & \text{if } K_1=K_2 \text{ and } P=0\\ \frac{1}{\sqrt{2}}[\chi_{K_1}(r_1)\chi_{K_2}(r_2)+(-1)^P\chi_{K_2}(r_1)\chi_{K_1}(r_2)] & \text{if } K_1\neq K_2 \text{ and } P=1,2.\end{cases} \tag{27}$$

The quantity $\eta_P=(-1)^P$ is the parity of the two-electron radial wave function: $\Xi^P_{K_1,K_2}(r_2,r_1)=\eta_P\Xi^P_{K_1,K_2}(r_1,r_2)$ and $\chi_{K_\mu}(r_\mu)$ are the eigenstates of the single-particle Schrödinger equation of a rectangular quantum well with infinitely high barriers. For the strong radial confinement we can use the wave functions

$$\chi_{K_\mu}(r_\mu)=\sqrt{\frac{2}{RW}}\sin\left\{\frac{\pi K_\mu}{W}\left[r_\mu-\left(R-\frac{W}{2}\right)\right]\right\}, \tag{28}$$

normalized by the relation

$$\int_{R-W/2}^{R+W/2}dr_\mu\,r_\mu\,\chi^*_{K_\mu}(r_\mu)\chi_{K'_\mu}(r_\mu)=\delta_{K_\mu,K'_\mu}, \tag{29}$$

with the associated eigenenergies

$$\mathcal{E}_{K_\mu}=\frac{\hbar^2\pi^2K_\mu^2}{2m_eW^2},\quad K_\mu=1,2,\,\ldots;\qquad \mu=1,2. \tag{30}$$

In the next step, we multiply both parts of the Schrödinger equation $H_0\Psi=E\Psi$ by $\Xi^{P}_{K_1,K_2}(r_1,r_2)$, integrate it over r_1 and r_2 and use Eq. (26). This gives

$$\sum_{P',K_1',K_2'}\left\{-\frac{\hbar^2}{2m_e}\left[\left\langle\left(\frac{1}{r_1^2}+\frac{1}{r_1^2}\right)\right\rangle_{P',K_1',K_2';P,K_1,K_2}\left(\frac{\partial^2}{\partial\gamma^2}+\frac{1}{4}\frac{\partial^2}{\partial\Theta^2}\right)+\left\langle\left(\frac{1}{r_1^2}-\frac{1}{r_1^2}\right)\right\rangle_{P',K_1',K_2';P,K_1,K_2}\frac{\partial}{\partial\gamma}\frac{\partial}{\partial\Theta}\right]\right.$$
$$\left.+\left\langle\frac{e^2}{4\pi\varepsilon_0\varepsilon_s\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}}\right\rangle_{P',K_1',K_2';P,K_1,K_2}+[\mathcal{E}^{\,\mathrm{rad}}_{K_1K_2}-E]\delta_{P',P}\delta_{K_1',K_1}\delta_{K_2',K_2}\right\}\psi_{P',K_1',K_2'}(\gamma,\Theta)=0, \tag{31}$$

where $\langle\,\rangle_{P',K_1',K_2';P,K_1,K_2}$ denotes the matrix element on the radial wave functions

$$\langle F(r_1,r_2)\rangle_{P',K_1',K_2';P,K_1,K_2}=\int_{R-W/2}^{R+W/2}dr_1\,r_1\int_{R-W/2}^{R+W/2}dr_2\,r_2F(r_1,r_2)\Xi^{P}_{K_1,K_2}(r_1,r_2)\Xi^{P'}_{K_1',K_2'}(r_1,r_2). \tag{32}$$

If the relative angular motion is much faster than the rotation of the two-electron system, these two motions can be decoupled by a *second adiabatic approximation*. This approximation, whose adequacy is formulated solely as a relation between the typical energies related to the two angular motions: $\hbar^2/(4m_eR^3)\ll\hbar\Omega$, allows one to neglect the terms with $\partial/\partial\gamma\partial/\partial\Theta$ in Eq. (31) in comparison to the other ones, i.e., to separate the variables γ and Θ. This leads to an important consequence: namely, the sets of equations (31) describing the states of the radial motion, possessing the opposite symmetry with respect to a permutation, occur decoupled.

As long as the above-stated criterion of the first adiabatic approximation is satisfied, the states of the radial motion are separated by the energy intervals of the order of the ground-state energy $\mathcal{E}^{\,\mathrm{rad}}_{1,1}=\hbar^2\pi^2/(m_eW^2)$. As a consequence, we can restrict the consideration to the matrix elements with $K_1'=K_1$, $K_2'=K_2$, describing transitions without changing the energy $\mathcal{E}^{\mathrm{rad}}_{K_1},K_2$ of the radial motion. This restriction, in view of the above consequence of the second adiabatic approximation, implies also $P'=P$. It is a fairly good approximation to restrict this average to the ground state of the radial motion, which has the quantum numbers $P'=P=0$, $K_1'=K_1=1$, $K_2'=K_2=1$:

$$\left\{-\frac{\hbar^2}{m_e}\left\langle\frac{1}{r^2}\right\rangle\left(\frac{\partial^2}{\partial\gamma^2}+\frac{1}{4}\frac{\partial^2}{\partial\Theta^2}\right)+\left\langle\frac{e^2}{4\pi\varepsilon_0\varepsilon_s\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}}\right\rangle+2\mathcal{E}_1-E\right\}\psi_{0,1,1}(\gamma,\Theta)=0, \tag{33}$$

where

$$\left\langle\frac{1}{r^2}\right\rangle=\int_{R-W/2}^{R+W/2}dr\;r\;\frac{\chi_1^2(r)}{r^2}\approx\frac{1}{R^2}, \tag{34}$$

$$\left\langle\frac{e^2}{4\pi\varepsilon_0\varepsilon_s\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}}\right\rangle=\frac{e^2}{4\pi\varepsilon_0\varepsilon_s}\int_{R-W/2}^{R+W/2}dr_1\,r_1\int_{R-W/2}^{R+W/2}dr_2\;r_2\frac{\chi_1^2(r_1)\chi_1^2(r_1)}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}}\approx\frac{e^2}{4\pi\varepsilon_0\varepsilon_sR\sqrt{2(1-\cos\gamma)}}. \tag{35}$$

As long as in the framework of the second adiabatic approximation the relative angular motion is separated from the angular motion of the two-electron system in the QR as a whole, we have

$$\psi_{P,K_1,K_2}(\gamma,\Theta)=\sum_p\sum_{j=0}^{\infty}\Phi_j^p(\gamma)Q_{\nu_\Upsilon}(\Theta), \tag{36}$$

where the wave function of the relative angular (ra) motion is a solution of the equation

$$\left\{-\frac{\hbar^2}{R^2m_e}\frac{\partial^2}{\partial\gamma^2}+\frac{e^2}{4\pi\varepsilon_0\varepsilon_sR\sqrt{2(1-\cos\gamma)}}-\mathcal{E}^{\,\mathrm{ra,p}}_{j}\right\}\Phi_j^p(\gamma)$$
$$=0. \tag{37}$$

In Eq. (36) Υ stands for $\{P,K_1,K_2,p,j\}$. The characteristic index p describes the possible symmetry types of the solutions $\Phi_j^p(\gamma)$ of the differential equation (37). Namely, in view of the 2π periodicity of the mean Coulomb potential as a function of γ, the wave function of the relative angular motion has the form $\Phi_j^p(\gamma)=\exp(ip\gamma)u_j^p(\gamma)$, where $u_j^p(\gamma+2\pi)=u_j^p(\gamma)$ and $-\frac{1}{2}<p\leqslant\frac{1}{2}$.

If the typical Coulomb energy between the two electrons $E_{\mathrm{Coul}}=e^2/(8\pi\varepsilon_0\varepsilon_sR)$ is much larger than the oscillation energy $\mathcal{E}^{\,\mathrm{osc}}_j=\mathcal{E}^{\,\mathrm{ra,p}}_j-E_{\mathrm{Coul}}$, the relative angular motion can be considered in the tight-binding-like approximation:

$$\Phi_j^p(\gamma)=\sum_{Q=-\infty}^{\infty}\exp(ip\gamma_Q)\phi_j(\gamma-\gamma_Q), \tag{38}$$

where $\phi_j(\gamma-\gamma_Q)$ is the wave function of a single well with a minimum at $\gamma=\gamma_Q=(2Q+1)\pi$, $Q=0,\pm1,\pm2,\ldots$. To evaluate the relative angular motion explicitly, we expand the mean Coulomb potential in a power series of the deviations $(\gamma-\gamma_Q)$ from the quasiequilibrium values γ_Q,

$$\frac{e^2}{4\pi\varepsilon_0\varepsilon_s R\sqrt{2(1-\cos\gamma)}}\approx\frac{e^2}{8\pi\varepsilon_0\varepsilon_s R}+\sum_{Q=-\infty}^{\infty}\frac{m_e\Omega^2R^2}{4}\times(\gamma-\gamma_Q)^2\Theta(\pi-|\gamma-\gamma_Q|), \tag{39}$$

where $\Omega^2\equiv e^2/(16\pi\varepsilon_0\varepsilon_s m_e R^3)$ and $\Theta(x)$ is the Heaviside unit step function $[\Theta(x)=1$ for $x>1$ and $\Theta(x)=0$ for $x<0]$. The harmonic approximation in Eq. (39) is valid if the typical excursions of the two electrons from their equilibrium angular positions are small compared with the interelectronic spacing ($2R$, which corresponds to $\gamma=\pi$). The Schrödinger equation of the relative angular motion of a single quantum well with minimum at γ_Q becomes the wave equation of a shifted harmonic oscillator with eigenfunctions and eigenenergies

$$\phi_j(\gamma-\gamma_Q)=C_j\,\exp\left[-\frac{1}{2}\left(\frac{\gamma-\gamma_Q}{\xi}\right)^2\right]H_j\left(\frac{\gamma-\gamma_Q}{\xi}\right), \tag{40}$$

$$\mathcal{E}_j^{\mathrm{osc}}=\hbar\Omega\left(j+\frac{1}{2}\right),\quad j=0,1,2,\ldots, \tag{41}$$

respectively.

