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Mean-field theory of strongly nonlinear random composites: Strong power-law nonlinearity and scaling behavior

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The effective response of random media consisting of two different kinds of strongly nonlinear materials with strong power-law nonlinearity is studied. Each component satisfies current density and electric-field relation of the form $\mathbf{J}=\chi|\mathbf{E}|^{\beta}\mathbf{E}$. A simple self-consistent mean-field theory, which leads to a simple way in determining the average local electric field in each constituent, is introduced. Each component is assumed to have a conductivity depending on the averaged local electric field. The averaged local electric field is then determined self-consistently. Numerical simulations of the system are carried out on random nonlinear resistor networks. Theoretical results are compared with simulation data, and excellent agreements are found. Results are also compared with the Hashin-Shtrikman lower bound proposed by Ponte Castaneda *et al.* [Phys. Rev. B **46**, 4387 (1992)]. It is found that the present theory, at small contrasts of χ between the two components, gives a result identical to that of Ponte Castaneda *et al.* up to second order of the contrast. The crossover and scaling behavior of the effective response near the percolation threshold as suggested by the present theory are discussed and demonstrated. [S0163-1829(96)02129-7]

I. INTRODUCTION

Much attention has been attracted to problems concerning the simultaneous presence of disorderness and nonlinearity.[1,2] A typical system in macroscopically inhomogeneous media is that of a random mixture of two kinds of materials with different nonlinear $\mathbf{J}-\mathbf{E}$ relations, where $\mathbf{J}$ is the current density and $\mathbf{E}$ is the electric field.[3] For strongly nonlinear composites, components with $\mathbf{J}-\mathbf{E}$ relations of the form $\mathbf{J}=\chi|\mathbf{E}|^{\beta}\mathbf{E}$ are randomly mixed. By suitably tuning the system parameters such as the volume fraction and the nonlinear susceptibility χ of the constituents, it is possible to control the effective nonlinear response of the nonlinear mixture.

Straley and Kenkel[4] studied the percolating effects in systems in which a strongly nonlinear conductor is mixed with an insulator. Using standard methods, such as scaling arguments and real-space renormalization group, in statistical physics, they studied the critical behavior of the effective response near the percolation threshold. They also established the uniqueness of the solution to the problem.[4,5] Meir and co-workers[6] carried out similar studies using series analysis. For systems consisting of two kinds of materials with the *same* nonlinearity but different conductivities, Blumenfeld and Bergman[7] developed a perturbative method, based on the difference of the conductivities to calculate the effective response. Yu and co-workers[8] have developed a variational method to calculate the local electric field in nonlinear components and an effective-medium approximation. Numerical simulations on random nonlinear networks have also been performed.[9] However, such calculations are often laborious, and may sometimes lead to unphysical results such as the recent report of the dependence of the percolation threshold on the nonlinearity of the problem.[10]

Recently, Hui and co-workers[11,12] have developed a simple mean-field theory, similar in idea to the mean-field theory developed in weakly nonlinear composites,[13,14] for the effective nonlinear response in strongly nonlinearity composites consisting of components with cubic nonlinearity. In this theory, each component is treated as a conductor with a conductivity depending on the local field squared. The local fields are then determined self-consistently. Results are found to be in good agreement with published simulation data. A similar idea has also been successfully applied to random mixtures of linear and nonlinear conductors.[12]

The aim of the present work is to generalize the self-consistent mean-field theory for the effective response in strongly nonlinear composites to systems with strong power-law nonlinear components of arbitrary nonlinear exponents. To establish the validity of our theory, results are compared with detailed simulation data and excellent agreements are found given the simplicity of the suggested theory. The theory also satisfies the Hashin-Shtrikman (HS) lower bound obtained by Ponte Castaneda, de Botton, and Li.[15] The crossover and scaling behavior of the effective response near the percolation threshold as suggested by the present theory are demonstrated and data collapse is evident.