In Eq. (40), $\xi\equiv\sqrt{2\hbar/(m_e\Omega)}/R$ is the typical width of the wave function and $H_j(x)$ is Hermite's polynomial. Within the accuracy of the tight-binding and the harmonic approximations, the wave function can be normalized on the interval $-\infty<\gamma<\infty$ with the result $C_j=1/\sqrt{2^j j!\,\pi^{1/2}\xi}$. The tunneling through the barrier of the Coulomb potential from well to well would lead to an explicit dependence of the energy levels $\mathcal{E}_j^{\mathrm{ra},p}$ on the characteristic index p and thus, would result in energy minibands. As shown in Ref. 22, the width $\Delta\mathcal{E}_j^{\mathrm{ra}}$ of the miniband $\mathcal{E}_j^{\mathrm{ra},p}$ is very small in comparison to the energy spacing $\mathcal{E}_j^{\mathrm{ra}}-\mathcal{E}_{j-1}^{\mathrm{ra}}$ Hence, the tunneling and the resulting hybridization of the energy levels can be neglected and thus the eigenenergies are independent of the characteristic index p.

The solution of the Schrödinger equation for the free rotation of the two-electron system as a whole,

$$\left(\frac{d^2}{d\Theta^2}+\nu_\Upsilon^2\right)Q_{\nu_\Upsilon}(\Theta)=0, \tag{42}$$

characterized by a rotational momentum

$$\nu_\Upsilon=\nu_\Upsilon(E)=\left\{\frac{4m_eR^2}{\hbar^2}\left[E-\mathcal{E}_{K_1}-\mathcal{E}_{K_2}-\frac{e^2}{8\pi\varepsilon_0\varepsilon_s R}-\mathcal{E}_j^{\mathrm{osc}}\right]\right\}^{1/2}, \tag{43}$$

is

$$Q_{\nu_\Upsilon}(\Theta)=\frac{1}{\sqrt{2\pi}}\exp(i\nu_\Upsilon\Theta). \tag{44}$$

The curves $\nu_\Upsilon(E)$ define the *channels*,[40–43,48,49] for the angular motion of the two-electron system as a whole. Note that the wave functions are orthonormalized according to

$$\int_{R-W/2}^{R+W/2}dr_1\,r_1\,\chi^*_{K_1}(r_1)\chi_{K_1'}(r_1)\int_{R-W/2}^{R+W/2}dr_2\;r_2\;\chi^*_{K_2}(r_2)\chi_{K_2}(r_2)\int_0^{2\pi}d\Theta\int_{\mathcal{D}}d\gamma\;\Psi^*_{P,K_1,K_2}(\gamma,\Theta)\Psi_{P',K_1',K_2'}(\gamma,\Theta)=\delta_{K_1,K_1'}\delta_{K_2,K_2'}\delta_{P,P'}, \tag{45}$$

where

$$\mathcal{D}=\begin{cases}-2\Theta\leqslant\gamma\leqslant2\Theta & \text{if } 0\leqslant\Theta<\pi\\ -2\pi\leqslant\gamma\leqslant2\pi & \text{if } \Theta=\pi\\ 2\Theta-4\pi\leqslant\gamma\leqslant4\pi-2\Theta & \text{if } \pi<\Theta\leqslant2\pi.\end{cases} \tag{46}$$

The symmetry of the two-electron wave function results in a strong correlation between the vibrational quantum number and that of the rotation. It is shown in Ref. 22 that $(\nu_\Upsilon-2\Phi/\Phi_0)=J$ with $J=0,\pm1,\pm2,\ldots$ is valid and that for $J=2m$ one obtains $\eta=(-1)^{P+j}$ and for $J=2m+1$ it follows that $\eta=-(-1)^{P+j}$, where $m=0,\pm1,\pm2,\ldots$. Thus, the parity of the orbital part of the two-electron wave function is $\eta=(-1)^{P+j+J}$. From these relations one obtains the only possible combinations of j and J.

(i) For $P=0$ and $P=2$:

$$\begin{aligned}&S=0\ \text{(para state)},\quad J=2m,\quad \text{and}\quad j=2k,\quad \text{or}\quad J=2m+1\quad \text{and}\quad j=2k+1;\\&S=1\ \text{(ortho state)},\quad J=2m+1,\quad \text{and}\quad j=2k,\quad \text{or}\quad J=2m\quad \text{and}\quad j=2k+1;\end{aligned} \tag{47}$$

(ii) for $P=1$:

$$\begin{aligned} &S=0 \ \text{(para state)}, \quad J=2m+1, \quad \text{and} \quad j=2k, \quad \text{or} \quad J=2m \quad \text{and} \quad j=2k+1; \\ &S=1 \ \text{(ortho state)}, \quad J=2m, \quad \text{and} \quad j=2k, \quad \text{or} \quad J=2m+1 \quad \text{and} \quad j=2k+1, \end{aligned} \tag{48}$$

where $k=0,1,2,\ldots$. In other words, for even values of P (symmetric radial wave functions), the para state is described by such an angular wave function where J and j have the *same parity*, while the ortho state has a wave function with J and j of *opposite parity*. The opposite is valid for odd P (antisymmetric radial wave functions). Further, for even J, p must be 0 and for odd J, p must be $\frac{1}{2}$.

We finally obtain the eigenenergies $E_{\Upsilon,\nu_\Upsilon} \equiv E_{K_1,K_2,j,J}(\Phi,2)$ of the two-electron QR:

$$E_{K_1,K_2,j,J}(\Phi,2)=\mathcal{E}_{K_1}+\mathcal{E}_{K_2}+\frac{e^2}{8\pi\varepsilon_0\varepsilon_s R}+\mathcal{E}_j^{\text{osc}}+\mathcal{E}_J^{\text{rot}}(\Phi), \tag{49}$$

where

$$\mathcal{E}_J^{\text{rot}}(\Phi)=\frac{\hbar^2}{4m_eR^2}\left(J+2\frac{\Phi}{\Phi_0}\right)^2 \tag{50}$$

is the rotational energy. These energy bands carry in the presence of a nonvanishing magnetic flux an equilibrium current, the *persistent current*,[21,40–43] which follows at $T=0$ K by $I_A^{(K_1,K_2,j,J)}(\Phi,2)=-dE_{K_1,K_2,j,J}(\Phi,2)/d\Phi$, and for a canonical ensemble at $T\neq 0$ K by $I_A(\Phi,2)=-dF(\Phi,2)/d\Phi$, where $F(\Phi,2)=-k_BT\ln Z(T,\Phi)$ is the Helmholtz free energy, k_B is the Boltzmann constant, and $Z(T,\Phi)$ is the partition function. Here, a canonical ensemble means a great number of absolutely identical rings (geometry, material) with two electrons in each ring, which can be in various two-electron states.

The energy bands of two interacting electrons in a narrow-width QR, $E_{j,J}(\Phi,2)\equiv E_{1,1,j,J}(\Phi,2)$, denoted by the label (J,j) are plotted in Fig. 1. It is seen that the energy bands of a certain symmetry become degenerate at $\Phi/\Phi_0=0$ and 0.5. In the case of a mesoscopic ring with impurities this degeneracy becomes lifted, and gaps between the energy bands open.[40–43] Further, the bands of the para and ortho states are mutually degenerate at $\Phi/\Phi_0=0.25$, which is valid also in the presence of (nonmagnetic) impurities. It is seen that the *minimal energy* in $-\frac{1}{2}<\Phi/\Phi_0\leqslant\frac{1}{2}$ belongs to different states:

$$\min\{E_{j,J}(\Phi)\}=\frac{\hbar^2\pi^2}{m_eW^2}+\frac{e^2}{8\pi\varepsilon_0\varepsilon_sR}+\frac{\hbar\Omega}{2}+\frac{\hbar^2}{m_eR^2}\begin{cases}\left(\dfrac{\Phi}{\Phi_0}+\dfrac{1}{2}\right)^2 & \text{for} \quad -\dfrac{1}{2}<\dfrac{\Phi}{\Phi_0}\leqslant-\dfrac{1}{4}\\ \left(\dfrac{\Phi}{\Phi_0}\right)^2 & \text{for} \quad -\dfrac{1}{4}\leqslant\dfrac{\Phi}{\Phi_0}\leqslant\dfrac{1}{4}\\ \left(\dfrac{\Phi}{\Phi_0}-\dfrac{1}{2}\right)^2 & \text{for} \quad \dfrac{1}{4}\leqslant\dfrac{\Phi}{\Phi_0}\leqslant\dfrac{1}{2}.\end{cases} \tag{51}$$

The persistent currents carried by the energy bands of Eqs. (49) and (51) are discussed in detail in Ref. [21]. It is shown that the persistent current shows bistability and hysteresis in dependence on the magnetic flux.

The above considered strongly correlated state of a two-electron Wigner molecule, i.e., a relatively rigid rotator, is based on the validity of the harmonic and the both adiabatic approximations. For typical semiconductor rings used in experiments, when charged with two electrons, the Wigner molecule in which the three types of motions (radial motions, relative angular motions, rotation) are adiabatically decoupled is the expected strongly correlated state. It becomes obvious that the first adiabatic approximation used above becomes valid if $\hbar\Omega\ll\hbar^2\pi^2/(m_eW^2)$, and the second adiabatic approximation becomes valid if $\hbar^2/(4m_eR^2)\ll\hbar\Omega$ is fulfilled. A nearly rigid rotator is formed if the mean Coulomb energy is much larger than the vibrational energy and no tunneling between the wells occurs, i.e., the harmonic approximation becomes valid. The harmonic approximation is applicable if $\hbar\Omega\ll e^2/(8\pi\varepsilon_0\varepsilon_sR)$, which can be expressed as $a_0^*\ll 2R$, where $a_0^*=4\pi\varepsilon_0\varepsilon_s\hbar^2/(m_ee^2)$ is the effective Bohr radius. These approximations are well fulfilled for typical semiconductor QR's used in experiments.