It should be pointed out that Bergman[16] has developed an elegant and fully self-consistent effective-medium approximation (EMA) for composites with power-law nonlinear $\mathbf{J}-\mathbf{E}$ relation. In that paper, the author reformulated the EMA in a manner such that the averaging procedure that must be used becomes unambiguous, and proposed scaling form for the effective response. The percolation threshold was found to be dependent on the nonlinear exponent β of the constituents. Being a geometrical property, p_c should re-

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late simply to the connectivity of the system. The variational approach of Lee, Yuen, and Yu[10] suffers from similar difficulty, and is complementary to Bergman's work in that they determined the amplitudes of the scaling functions. The present work gives percolation thresholds which are independent of the nonlinear exponents.

The plan of the paper is as follows. Our mean-field theory for arbitrary nonlinear exponent is presented in Sec. II. In Sec. III, the effective-medium approximation for treating linear random composites is applied within the context of our mean-field theory as a specific example and results are compared to simulation data. In Sec. IV, results are compared with the HS lower bound for small contrasts of the components. In Sec. V, the crossover and scaling behavior near the percolation threshold are discussed in detail for the cases of cubic nonlinearity and arbitrary nonlinear exponents. In Sec. VI, results are summarized and possible extensions of the present work are discussed.

II. FORMALISM

Consider a macroscopically inhomogeneous medium consisting of strongly nonlinear conductors with arbitrary nonlinearity. The current density **J** and electric field **E** at position **x** within the medium are related by

$$\mathbf{J}(\mathbf{x})=\chi(\mathbf{x})|E(\mathbf{x})|^{\beta}\mathbf{E}(\mathbf{x}), \tag{1}$$

where $\chi(\mathbf{x})$ describes the strength of the nonlinear response and will be termed nonlinear susceptibility, and β the nonlinear exponent. An external field $\mathbf{E}_0$ is applied to the composite, which is assumed to occupy a volume V enclosed by surface S. Such an external field can be applied by imposing suitable boundary condition on the surface S. The effective response is defined in such a way that if the medium *were* uniform, the $\mathbf{J}-\mathbf{E}$ response is identical, on the average, to that of the inhomogeneous media. The effective nonlinear response is characterized by an effective nonlinear susceptibility χ_e defined by

$$\langle \mathbf{J}(\mathbf{x})\rangle=\chi_e|E_0|^{\beta}\mathbf{E}_0, \tag{2}$$

where $\langle\cdots\rangle$ denotes a volume average. In general, χ_e will depend on the physical properties and concentration of the constituents, as well as the microgeometry within the composite.

Consider a two-component composite consisting of a nonlinear component a with concentration p and nonlinear component b with concentration $1-p$. The positional-dependent $\chi(\mathbf{x})$ in Eq. (1) takes on the value χ_a (χ_b) for **x** in regions occupied by material a (b). As some standard methods are well-developed for treating linear inhomogeneous media, a mean-field theory is set up to make these methods applicable in strongly nonlinear composites as well. We approximate the $\mathbf{J}-\mathbf{E}$ relation for **x** in regions occupied by component a as

$$\mathbf{J}(\mathbf{x})=\chi_a\langle|E(\mathbf{x})|^{\beta}\rangle_a\mathbf{E}(\mathbf{x})\equiv\sigma_a\mathbf{E}(\mathbf{x}), \tag{3}$$

where $\langle|E|^{\beta}\rangle_a$ is the volume average of $|E|^{\beta}$ taken over regions occupied by component a and will be determined self-consistently. Similarly, for **x** in regions occupied by component b,

$$\mathbf{J}(\mathbf{x})=\chi_b\langle|E(\mathbf{x})|^{\beta}\rangle_b\mathbf{E}(\mathbf{x})\equiv\sigma_b\mathbf{E}(\mathbf{x}), \tag{4}$$

where $\langle|E|^{\beta}\rangle_b$ is now an average over volume occupied by component b. We can then treat the components as *linear*, but with field-dependent conductivities σ_a and σ_b. The basic idea of our mean-field theory is to apply standard technique[3] in handling linear random composites to obtain an expression for the effective response. The averaged electric fields in regions occupied by components a and b are then determined by imposing self-consistency conditions.