III. OPTICAL ABSORPTION

A. General

In an optical absorption experiment the attenuation of a light beam, which passes through a sample of thickness d, is measured. We assume the incident wave to be a monochromatic plane wave in an isotropic homogeneous medium with, in general, a complex dielectric function $\varepsilon(\omega)=\text{Re}\varepsilon(\omega)+i\,\text{Im}\varepsilon(\omega)$, related by $\sqrt{\varepsilon(\omega)}=n(\omega)+i\kappa(\omega)$, $\text{Re}\varepsilon(\omega)=n^2(\omega)-\kappa^2(\omega)$, and $\text{Im}\varepsilon(\omega)=2n(\omega)\kappa(\omega)$ to the refractive index $n(\omega)$ and the extinction coefficient $\kappa(\omega)$. The vector potential of this wave is given by

$$\mathbf{A}_1(\mathbf{x}_\mu,t)=A_1^{(0)}\mathbf{e}\,e^{i(\mathbf{k}\cdot\mathbf{x}_\mu-\omega t)}+\text{c.c.}, \tag{52}$$

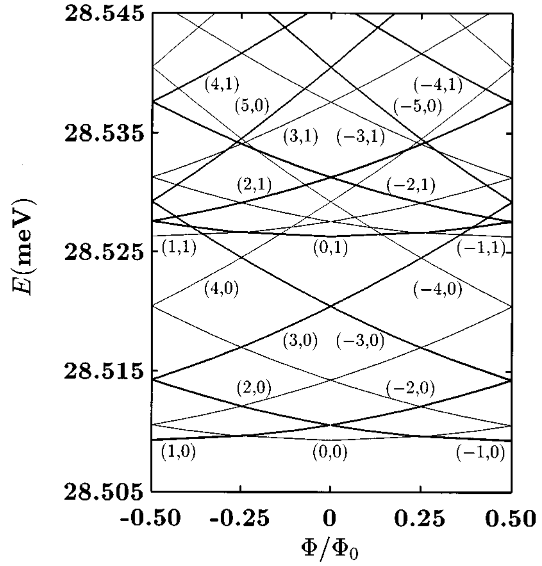

FIG. 1. The energy bands $E_{j,J}(\Phi,2) \equiv E_{1,1,j,J}(\Phi,2)$ of the para state ($S=0$, thin solid lines) and ortho state ($S=1$, heavy solid lines) labeled by (J,j) for a two-electron quantum ring synthesized from a GaAs-Ga$_{1-x}$Al$_x$As heterostructure. Parameters used in the calculation for the GaAs ring of radius R = 480 nm and width W = 20 nm: ε_s = 12.87 and m_e = 0.06624m_0; m_0 denotes bare electron mass.

where $\mathbf{k}=(k_x,k_y,k_z)$ is the wave vector, $\mathbf{e}$ is the (in general) complex polarization vector, $A_1^{(0)}$ is the real amplitude of the plane wave, and c.c. means complex conjugate of the first summand.

This plane wave propagates in a three-dimensional (3D) bulk medium with the dispersion relation $k=|\mathbf{k}| = \varepsilon(\omega)\omega/c$, assuming that $\mathbf{k}$ is a complex vector and the (angular) frequency ω is real if $\varepsilon(\omega)$ is complex; and c is the vacuum speed of light. From Eq. (52) it follows that

$$\mathbf{E}_1(\mathbf{x}_\mu,t) = -\frac{\partial}{\partial t}\mathbf{A}_1(\mathbf{x}_\mu,t) = i\omega[A_1^{(0)}\mathbf{e}\, e^{i(\mathbf{k}\cdot\mathbf{x}_\mu-\omega t)} - \text{c.c.}] \tag{53}$$

for the electric field, where $A_1^{(0)} = -(i/\omega)E_1^{(0)}$, and

$$\begin{aligned}\mathbf{B}_1(\mathbf{x}_\mu,t) &= \boldsymbol{\nabla}_{\mathbf{x}_\mu}\times\mathbf{A}_1(\mathbf{x}_\mu,t)\\ &= i\sqrt{\varepsilon(\omega)}\,\frac{\omega}{c}[A_1^{(0)}\mathbf{e}_B e^{i(\mathbf{k}\cdot\mathbf{x}_\mu-\omega t)} - \text{c.c.}]\end{aligned} \tag{54}$$

for the magnetic induction, where $A_1^{(0)} = -[ic/(\varepsilon(\omega)\omega)]B_1^{(0)}$ and $\mathbf{e}_B=\mathbf{e}\times\mathbf{e}_k$, with $\mathbf{e}_k=\mathbf{k}/|\mathbf{k}|$. The intensity I of the incident light is given by the time-averaged Poynting vector $\mathbf{S}=\mathrm{Re}\mathbf{E}\times\mathrm{Re}\mathbf{H}$ according to $\overline{\mathbf{S}_1(\mathbf{x}_\mu,t)} = \frac{1}{2}\overline{[\mathrm{Re}\mathbf{E}_1(\mathbf{x}_\mu,t)} \times \mathrm{Re}\mathbf{H}_1(\mathbf{x}_\mu,t)] \equiv I_1(\mathbf{x}_\mu,\omega)\mathbf{e}_k$, with $I_1(\mathbf{x}_i,\omega) = [n(\omega)\omega^2/(2\mu_0 c)](A_1^{(0)})^2 e^{-2(\omega/c)\kappa(\omega)\mathbf{e}_\mathbf{k}\cdot\mathbf{x}_\mu}$, where μ_0 is the permeability in vacuum and $\mathbf{H}$ is the magnetic field, assuming a nonmagnetic medium, so that $\mathbf{B}=\mu_0\mathbf{H}$ is valid. For a homogeneous 3D medium the *absorption coefficient* $\alpha(\omega)$, which describes the fractional decrease in the intensity with distance, is defined as $\alpha(\omega) = -(1/I)dI/dx$ if the wave propagates along the x axis. Then, it follows that $\alpha(\omega)=2(\omega/c)\kappa(\omega)$. Here, we are interested in the optical absorption of QR's, which are synthesized on semiconductor heterostructures, being multilayer structures in general. In this case there exists no unique first-principles derivation of an absorption coefficient.[50] Hence, it is more profitable to calculate the energy that is absorbed per unit time and unit area (*power absorption*) by the nanostructures under consideration.

The incident electromagnetic wave will induce transitions between the *initial states* $|i\rangle$ and the *final states* $|f\rangle$ of the two-electron system, where $|\nu\rangle$ denotes an eigenstate of H_0, Eq. (3), with energy E_ν given in Eq. (49). The transition probability from the state $|i\rangle$ to the state $|f\rangle$ under the perturbation H_{int} is

$$\begin{aligned}P_{if}(t) = \Bigg| &\frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{fi}t'} H_{fi}^{\text{int}}(t')\\ &+ \left(\frac{1}{i\hbar}\right)^2 \sum_{n(\neq i)} \int_0^t dt' e^{i\omega_{fn}t'} H_{fn}^{\text{int}}(t')\\ &\times \int_0^{t'} dt'' e^{i\omega_{ni}t''} H_{ni}^{\text{int}}(t'') + \cdots \Bigg|^2,\end{aligned} \tag{55}$$

where $H_{\nu\nu'}^{\text{int}}(t) = \langle\nu|H_{\text{int}}(t)|\nu'\rangle$ and $\omega_{\nu\nu'}=(E_\nu-E_{\nu'})/\hbar$ is the Bohr angular frequency. As far as the separation of the energy bands of QR's is in the range of meV, FIR light is used in the optical absorption spectroscopy. To calculate the power absorption, we identify the interaction Hamiltonian H_{int} as H_1, given by Eq. (6), and use the first summand of Eq. (55) only. Then for the *transition rate* (transition probability per unit time) from the state $|i\rangle$ to the state $|f\rangle$ it follows that

$$W_{if} = \frac{2\pi}{\hbar^2}\frac{e^2}{m_e^2}\left|\langle f|\left(\sum_{\mu=1}^{2} A_1{}^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}_\mu}\,\mathbf{e}\cdot\mathbf{p}_\mu\right)|i\rangle\right|^2 \delta(\omega_{fi}-\omega), \tag{56}$$

where we have used the fact that $\omega_{fi}>0$. Because in the usual experiments the light is incident from vacuum to the sample, the amplitude $A_1^{(0)}$ present inside the ring can be calculated from the amplitude outside the specimen by standard techniques, e.g., with the transfer-matrix method. Under typical experimental conditions, the wavelength of the incident light is much larger than the characteristic dimensions of the electron system in semiconductor nanostructures ($\lambda_{\text{el}}\approx 10$ nm $\ll \lambda_{\text{light}}\approx 10$ μm). In this case one can expand the exponential $e^{i\mathbf{k}\cdot\mathbf{x}_\mu}$ leaving only the zeroth order, i.e., $e^{i\mathbf{k}\cdot\mathbf{x}_\mu}\approx 1$. This is the *electric dipole approximation.* Notice that the next term $\propto i\mathbf{k}\cdot\mathbf{x}_\mu$ results in *electric quadrupole transitions,* and H_2 of Eq. (7) in *magnetic dipole transitions.* Both of them are beyond the scope of this paper. Using the electric dipole approximation and $\mathbf{p}_\mu=(m_e/i\hbar)[\mathbf{x}_\mu,H_0]$ in Eq. (56) and defining the electric dipole operator of the two electrons by $\mathbf{d}=-e(\mathbf{x}_1+\mathbf{x}_2)$, we obtain

$$W_{if} = \frac{2\pi}{\hbar^2}(A_1^{(0)})^2\omega_{fi}^2|\langle f|\mathbf{e}\cdot\mathbf{d}|i\rangle|^2\delta(\omega_{fi}-\omega). \tag{57}$$

Then the power absorption $\mathcal{P}_{\text{abs}}$ by the electronic transition from $|i\rangle$ to $|f\rangle$ is $\mathcal{P}_{\text{abs}}(\omega)=\eta_{\text{QR}}\hbar\omega W_{if}$, where η_{QR} is the number of QR's per unit area in the x-y plane. In Eq. (57) it is assumed that the initial state is always occupied and the final one is free. For finite temperatures, one has to take into account the thermal occupation of the energy bands. In this case the incident wave can also induce the transition $|i\rangle$ to $|f\rangle$ (stimulated emission), for which $\omega_{fi}<0$ is valid, and one has $\mathcal{P}_{\rm abs}(\omega)=\eta_{\rm QR}\hbar\omega\Sigma_f\Sigma_i\{w_{\rm el}(i)[1-w_{\rm el}(f)]W_{if}-w_{\rm el}(f)[1-w_{\rm el}(i)]W_{fi}\}$, where $w_{\rm el}(i)$ is the probability that the system is in the two-particle state $|i\rangle$ and $W_{if}=W_{fi}$ is valid. Thus, we obtain the following for the power absorption of an ensemble of (uncoupled) QR's per unit area:

$$\mathcal{P}_{\rm abs}(\omega)=\frac{2\pi\omega}{\hbar}\eta_{\rm QR}(A_1^0)^2\sum_f\sum_i\omega_{fi}^2[w_{\rm el}(i)-w_{\rm el}(f)]\times|\langle f|\mathbf{e}\cdot\mathbf{d}|i\rangle|^2\delta(\omega_{fi}-\omega). \tag{58}$$

It is important to note that Eq. (58) with the two-electron states of Sec. II B is valid only for such a temperature range as long as both the adiabatic and the harmonic approximation are valid. For higher temperatures the different motions become coupled due to the thermal transitions.[43] Thus, the range of adequacy of Eq. (58) is at low temperatures, where the final states are practically free: $\omega_{\rm el}(f)\approx0$. The *oscillator strength* of the electronic transition $|\nu'\rangle\rightarrow|\nu\rangle$ is defined by $f_{\nu\nu'}=(2m_e/\hbar)\omega_{\nu\nu'}|\langle\nu|\mathbf{e}\cdot(\mathbf{x}_1+\mathbf{x}_2)|\nu'\rangle|^2$, which can be written as the matrix element of the total momentum operator with $\langle\nu|\mathbf{e}\cdot(\mathbf{p}_1+\mathbf{p}_2)|\nu'\rangle=m_e\omega_{\nu\nu'}\langle\nu|\mathbf{e}\cdot(\mathbf{x}_1+\mathbf{x}_2)|\nu'\rangle$. The oscillator strength satisfies the f-sum rule $\Sigma_\nu f_{\nu\nu'}=1$.