The effective response σ_e of a composite consisting of components characterized by σ_a and σ_b can be represented by

$$\sigma_e=\sigma_e(\sigma_a,\sigma_b,p), \tag{5}$$

where, in general, the explicit form of σ_e depends on the microgeometry of the composite. Subsequently, the averaged local fields $\langle E^2\rangle_a$ and $\langle E^2\rangle_b$ can be determined self-consistently from σ_e. Treating the components as linear conductors, the average local fields and the external field are related by[3,14,17]

$$\langle E^2\rangle_a=\frac{1}{p}\frac{\partial\sigma_e}{\partial\sigma_a}E_0^2, \tag{6}$$

and

$$\langle E^2\rangle_b=\frac{1}{1-p}\frac{\partial\sigma_e}{\partial\sigma_b}E_0^2. \tag{7}$$

These expressions follow from the formula of σ_e in terms of the field distribution[3,17]

$$\sigma_e=\frac{1}{V|E_0|^2}\int_V\sigma(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2d^3x, \tag{8}$$

where $\sigma(\mathbf{x})$ takes on σ_a (σ_b) for **x** in regions occupied by component a (b). For the strongly nonlinear composite, a local field-dependent conductivity is introduced, hence $\sigma_a=\chi_a\langle|E|^{\beta}\rangle_a$, $\sigma_b=\chi_b\langle|E|^{\beta}\rangle_b$, and $\sigma_e=\chi_e|E_0|^{\beta}$. By applying the following decoupling approximations developed in treating weakly nonlinear composite,[13,14,17] $\langle|E|^{\beta}\rangle_{a,b}\approx\langle|E|^2\rangle_{a,b}^{\beta/2}$, the right-hand sides of Eqs. (6) and (7) are thus functions of $\langle|E|^2\rangle_a$ and $\langle|E|^2\rangle_b$. Equations (6) and (7) form a set of coupled self-consistent equations, which can be solved for $\langle E^2\rangle_a/E_0^2$ and $\langle E^2\rangle_b/E_0^2$. Substituting the values for the averaged local fields into Eq. (5) gives the effective nonlinear response χ_e. Equation (5) together with Eqs. (6) and (7) thus provide a straightforward way to estimate the effective response in strong power-law nonlinear composites with arbitrary nonlinear exponents. The decoupling scheme amounts to neglecting the fluctuations of the electric field in the same component at different places in the composite. Although such fluctuations may be important near the percolation threshold, the decoupling scheme is believed to give a simple and direct way for estimating the effective response away from the percolation threshold. Near the threshold, it is expected that the mean-field theory captures the qualitative features, but predicts the critical behavior inaccurately. The validity of our approximation is best seen from the good agreements with simulation results except in the vicinity of the threshold.

III. EFFECTIVE-MEDIUM APPROXIMATION AND NUMERICAL SIMULATION

The effective-medium approximation (EMA) is one of the most useful approximations in handling linear random composites.[3] Within EMA, σ_e is given by the solution to the expression

$$p\,\frac{\sigma_a-\sigma_e}{\sigma_a+g\sigma_e}+(1-p)\,\frac{\sigma_b-\sigma_e}{\sigma_b+g\sigma_e}=0, \tag{9}$$

where g is a geometrical factor given by $g=1$ and 2 in two (2D) and three dimensions, respectively. Thus, in 2D, we obtain after substituting back the χ's into the σ's,

$$\chi_e=\frac{\sigma_e}{E_0^\beta}=\frac{1}{2E_0^\beta}\left\{(1-2p)(\chi_b\langle|E|^2\rangle_b^{\beta/2}-\chi_a\langle|E|^2\rangle_a^{\beta/2})+\sqrt{(1-2p)^2(\chi_b\langle|E|^2\rangle_b^{\beta/2}-\chi_a\langle|E|^2\rangle_a^{\beta/2})^2+4\chi_a\chi_b\langle|E|^2\rangle_a^{\beta/2}\langle|E|^2\rangle_b^{\beta/2}}\right\}. \tag{10}$$