The *relative* absorption $K(\omega)$ may be defined by $K(\omega)=\mathcal{P}_{\rm abs}(\omega)/I_1$, where $I_1=\omega^2(A_1^{(0)})^2/(2\mu_0c)$ is the intensity of the incident wave, propagating in vacuum as usual in experiments. Further, from Eq. (58) an absorption coefficient may be defined by $\alpha(\omega)=\mathcal{P}_{\rm abs}(\omega)/(I_1d)$, where d is the thickness of the sample. But in general this gives no definite physical quantity for layered systems.

B. Dipole selection rules

Let us examine the *electric dipole selection rules* of a system of independent rings in the x-y plane; i.e., no tunneling and Coulomb coupling between the rings is assumed. For this case, we identify the states $|i\rangle$ and $|f\rangle$ by those of the QR, given in Eqs. (24), (25), (27), (28), (36), (38), (40), and (44): $|i\rangle=|P,K_1,K_2\rangle\otimes|p,j\rangle\otimes|J\rangle\otimes|S,M_S\rangle$ and $|f\rangle=|P',K_1',K_2'\rangle\otimes|p',j'\rangle\otimes|J'\rangle\otimes|S',M_S'\rangle$. Further, we assume a perpendicular incident plane wave (see the inset of Fig. 1) propagating in vacuum along the negative z axis with the wave vector $\mathbf{k}=(0,0,-k)$. The dipole operator of the two electrons is $\mathbf{d}=-2eR\cos(\gamma/2)(\cos\Theta,\sin\Theta,0)$. In the following we consider the dipole matrix element $\langle f|\mathbf{e}\cdot\mathbf{d}|i\rangle$ for different polarizations of the incident monochromatic plane wave.

1. Elliptically polarized light

In this case we have $\mathbf{e}=a_xe^{i\alpha_x}\mathbf{e}_x+a_ye^{i\alpha_y}\mathbf{e}_y$, with $a_x^2+a_y^2=1$, from which it follows $\mathbf{e}\cdot\mathbf{d}=-2eR\cos(\gamma/2)(a_xe^{i\alpha_x}\cos\Theta+a_ye^{i\alpha_y}\sin\Theta)$. We obtain

$$\langle P',K_1',K_2';p',j';J';S',M_S'|\mathbf{e}\cdot\mathbf{d}|P,K_1,K_2;p,j;J;S,M_S\rangle=C_e\delta_{P',P}\delta_{K_1',K_1}\delta_{K_2',K_2}(\delta_{p',p+1/2}+\delta_{p',p-1/2})\delta_{j',j+(2\ell+1)}\times[a_xe^{i\alpha_x}(\delta_{J',J+1}+\delta_{J',J-1})-ia_ye^{i\alpha_y}(\delta_{J',J+1}-\delta_{J',J-1})]\delta_{S',S}\delta_{M_S',M_S}, \tag{59}$$

where

$$C_e=-i(-1)^{J+\ell}eR\sqrt{\frac{2^jj!}{2^{j'}j'!}}\left(\frac{\xi}{2}\right)^{2\ell+1}\exp\left[-\left(\frac{\xi}{4}\right)^2\right]L_j^{2\ell+1}\left(\frac{\xi}{8}\right)^2, \tag{60}$$

$\ell=0,\pm1,\pm2,\ldots$ under the condition that $j'>0$ and $L_j^\ell(\xi)$ is the associated Laguerre polynomial. Note the correlation between p and J discussed above. Thus, the dipole selection rules for elliptically polarized light are

$$\Delta P=0,\quad \Delta K_1=0,\quad \Delta K_2=0,\quad \Delta p=\pm\tfrac{1}{2},\quad \Delta j=2\ell+1,\quad \Delta J=\pm1,\quad \Delta S=0,\quad \Delta M_S=0. \tag{61}$$

Hence, the optical transitions are between the energy bands of the para state and between that of the ortho state.

2. Circularly polarized light

For circularly polarized light we have $a_x=a_y$, $\alpha_y-\alpha_x=\pm\pi/2$, and assume $\alpha_x=0$: $\mathbf{e}\equiv\mathbf{e}_\pm=(1/\sqrt{2})(\mathbf{e}_x\pm i\mathbf{e}_y)$, from where it follows $\mathbf{e}\cdot\mathbf{d}=-\sqrt{2}eR\cos(\gamma/2)\exp(\pm i\Theta)$. We obtain

$$\langle P',K_1',K_2';p',j';J';S',M_S'|\mathbf{e}\cdot\mathbf{d}|P,K_1,K_2;p,j;J;S,M_S\rangle=C_c\delta_{P',P}\delta_{K_1',K_1}\delta_{K_2',K_2}(\delta_{p',p+1/2}+\delta_{p',p-1/2})\times\delta_{j',j+(2\ell+1)}\delta_{J',J\pm1}\delta_{S',S}\delta_{M_S',M_S}, \tag{62}$$

where

$$C_c=-i(-1)^{J+\ell}\sqrt{2}eR\sqrt{\frac{2^jj!}{2^{j'}j'!}}\left(\frac{\xi}{2}\right)^{2\ell+1}\exp\left[-\left(\frac{\xi}{4}\right)^2\right]L_j^{2\ell+1}\left(\frac{\xi^2}{8}\right). \tag{63}$$

From Eq. (62), the dipole selection rules for circularly polarized light are the same as for elliptically polarized light given by Eq. (61).

3. Linearly polarized light

For the incident wave linearly polarized in the x direction (without loss of generality) we have $\mathbf{e}\equiv\mathbf{e}_x$ and $\mathbf{e}\cdot\mathbf{d}=-2eR\cos(\gamma/2)\cos\Theta$. From this we obtain

$$\langle P',K_1',K_2';p',j';J';S',M_S'|\mathbf{e}\cdot\mathbf{d}|P,K_1,K_2;p,j;J;S,M_S\rangle=C_l\delta_{P',P}\delta_{K_1',K_1}\delta_{K_2',K_2}(\delta_{p',p+1/2}+\delta_{p',p-1/2})\times\delta_{j',j+(2\ell+1)}(\delta_{J',J+1}+\delta_{J',J-1})\delta_{S',S}\delta_{M_S',M_S}, \tag{64}$$

where $C_l=C_e$, which results in the dipole selection rules given by Eq. (61).

C. Power absorption

Now we calculate the power absorption at low temperatures, assuming as an example the incident light to be circularly polarized. The Bohr angular frequencies for a narrow-width QR are

$$\omega_{fi}\equiv\omega_{\ell J}^{\pm}=(2\ell+1)\Omega+\frac{\hbar}{4m_eR^2}\left(1\pm2J\pm4\frac{\Phi}{\Phi_0}\right). \tag{65}$$

In the case of low temperatures, i.e., if $k_BT\ll\hbar\Omega$ is valid, there are no oscillations excited. Then the initial state $|i\rangle$ is characterized by $j=0$ and $K_1=K_2=1$, from where it follows that we have for an initial para state $J=2m$, and $J=2m+1$ if the initial state is an ortho state. To calculate the power absorption, we consider a canonical ensemble of QR's, all charged by two electrons at low temperatures. In this case the probability distribution functions are $w_{\mathrm{el}}(i)=\exp[-E_i/(k_BT)]/Z$ and $w_{\mathrm{el}}(f)\approx0$, where the partition function $Z(T,\Phi)$ has the form[43]

$$Z(T,\Phi)=\exp\left(-\frac{2\mathcal{E}_1+e^2/(8\pi\varepsilon_0\varepsilon_sR)+\hbar\Omega/2}{k_BT}\right)Z_{\mathrm{rot}}(T,\Phi) \tag{66}$$

with

$$Z_{\mathrm{rot}}(T,\Phi)=g_{\mathrm{para}}\,Z_{\mathrm{even}}^{\mathrm{rot}}(T,\Phi)+g_{\mathrm{ortho}}\,Z_{\mathrm{odd}}^{\mathrm{rot}}(T,\Phi), \tag{67}$$

where

$$Z_{\mathrm{even}}^{\mathrm{rot}}(T,\Phi)\equiv\sum_{m=-\infty}^{\infty}\exp\left[-\frac{\hbar^2(2m+2\Phi/\Phi_0)^2}{4m_eR^2k_BT}\right]=\exp\left[-\frac{\hbar^2}{m_eR^2k_BT}(\Phi/\Phi_0)^2\right]\vartheta_3\left(\frac{i\hbar^2}{m_eR^2k_BT}\frac{\Phi}{\Phi_0},e^{-\hbar^2/(m_eR^2k_BT)}\right)\approx\sqrt{\frac{\pi m_eR^2}{\hbar^2}k_BT}, \tag{68}$$

$$Z_{\mathrm{odd}}^{\mathrm{rot}}(T,\Phi)\equiv\sum_{m=-\infty}^{\infty}\exp\left[-\frac{\hbar^2(2m+1+2\Phi/\Phi_0)^2}{4m_eR^2k_BT}\right]=\exp\left[-\frac{\hbar^2}{m_eR^2k_BT}\left(\frac{\Phi}{\Phi_0}\right)^2\right]\vartheta_2\left(\frac{i\hbar^2}{m_eR^2k_BT}\frac{\Phi}{\Phi_0},e^{-\hbar^2/(m_eR^2k_BT)}\right)\approx\sqrt{\frac{\pi m_eR^2}{\hbar^2}k_BT}. \tag{69}$$

Thus, we have

$$Z_{\mathrm{rot}}(T,\Phi)\approx[g_{\mathrm{para}}+g_{\mathrm{ortho}}]\sqrt{\frac{\pi m_eR^2}{\hbar^2}k_BT}=4\sqrt{\frac{\pi m_eR^2}{\hbar^2}k_BT}. \tag{70}$$

Herein, we have used the asymptotic representations of the ϑ functions $\vartheta_2(z,q)$ and $\vartheta_3(z,q)$,[51] $\mathcal{E}_{J=1}^{\mathrm{rot}}\ll k_BT(\ll\hbar\Omega)$. The statistical weights of the para state and of the ortho state are $g_{\mathrm{para}}=1$ and $g_{\mathrm{ortho}}=3$, respectively. From Eqs. (58), (62), (66), and (70) it follows for the power absorption

$$\mathcal{P}_{\mathrm{abs}}(\omega)=\frac{4\pi e^2R\omega\eta_{\mathrm{QR}}}{\hbar}\frac{(A_1^{(0)})^2|C_c|^2}{Z_{\mathrm{rot}}(T,\Phi)}\sum_{\ell=0,1,2,\ldots}\left\{\sum_{J=0,\pm 2,\pm 4,\ldots} g_{\mathrm{para}}e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[(\omega_{\ell J}^{+})^2\delta(\omega-\omega_{\ell J}^{+})+(\omega_{\ell J}^{-})^2\delta(\omega-\omega_{\ell J}^{-})]\right.$$

$$\left.+\sum_{J=\pm 1,\pm 3,\pm 5\ldots} g_{\mathrm{ortho}}e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[(\omega_{\ell J}^{+})^2\delta(\omega-\omega_{\ell J}^{+})+(\omega_{\ell J}^{-})^2\delta(\omega-\omega_{\ell J}^{-})]\right\}. \tag{71}$$

In Fig. 2 we have plotted the energy bands of the para state and in Fig. 3 those of the ortho state, where each band of the ortho state is threefold degenerate ($M_s=0,\pm 1$). The arrows indicate the lowest optical transitions, which are (electric) dipole allowed: $\Delta j=1$ and $\Delta J=\pm 1$. The absorption peaks arise as doublets with spacing depending on the magnetic flux. If we consider, for example, the two lowest transitions of Fig. 2, $(0,0)\rightarrow(\pm 1,1)$, it becomes obvious that for $\Phi=0$ one peak arises, which separates into two peaks for $\Phi\neq 0$. The spacing between the two peaks increases if $|\Phi|$ increases. Whereas the higher-frequency peak arises for $\Phi>0$ from the transition $(0,0)\rightarrow(1,1)$, it arises for $\Phi<0$ from the transition $(0,0)\rightarrow(-1,1)$. For the ortho state (Fig. 3) the lowest-frequency doublet, arising from the transitions $(-1,0)\rightarrow(0,1)$ and $(-2,1)$, shows a maximum separation for $\Phi/\Phi_0=-\frac{1}{2}$ and occurs as one peak at $\Phi/\Phi_0=\frac{1}{2}$. The opposite is valid for the doublet, which is connected with the transitions $(1,0)\rightarrow(0,1)$ and $(2,1)$. It becomes obvious that the measured absorption spectrum allows one in dependence on the magnetic flux threading the ring, to determine if the two-electron system is in a para or in an ortho state. Thus, the study of the absorption spectrum of QR's gives complementary results to the observation of the persistent current. While the persistent current as a function of Φ gives a physical representation of the derivative of the energy bands with respect to Φ and thus gives continuous characteristics of the band structure, the absorption spectra form discrete characteristics of the same band structure according to the dipole-allowed transitions.

IV. RAMAN SCATTERING

A. General formulation

In this section we consider the inelastic light scattering on an ensemble of QR's. The Raman scattering cross section involves two terms to be treated by perturbation theory, the $\mathbf{A}_1\cdot\mathbf{p}$ term in the second order and the $\mathbf{A}_1^2$ term in first order. The *differential cross section* for scattering of a photon (corresponding to a plane wave) from the state $\omega_L,\mathbf{k}_L,\mathbf{e}_L$ to the state $\omega_S,\mathbf{k}_S,\mathbf{e}_S$ and the concomitant transition of the two-electron system from the state $|i\rangle$ to the state $|f\rangle$ is given by[46]

$$\frac{d^2\sigma}{d\omega d\Omega}=\left(\frac{\varepsilon(\omega_S)}{\varepsilon(\omega_L)}\right)^{1/2}\frac{\omega_S}{\omega_L}r_0^2\sum_f\sum_i w_{\mathrm{el}}(i)|\langle f|\mathcal{M}(\mathbf{q})|i\rangle|^2 \times\delta(\omega_{fi}-\omega), \tag{72}$$

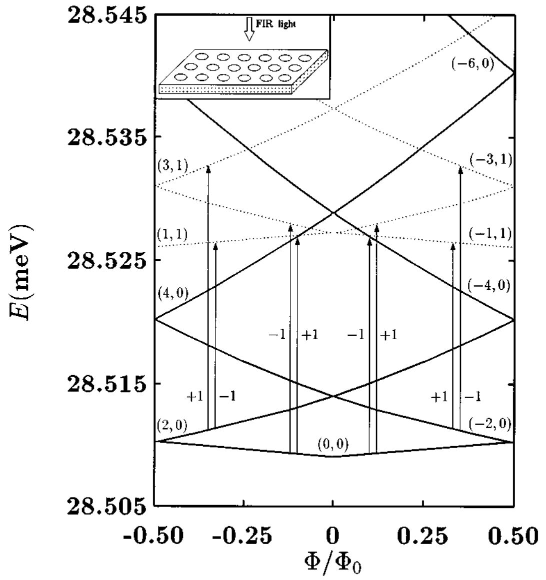

FIG. 2. The optical transitions within the energy bands $E_{j,J}(\Phi,2)$ of the para state represented in dependence on the magnetic flux Φ for a two-electron ring. These bands are labeled by (J,j). The arrows indicate the lowest optical transitions with $\Delta J=\pm 1$ and $\Delta j=1$. The same transitions are induced by $\mathcal{M}_1(\mathbf{q})$ in the polarized Raman scattering.

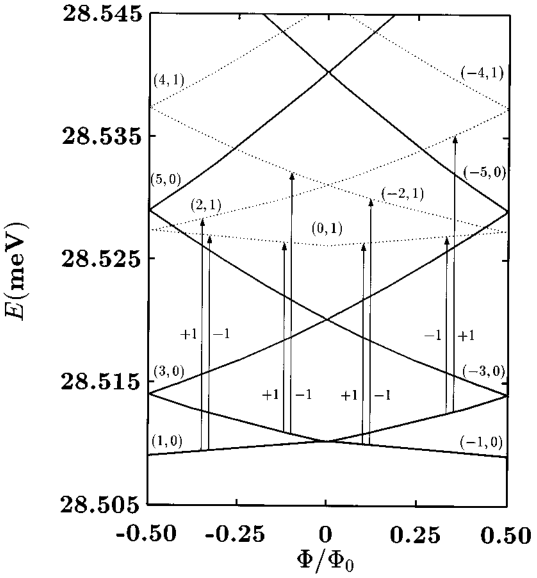

FIG. 3. The optical transitions within the energy bands $E_{j,J}(\Phi,2)$ of the ortho state represented in dependence on the magnetic flux Φ for a two-electron ring. The arrows indicate the lowest optical transitions with $\Delta J=\pm 1$ and $\Delta j=1$. The same transitions are induced by $\mathcal{M}_1(\mathbf{q})$ in the polarized Raman scattering and by $\mathcal{M}_2^{(a)}(\mathbf{q})$ in the depolarized Raman scattering. Denotations and parameters are the same as in Fig. 1.

where $\omega=\omega_L-\omega_S$ is the scattering frequency, $\mathbf{q}=\mathbf{k}_L-\mathbf{k}_S$ is the scattering wave vector, and $r_0\equiv e^2/(4\pi\varepsilon_0 m_0 c^2)$ is the *classical* electron radius. Assuming that the many-particle states $|i\rangle$ and $|f\rangle$ are within the conduction band of the semiconductor, the scattering tensor $\mathcal{M}(\mathbf{q})$ is given by Eq. (12).[46,47] The explicit expression of the light scattering cross section depends on the details of the electron energy band structure of the host semiconductor as well as on the scattering mechanism. Considering in this paper resonant Raman scattering on QR's synthesized on GaAs-Ga$_{1-x}$Al$_x$As heterostructures, we can use the results of Refs. 46 and 52:

$$\gamma_{\alpha\beta}=(\mathbf{e}_L\cdot\overleftrightarrow{\mathbf{A}}\cdot\mathbf{e}_S^*)\xi_{\alpha\beta}-i(\mathbf{e}_L\times\mathbf{e}_S^*)\cdot\overleftrightarrow{\mathbf{B}}\cdot\boldsymbol{\zeta}_{\alpha\beta}, \qquad (73)$$

where

$$\boldsymbol{\zeta}_{\alpha\beta}=\langle\alpha|e^{i\mathbf{q}\cdot\mathbf{x}}\boldsymbol{\sigma}|\beta\rangle=\xi_{\alpha\beta}\langle m_\alpha|\boldsymbol{\sigma}|m_\beta\rangle, \qquad (74)$$

$$\overleftrightarrow{\mathbf{A}}=\overleftrightarrow{\mathbf{I}}\left[1+\frac{2P^2}{3m_0}\left(\frac{E_{v_h}}{E_{v_h}^2-\hbar^2\omega_L^2}+\frac{E_{v_l}}{E_{v_l}^2-\hbar^2\omega_L^2}+\frac{E_{v_s}}{E_{\mathrm{SO}}^2-\hbar^2\omega_L^2}\right)\right], \qquad (75)$$

$$\overleftrightarrow{\mathbf{B}}=\overleftrightarrow{\mathbf{I}}\frac{P^2}{3m_0}\hbar\omega_L\left(\frac{1}{E_{v_h}^2-\hbar^2\omega_L^2}+\frac{1}{E_{v_l}^2-\hbar^2\omega_L^2}-\frac{2}{E_{\mathrm{SO}}^2-\hbar^2\omega_L^2}\right)+(\mathbf{e}_p\otimes\mathbf{e}_p-\overleftrightarrow{\mathbf{I}})\frac{P^2}{m_0}\hbar\omega_L\left(\frac{1}{E_{v_h}^2-\hbar^2\omega_L^2}+\frac{1}{E_{v_l}^2-\hbar^2\omega_L^2}\right). \qquad (76)$$

Herein, $\overleftrightarrow{\mathbf{I}}$ is the unit dyadic tensor, $\mathbf{e}_p$ is the unit vector along the direction of the momentum, and $P=\langle S|p_x|X\rangle=\langle S|p_y|Y\rangle=\langle S|p_z|Z\rangle$ is the interband matrix element of the momentum operator in the Kane model[53] with $|S\rangle$ the s-like wave function of the conduction band and $|X\rangle$, $|Y\rangle$, and $|Z\rangle$ are associated with the valence bands. The anisotropic term in the right-hand side of Eq. (76) is small[46] and is neglected in the following. In Eqs. (75), and (76), E_{v_h}, E_{v_l}, and E_{SO} are the energy differences between the conduction band and the heavy-hole, light-hole, and split-off valence bands, respectively. The spin dependence of $\mathcal{M}(\mathbf{q})$ is due to the spin-orbit coupling in the valence band. In the bulk (3D) semiconductor GaAs, the heavy- and light-hole valence bands are degenerate at the Γ point and thus $E_{v_h}=E_{v_l}=E_0$ is valid. Further, the split-off gap is $E_{\mathrm{SO}}=E_0+\Delta_0$, where Δ_0 is the spin-orbit splitting of the top valence-band states. It is important to note that for a semiconductor heterostructure according to the size quantization in the growth direction (along the z axis), the energy gaps increase by the energy of the lowest quantized levels in the quantum well. Without loss of generality this additional energy is assumed to be incorporated into E_{v_l} and E_{SO} if necessary. If in the valence bands a quantum well is formed, this results in a lifting of the degeneracy of the heavy- and light-hole valence bands at the Γ point.

We are interested in *resonant* inelastic light scattering. Under the resonance condition between the degenerate valence band ($\hbar\omega_L\approx E_0$), we find from Eqs. (12), (14), and (73) – (76)

$$\mathcal{M}(\mathbf{q})=\frac{P^2}{3m_0}\frac{1}{E_0-\hbar\omega_L}\times\sum_{\mu=1}^{2}e^{i\mathbf{q}\cdot\mathbf{x}_\mu}[2\mathbf{e}_L\cdot\mathbf{e}_S^*-i(\mathbf{e}_L\times\mathbf{e}_S^*)\cdot\boldsymbol{\sigma}_\mu], \qquad (77)$$

and for the resonance between the split-off valence band and the conduction band it follows that

$$\mathcal{M}(\mathbf{q})=\frac{P^2}{3m_0}\frac{1}{E_0+\Delta_0-\hbar\omega_L}\times\sum_{\mu=1}^{2}e^{i\mathbf{q}\cdot\mathbf{x}_\mu}[\mathbf{e}_L\cdot\mathbf{e}_S^*+i(\mathbf{e}_L\times\mathbf{e}_S^*)\cdot\boldsymbol{\sigma}_\mu]. \qquad (78)$$

We will further consider as an example in more detail the case of the resonance with the split-off valence band in the limit $\mathbf{q}\cdot\mathbf{x}_\mu\ll 1$. This means that the following inequality for the wavelength of the net excitation in the QR, $\lambda_{\mathrm{excitation}}\ll 2R$, is assumed to be fulfilled. The case of the resonance with the degenerate valence band (or nondegenerate if the valence band forms a quantum well) is qualitatively quite analogous. Then we write

$$\mathcal{M}(\mathbf{q})=\mathcal{M}_0(\mathbf{q})+\mathcal{M}_1(\mathbf{q})+\mathcal{M}_2(\mathbf{q}), \qquad (79)$$

where

$$\mathcal{M}_0(\mathbf{q})=D_{\mathrm{SO}}\left[2\mathbf{e}_L\cdot\mathbf{e}_S^*+\frac{i}{\hbar}(\mathbf{e}_L\times\mathbf{e}_S^*)\cdot\mathbf{S}\right], \qquad (80)$$

$$\mathcal{M}_1(\mathbf{q})=iD_{\mathrm{SO}}(\mathbf{e}_L\cdot\mathbf{e}_S^*)[\mathbf{q}\cdot(\mathbf{x}_1+\mathbf{x}_2)], \qquad (81)$$

$$\mathcal{M}_2(\mathbf{q})=iD_{\mathrm{SO}}\sum_{\mu=1}^{2}(\mathbf{q}\cdot\mathbf{x}_\mu)[(\mathbf{e}_L\times\mathbf{e}_S^*)\cdot\boldsymbol{\sigma}_\mu], \qquad (82)$$

with

$$D_{\mathrm{SO}}=\frac{P^2}{3m_0}\frac{1}{E_0+\Delta_0-\hbar\omega_L}. \tag{83}$$

It is obvious that the operator $\mathcal{M}_0(\mathbf{q})$ does not change the orbital part of the wave function, but can change the spin. The operator $\mathcal{M}_1(\mathbf{q})$ is connected with the center-of-mass motion of the electrons and can change the orbital quantum numbers only. On the other hand, the operator $\mathcal{M}_2(\mathbf{q})$ depends on the coordinates of the electrons and includes the Pauli spin vector operator, and hence can change the orbital as well as the spin quantum numbers simultaneously.

B. Polarized spectrum

Let us discuss the polarized spectrum for which $\mathbf{e}_L\cdot\mathbf{e}_S^*=1$ and $\mathbf{e}_L\times\mathbf{e}_S^*=0$ is valid. In this case we have $\mathcal{M}_0(\mathbf{q})=2D_{\mathrm{SO}}$, $\mathcal{M}_2(\mathbf{q})=0$, and

$$\mathcal{M}_1(\mathbf{q})=i\ 2RD_{\mathrm{SO}}\cos\frac{\gamma}{2}(q_x\cos\Theta+q_y\sin\Theta). \tag{84}$$

From this we obtain

$$\langle P',K_1',K_2';p',j';J';S',M_S'|\mathcal{M}_1(\mathbf{q})|P,K_1,K_2;p,j;J;S,M_S\rangle=C_{\mathrm{pol}}\delta_{P'P}\delta_{K_1'K_1}\delta_{K_2'K_2}(\delta_{p',p+1/2}+\delta_{p',p-1/2})\delta_{j',j+(2\ell+1)}\times[q_x(\delta_{J',J+1}+\delta_{J',J-1})-iq_y(\delta_{J',J+1}-\delta_{J',J-1})]\delta_{S'S}\delta_{M_S'M_S} \tag{85}$$

with

$$C_{\mathrm{pol}}=-(-1)^{J+\ell}RD_{\mathrm{SO}}\sqrt{\frac{2^j j!}{2^{j'}j'!}}\left(\frac{\xi}{2}\right)^{2\ell+1}\exp\left[-\left(\frac{\xi}{4}\right)^2\right]L_j^{2\ell+1}\left(\frac{\xi^2}{8}\right). \tag{86}$$

Thus, the *selection rules* for polarized Raman scattering spectra are identical with those for the absorption light and give rise to transitions between the energy bands of the para state and between the energy bands of the ortho state. The differential cross section reads

$$\frac{d^2\sigma}{d\omega d\Omega}=\left(\frac{\varepsilon(\omega_S)}{\varepsilon(\omega_L)}\right)^{1/2}\frac{\omega_S}{\omega_L}r_0^2\frac{|C_{\mathrm{pol}}|^2}{Z_{\mathrm{rot}}(T,\Phi)}(q_x^2+q_y^2)\sum_{\ell=0,1,2,\dots}\left\{\sum_{J=0,\pm2,\pm4,\dots}g_{\mathrm{para}}e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[\delta(\omega-\omega_{\ell J}^+)+\delta(\omega-\omega_{\ell J}^-)]+\sum_{J=\pm1,\pm3,\pm5,\dots}g_{\mathrm{ortho}}e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[\delta(\omega-\omega_{\ell J}^+)+\delta(\omega-\omega_{\ell J}^-)]\right\}. \tag{87}$$

As in the power absorption we have assumed that the initial state is characterized by $K_1=K_2=0$ and $j=0$.

C. Depolarized spectrum

In this case we have $\mathbf{e}_L\cdot\mathbf{e}_S^*=0$ and $\mathbf{e}_L\times\mathbf{e}_S^*\equiv\mathbf{b}\neq0$. Then, it follows that $\mathcal{M}_1(\mathbf{q})=0$, $\mathcal{M}_0(\mathbf{q})=(i/r\hbar)D_{\mathrm{SO}}\mathbf{b}\cdot\mathbf{S}$, and

$$\begin{aligned}\mathcal{M}_2(\mathbf{q})&=\frac{2i}{\hbar}RD_{\mathrm{SO}}\left\{\left[q_x\cos\Theta\cos\frac{\gamma}{2}+q_y\sin\Theta\sin\frac{\gamma}{2}\right]\mathbf{b}\cdot\mathbf{S}+\left[q_y\cos\Theta\sin\frac{\gamma}{2}-q_x\sin\Theta\sin\frac{\gamma}{2}\right]\mathbf{b}\cdot\Delta\mathbf{S}\right\}\\&\equiv\mathcal{M}_2^{(a)}(\mathbf{q})+\mathcal{M}_2^{(a)}(\mathbf{q}),\end{aligned} \tag{88}$$

where $\Delta\mathbf{S}=(\hbar/2)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$. Let us start to investigate the Raman scattering matrix element due to $\mathcal{M}_0(\mathbf{q})$. The only nonzero matrix elements between the spin states are

$$\langle1,0|S_x|1,1\rangle=\langle1,1|S_x|1,0\rangle=\frac{\hbar}{\sqrt{2}},\quad\langle1,0|S_x|1,-1\rangle=\langle1,-1|S_x|1,0\rangle=\frac{\hbar}{\sqrt{2}},\quad\langle1,0|S_y|1,1\rangle=-\langle1,1|S_y|1,0\rangle=\frac{i\hbar}{\sqrt{2}},$$

$$\langle1,0|S_y|1,-1\rangle=-\langle1,-1|S_y|1,0\rangle=-\frac{i\hbar}{\sqrt{2}},\quad\langle1,1|S_z|1,1\rangle=-\langle1,-1|S_z|1,-1\rangle=\hbar. \tag{89}$$

Thus, if $\mathbf{b}\cdot\mathbf{S}$ has nonvanishing x and y components, Raman scattering induces electronic transitions with a spin flip of both electrons, but keeping the total spin of both electrons fixed. The only process with nonvanishing matrix elements is between the energy bands of the ortho state with the selection rules:

$$\Delta P=0,\quad\Delta K_1=0,\quad\Delta K_2=0,\quad\Delta p=0,\quad\Delta j=0,\quad\Delta J=0,\quad\Delta S=0,\quad\Delta M_S=0,\pm1. \tag{90}$$

In an AB geometry these state are degenerate and hence no inelastic light scattering process is involved. But for the case where the magnetic field is present in the ring, i.e., it penetrates the electron path, this degeneracy becomes lifted. For a narrow-width ring the eigenenergies of two interacting electrons, Eq. (49), are shifted by the Zeeman term: $E_{j,J,M_s}(\Phi,2)=E_{j,J}(\Phi,2)+g^*\mu_B B M_s$. Assuming, e.g., $\mathbf{b}\cdot\mathbf{S}=S_x$, the differential scattering cross section reads

$$\left.\frac{d^2\sigma}{d\omega d\Omega}\right|_{\mathcal{M}_0}=\left(\frac{\varepsilon(\omega_S)}{\varepsilon(\omega_L)}\right)^{1/2}\frac{\omega_S}{\omega_L}r_0^2\frac{D_{\mathrm{SO}}^2}{2Z_{\mathrm{odd}}^{\mathrm{rot}}(T,\Phi)}\sum_{J=\pm1,\pm3,\pm5,\ldots}\left\{e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}\sum_{M_{S'}=\pm1}\delta(\omega-\omega_{M_{S'}0})\right.$$

$$\left.+e^{-[\mathcal{E}_J^{\mathrm{rot}}(\Phi)+g^*\mu_BB]/(k_BT)}\sum_{M_{S'}=0,-1}\delta(\omega-\omega_{M_{S'}M_S})+e^{-[\mathcal{E}_J^{\mathrm{rot}}(\Phi)-g^*\mu_BB]/(k_BT)}\sum_{M_{S'}=0,+1}\delta(\omega-\omega_{M_{S'}M_S})\right\},\tag{91}$$

where $\omega_{M_{S'}M_S}=2g^*\mu_BB(M_{S'}-M_S)/\hbar$.

We consider now the inelastic light scattering processes induced by $\mathcal{M}_2(\mathbf{q})$. It becomes obvious that $\mathcal{M}_2(\mathbf{q})$ induces the same transitions within the spin space as $\mathcal{M}_0(\mathbf{q})$, but now associated by transitions in the orbital Hilbert space. Assuming, for example, again $\mathbf{b}\cdot\mathbf{S}=S_x$, the Raman scattering matrix element is easily calculated to be

$$\begin{aligned}\langle P',K_1',K_2';p',j';J';S',M_S'|\mathcal{M}_2^{(a)}(\mathbf{q})|P,K_1,K_2;p,j;J;S,M_S\rangle={}&C_{\mathrm{depol}}^{(a)}\delta_{P',P}\delta_{K_1',K_1}\delta_{K_2',K_2}(\delta_{p',p+1/2}+\delta_{p',p-1/2})\delta_{j',j+(2\ell+1)}\\&\times[q_x(\delta_{J',J+1}+\delta_{J',J-1})-iq_y(\delta_{J',J+1}-\delta_{J',J-1})]\\&\times\delta_{S'S}(\delta_{M_S',M_S}+\delta_{M_S',M_S\pm1}),\end{aligned}\tag{92}$$

with

$$C_{\mathrm{depol}}^{(a)}=-(-1)^{J+\ell}\frac{RD_{\mathrm{SO}}}{\sqrt{2}}\sqrt{\frac{2^jj!}{2^{j'}j'!}}\left(\frac{\xi}{2}\right)^{2\ell+1}\exp\left[-\left(\frac{\xi}{4}\right)^2\right]L_j^{2\ell+1}\left(\frac{\xi}{8}\right)^2.\tag{93}$$

This gives the selection rules

$$\Delta P=0,\quad\Delta K_1=0,\quad\Delta K_2=0,\quad\Delta p=\pm\tfrac{1}{2},\quad\Delta j=2\ell+1,\quad\Delta J=\pm1,\quad\Delta S=0,\quad\Delta M_S=0,\pm1,\tag{94}$$

i.e., results in transitions within the energy bands of the ortho state (*ortho-ortho transitions*). This is true because the matrix elements of **S** between para states vanish. The scattering tensor $\mathcal{M}_2^{(a)}(\mathbf{q})$ gives rise to the differential scattering cross section

$$\left.\frac{d^2\sigma}{d\omega d\Omega}\right|_{\mathcal{M}_2^{(a)}}=\left(\frac{\varepsilon(\omega_S)}{\varepsilon(\omega_L)}\right)^{1/2}\frac{\omega_S}{\omega_L}r_0^2\frac{|C_{\mathrm{depol}}^{(a)}|^2}{Z_{\mathrm{odd}}^{\mathrm{rot}}(T,\Phi)}(q_x^2+q_y^2)\sum_{\ell=0,1,2}\sum_{J=\pm1,\pm3,\pm5,\ldots}\{e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[\delta(\omega-\omega_{lJ}^+)+\delta(\omega-\omega_{lJ}^-)]\},\tag{95}$$

where $\omega_{\ell J}^{\pm}$ is given by Eq. (65). Equation (95) is valid for an Aharonov-Bohm geometry. The calculation for the case of a magnetic field penetrating the electron paths in the ring is straightforward. The electronic transitions induced by $\mathcal{M}_2^{(a)}$ are the same as depicted in Fig. 3.

To calculate the matrix elements of $\mathcal{M}_2^{(b)}$ we find the only nonvanishing matrix elements of $\Delta\mathbf{S}$ to be

$$\langle0,0|\Delta S_x|1,1\rangle=\langle1,1|\Delta S_x|0,0\rangle=-\frac{\hbar}{\sqrt{2}},\quad\langle0,0|\Delta S_x|1,-1\rangle=\langle1,-1|\Delta S_x|0,0\rangle=\frac{\hbar}{\sqrt{2}},$$

$$\langle0,0|\Delta S_y|1,1\rangle=-\langle1,1|\Delta S_y|0,0\rangle=-\frac{i\hbar}{\sqrt{2}},\quad\langle0,0|\Delta S_y|1,-1\rangle=-\langle1,-1|\Delta S_y|0,0\rangle=-\frac{i\hbar}{\sqrt{2}},\quad\langle0,0|\Delta S_z|1,0\rangle=\langle1,0|\Delta S_z|0,0\rangle=\hbar.\tag{96}$$

Then, we obtain assuming as an example $\mathbf{b}\cdot\Delta\mathbf{S}=\Delta S_x$:

$$\begin{aligned}\langle P',K_1',K_2';p',j';J';S',M_S'|\mathcal{M}_2^{(b)}(\mathbf{q})|P,K_1,K_2;p,j;J;S,M_S\rangle={}&C_{\mathrm{depol}}^{(a)}\delta_{P',P}\delta_{K_1',K_1}\delta_{K_2',K_2}(\delta_{p',p+1/2}+\delta_{p',p-1/2})\delta_{j',j+(2\ell+1)}\\&\times[iq_x(\delta_{J',J+1}-\delta_{J',J-1})+q_y(\delta_{J',J+1}+\delta_{J',J-1})]\\&\times\delta_{S',S\pm1}(\delta_{M_S',M_S}-\delta_{M_S',M_S\pm1}),\end{aligned}\tag{97}$$

with

$$C_{\mathrm{depol}}^{(b)}=-(-1)^{J+\ell}\frac{RD_{\mathrm{SO}}}{\sqrt{2}}\sqrt{\frac{2^jj!}{2^{j'}j'!}}\left(\frac{\xi}{2}\right)^{2\ell+1}\exp\left[-\left(\frac{\xi}{4}\right)^2\right]L_j^{2\ell}\left(\frac{\xi}{8}\right)^2.\tag{98}$$

This gives the selection rules

$$\Delta P=0,\quad \Delta K_1=0,\quad \Delta K_2=0,\quad \Delta p=\pm\tfrac{1}{2},\quad \Delta j=2\ell,\quad \Delta J=\pm1,\quad \Delta S=\pm1,\quad \Delta M_S=0,\pm1. \tag{99}$$

Hence, the last process provides spin-flip transitions between the energy bands of the para and ortho state and vice versa (*para-ortho transitions*). The differential scattering cross section reads

$$\left.\frac{d^2\sigma}{d\omega d\Omega}\right|_{\mathcal{M}_2^{(b)}}=\left(\frac{\varepsilon(\omega_S)}{\varepsilon(\omega_L)}\right)^{1/2}\frac{\omega_S}{\omega_L}r_0^2\frac{|C_{\mathrm{depol}}^{(b)}|^2}{Z_{\mathrm{odd}}^{\mathrm{rot}}(T,\Phi)}(q_x^2+q_y^2)\sum_{\ell=0,1,2}\left\{\sum_{J=0,\pm2,\pm4,\ldots}g_{\mathrm{para}}\,e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[\delta(\omega-\tilde{\omega}_{\ell J}^{+})+\delta(\omega-\tilde{\omega}_{\ell J}^{-})]\right.$$

$$\left.+\sum_{J=\pm1,\pm3,\pm5,\ldots}g_{\mathrm{ortho}}\,e^{-\mathcal{E}_J^{\mathrm{rot}}(\Phi)/(k_BT)}[\delta(\omega-\tilde{\omega}_{\ell J}^{+})+\delta(\omega-\tilde{\omega}_{\ell J}^{-})]\right\}, \tag{100}$$

where

$$\tilde{\omega}_{\ell J}^{\pm}=2\ell\Omega+\frac{\hbar}{4m_eR^2}\left(1\pm2J\pm4\frac{\Phi}{\Phi_0}\right). \tag{101}$$

The first term describes the transition from the energy bands of the para state to those of the ortho state and the second term describes the opposite transitions. In Fig. 4 we have plotted the lowest transitions between the energy bands of para and ortho and ortho and para states. The total scattering cross section for depolarized Raman scattering is just the sum of Eqs. (95) and (100).

If φ is the ray angle measured from the z axis of the incident wave and γ that of the scattered wave, the scattering wave vector is given by

$$\mathbf{q}=(k_L\sin\varphi-k_S\sin\gamma)\mathbf{e}_x-(k_L\cos\varphi-k_S\cos\gamma)\mathbf{e}_z, \tag{102}$$

where $k_L=|\mathbf{k}_L|$ and $k_S=|\mathbf{k}_S|$. This gives for a *tilted backscattering geometry* $(\gamma=-\varphi)$: $\mathbf{q}=(k_L+k_S)\sin\varphi\,\mathbf{e}_x-(kL+kS)\cos\varphi\,\mathbf{e}z$ and for a *conventional backscattering geometry* $(\gamma=0°)$: $\mathbf{q}=-(k_L+k_S)\mathbf{e}_z$. Further, a *tilted right-angle geometry* $(\gamma+\varphi=90°)$ gives $\mathbf{q}=(k_L\sin\varphi-k_S\cos\varphi)\,\mathbf{e}_x-(k_L\cos\varphi+k_S\sin\varphi)\,\mathbf{e}_z$ and for the *conventional right-angle geometry* $(\varphi=0°)$: $\mathbf{q}=-k_S\,\mathbf{e}_x-k_L\mathbf{e}_z$ is valid. Then it follows for the *polarized spectra*: (i) if $\mathbf{e}_L=\mathbf{e}_y$ we have $\mathbf{e}_S=\mathbf{e}_y$ or (ii) if $\mathbf{e}_L=\cos\varphi\,\mathbf{e}_x+\sin\varphi\,\mathbf{e}_z$ we have $\mathbf{e}_S=-\cos\gamma\,\mathbf{e}_x+\sin\gamma\,\mathbf{e}_z$; and for the *depolarized spectra*: (i) if $\mathbf{e}_L=\mathbf{e}_y$ we have $\mathbf{e}_S=-\cos\gamma\,\mathbf{e}_x+\sin\gamma\,\mathbf{e}_z$ or (ii) if $\mathbf{e}_L=\cos\varphi\,\mathbf{e}_x+\sin\varphi\,\mathbf{e}_z$ we have $\mathbf{e}_S=\mathbf{e}_y$.

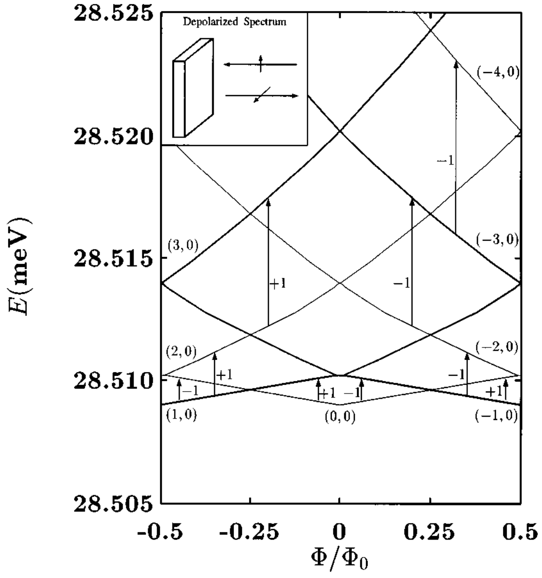

FIG. 4. The lowest Raman scattering transitions of the depolarized spectrum induced by $\mathcal{M}_2^{(b)}(\mathbf{q})$ between the energy bands $E_{j,J}(\Phi,2)$ of the para state (heavy solid lines) and that of the ortho state (thin solid lines) represented in dependence on the magnetic flux Φ for a two-electron ring. Denotations and parameters are the same as in Fig. 1.

Let us give the Raman scattering matrix element $\langle f|\mathcal{M}(\mathbf{q})|i\rangle$ for the two conventional geometries: (i) *backscattering geometry*: $\mathbf{k}_L=(0,0,-k_L)$, and $\mathbf{k}_S=(0,0,k_S)$, and (ii) *right-angle geometry*: $\mathbf{k}_L=(0,0,-k_L)$ and $\mathbf{k}_S=(k_S,0,0)$. In both cases, we consider *polarized* $(\mathbf{e}_L\|\mathbf{e}_S)$ and *depolarized* $(\mathbf{e}_L\perp\mathbf{e}_S)$ Raman scattering.

D. Backscattering geometry

1. Polarized spectrum

In this case, which has the usual notation $z(y,y)\overline{z}$, we have $\mathbf{e}_L=\mathbf{e}_y$ and $\mathbf{e}_S=\mathbf{e}_y$, i.e., $\mathbf{e}_L\cdot\mathbf{e}_S^*=1$, $\mathbf{e}_L\times\mathbf{e}_S^*=0$, and $\mathbf{q}=-(k_L+k_S)\mathbf{e}_z$, from where it follows that $\mathcal{M}_0(\mathbf{q})=2D_{\mathrm{SO}}$. Hence, $\mathcal{M}_0(\mathbf{q})$ gives no inelastic scattering process. Further, we have $\mathcal{M}_1(\mathbf{q})=(i/e)D_{\mathrm{SO}}(k_L+k_S)\mathbf{e}_z\cdot\mathbf{d}=0$ because $\mathbf{d}=(d_x,d_y,0)$, and also $\mathcal{M}_2(\mathbf{q})=0$.

2. Depolarized spectrum

In this case the scattering notation is $z(x,y)\overline{z}$, and we have $\mathbf{e}_L=\mathbf{e}_y$ and $\mathbf{e}_S=-\mathbf{e}_x$, i.e., $\mathbf{e}_L\cdot\mathbf{e}_S^*=0$, $\mathbf{e}_L\times\mathbf{e}_S^*=\mathbf{e}_z$, and $\mathbf{q}=-(k_L+k_S)\mathbf{e}_z$. Thus, it follows that $\mathcal{M}_0(\mathbf{q})=(2i/\hbar)D_{\mathrm{SO}}S_z$, where $S_z=(\hbar/2)(\sigma_{z1}+\sigma_{z2})$ is the z component of the total spin operator. Because of the relation $S_z|S,M_S\rangle=\hbar M_S|S,M_S\rangle$, the operator $\mathcal{M}_0(\mathbf{q})$ does not result in an inelastic light scattering process. Further, it is obvious that in this case $\mathcal{M}_1(\mathbf{q})=0$ and $\mathcal{M}_2(\mathbf{q})=0$ is valid.

E. Right-angle geometry

1. Polarized spectrum

For the right-angle geometry, i.e., $\overline{x(y,y)z}$, we have $\mathbf{e}_L=\mathbf{e}_y$ and $\mathbf{e}_S=\mathbf{e}_y$, i.e., $\mathbf{e}_L\cdot\mathbf{e}_S^*=1$, $\mathbf{e}_L\times\mathbf{e}_S^*=0$, and $\mathbf{q}=-k_S\mathbf{e}_x-k_L\mathbf{e}_z$. Then it follows that $\mathcal{M}_0(\mathbf{q})=2D_{\mathrm{SO}}$, $\mathcal{M}_1(\mathbf{q})$ of Eq. (84) with $q_x=-k_S$ and $q_y=0$, and

$\mathcal{M}_2(\mathbf{q})=0$. Note that for the considered case the scattering matrix element is quite similar to that considered above in Sec. IV B with the only distinction being that instead of $q_y \neq 0$ we have $q_z \neq 0$. From this, we obtain the Raman scattering matrix element of Eq. (85) and the scattering cross section of Eq. (87).

2. Depolarized spectrum

In this scattering geometry, where $\overline{x}(z,y)\overline{z}$, is valid we have $\mathbf{e}_L=\mathbf{e}_y$ and $\mathbf{e}_S=\mathbf{e}_z$, i.e., $\mathbf{e}_L \cdot \mathbf{e}_S^*=0$, $\mathbf{e}_L \times \mathbf{e}_S^* = -\mathbf{e}_z$, and $\mathbf{q}=-k_S\,\mathbf{e}_x - k_L \mathbf{e}_z$, we derive $\mathcal{M}_0(\mathbf{q})=-(2i/\hbar)D_{\mathrm{SO}}S_z$, which gives no inelastic scattering process, $\mathcal{M}_1(\mathbf{q})=0$ and $\mathcal{M}_2(\mathbf{q})$ of Eq. (88) with $q_x=-k_S$ and $q_y=0$. The Raman scattering matrix element is given by Eqs. (92) and (97) and the differential scattering cross section by Eqs. (95) and (100).

V. CONCLUSION

We have shown that an ensemble of QR's in the presence of an Aharonov-Bohm flux with two interacting electrons in each ring having the energy spectrum

$$E_{K_1,K_2,j,J}(\Phi)=\mathcal{E}_{K_1}+\mathcal{E}_{K_2}+\frac{e^2}{8\pi\varepsilon_0\varepsilon_s R}+\mathcal{E}_j^{\mathrm{osc}} + \frac{\hbar^2}{4m_e R^2}\left(J+2\frac{\Phi}{\Phi_0}\right)^2$$

absorbs FIR light with the dipole selection rules $\Delta J=\pm 1$ and $\Delta j=\pm 1, \pm 3, \pm 5, \ldots$. Thus, transitions take place between the energy bands of the para state and between those of the ortho state. Further, when applied to a Raman scattering experiment, the scattering processes are accompanied by the same selection rules if the polarized spectrum is measured. In the depolarized inelastic light scattering process two types of transitions contribute: (i) between the energy bands of the ortho state (ortho-ortho transitions) with $\Delta J=\pm 1$, $\Delta j=\pm 1, \pm 3, \pm 5, \ldots$, and $\Delta M_S=0,\pm 1$ and (ii) between the energy bands of the para and ortho state (para-ortho transitions) and vice versa with $\Delta J=\pm 1$, $\Delta j=0,\pm 2,\pm 4,\ldots$, $\Delta S=\pm 1$, and $\Delta M_S=0,\pm 1$. The peaks arise as doublets according to the transitions with $\Delta J=1$ and -1. The spacing between these two peaks depends on the magnetic flux.

Absorption and Raman scattering on the one hand, and persistent currents on the other hand, are manifestations of the same band structure of QR's. The persistent current as a function of Φ gives a physical representation of the derivative of the energy bands with respect to Φ and is a continuous characteristic of the band structure. As distinct from that, the absorption and the Raman scattering select only some points in the Φ scale, at which the transitions are allowed according to the selection rules discussed in this paper and hence form discrete characteristics of the same band structure. Thus, they are complementary to each other.

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[*] Permanent address: Department of Theoretical Physics, State University of Moldova, Strada Mateevici 60, 277009 Kishinev, Republic of Moldova.

[†] Permanent address: Institute of Semiconductor Physics, Siberian Branch of the Russian Academy of Sciences, Prospekt Lavrentyeva 13, 630090 Novosibirsk, Russia.

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