From Eqs. (6) and (7), we have

$$\langle E^2\rangle_a=\frac{E_0^2}{2p}\left\{(2p-1)+\frac{2\chi_b\langle|E|^2\rangle_b^{\beta/2}-(1-2p)^2(\chi_b\langle|E|^2\rangle_b^{\beta/2}-\chi_a\langle|E|^2\rangle_a^{\beta/2})}{\sqrt{(1-2p)^2(\chi_b\langle|E|^2\rangle_b^{\beta/2}-\chi_a\langle|E|^2\rangle_a^{\beta/2})^2+4\chi_a\chi_b\langle|E|^2\rangle_a^{\beta/2}\langle|E|^2\rangle_b^{\beta/2}}}\right\} \tag{11}$$

and

$$\langle E^2\rangle_b=\frac{E_0^2}{2(1-p)}\left\{(1-2p)+\frac{2\chi_a\langle|E|^2\rangle_a^{\beta/2}+(1-2p)^2(\chi_b\langle|E|^2\rangle_b^{\beta/2}-\chi_a\langle|E|^2\rangle_a^{\beta/2})}{\sqrt{(1-2p)^2(\chi_b\langle|E|^2\rangle_b^{\beta/2}-\chi_a\langle|E|^2\rangle_a^{\beta/2})^2+4\chi_a\chi_b\langle|E|^2\rangle_a^{\beta/2}\langle|E|^2\rangle_b^{\beta/2}}}\right\}. \tag{12}$$

Equations (11) and (12) can be solved simultaneously for $\langle|E|^2\rangle_a^{\beta/2}/|E_0|^\beta$ and $\langle|E|^2\rangle_b^{\beta/2}/|E_0|^\beta$ for given values of p and χ_a/χ_b. Substituting the results back into Eq. (10) gives the effective nonlinear response χ_e.

To establish the validity of our mean-field theory for arbitrary nonlinearity, numerical simulations for nonlinear conductance networks were performed for various β. The simulations were carried out using the algorithm of Lee and Yu.[9] Consider a two-dimensional square network with a fraction p of inclusion conductors with nonlinear susceptibility χ_a, and a fraction $1-p$ of host conductors with nonlinear susceptibility χ_b. The nonlinear conductors satisfy the following current-voltage (I-V) relationship

$$I=\chi|V|^\beta V, \tag{13}$$

where χ takes on χ_a or χ_b. The effective response of the random network is calculated and then compared with the predictions by the mean-field theory. The effective response of the random network is defined in a way that if the resistor network *were* homogeneous, it would be represented by a full network of identical conductors with I-V relations of the form

$$I=\chi_e|V|^\beta V, \tag{14}$$

where χ_e is the effective nonlinear susceptibility of the equivalent homogeneous network. A unit voltage is applied across the top and bottom bars of the network. Kirchhoff equations for the voltages at each node are solved self-consistently and convergence is achieved. In the present study, χ_a and χ_b are kept finite so that we are away from the percolation threshold. Figure 1 shows the simulation data together with results of the mean-field theory coupled with EMA for β=1,2,3,4, respectively. For each case, four different values of the ratios of the nonlinear susceptibilities χ_b/χ_a of the components are considered. Each data point is obtained by averaging 200 different configurations corresponding to the same concentration p. In general, the agreement is good, especially given the simplicity of the theory. The agreement is better than results obtained by variational calculations.[18] Deviations from the simulation data become apparent only for high contrast between the two components, and this is expected in view of the fact that the mean-field approximation is more valid when the fluctuations of the local field are small. It should be noted that even for the highest contrast $\chi_b/\chi_a=1000$ studied, the agreement between theory and simulations is reasonably good. The reason for the good agreement is that the present theory determines $\langle E^2\rangle$ rather than $\langle E\rangle$ as done in previous works based on the variational method.[9,18]

IV. COMPARISON WITH THE HASHIN-SHTRIKMAN BOUND

It is instructive to compare the present theory with the Hashin-Shtrikman lower bound proposed by Ponte Castaneda, de Botton, and Li.[15] Consider an isotropic random composite in arbitrary dimension d. Within EMA, the factor g in Eq. (9) is given by $g=d-1$. For convenience, we set $\chi_a=1$, $\chi_b=1+\delta$ with $\delta\to0$, and work out the small contrast limit. From Eqs. (6), (7) and (9), we obtain

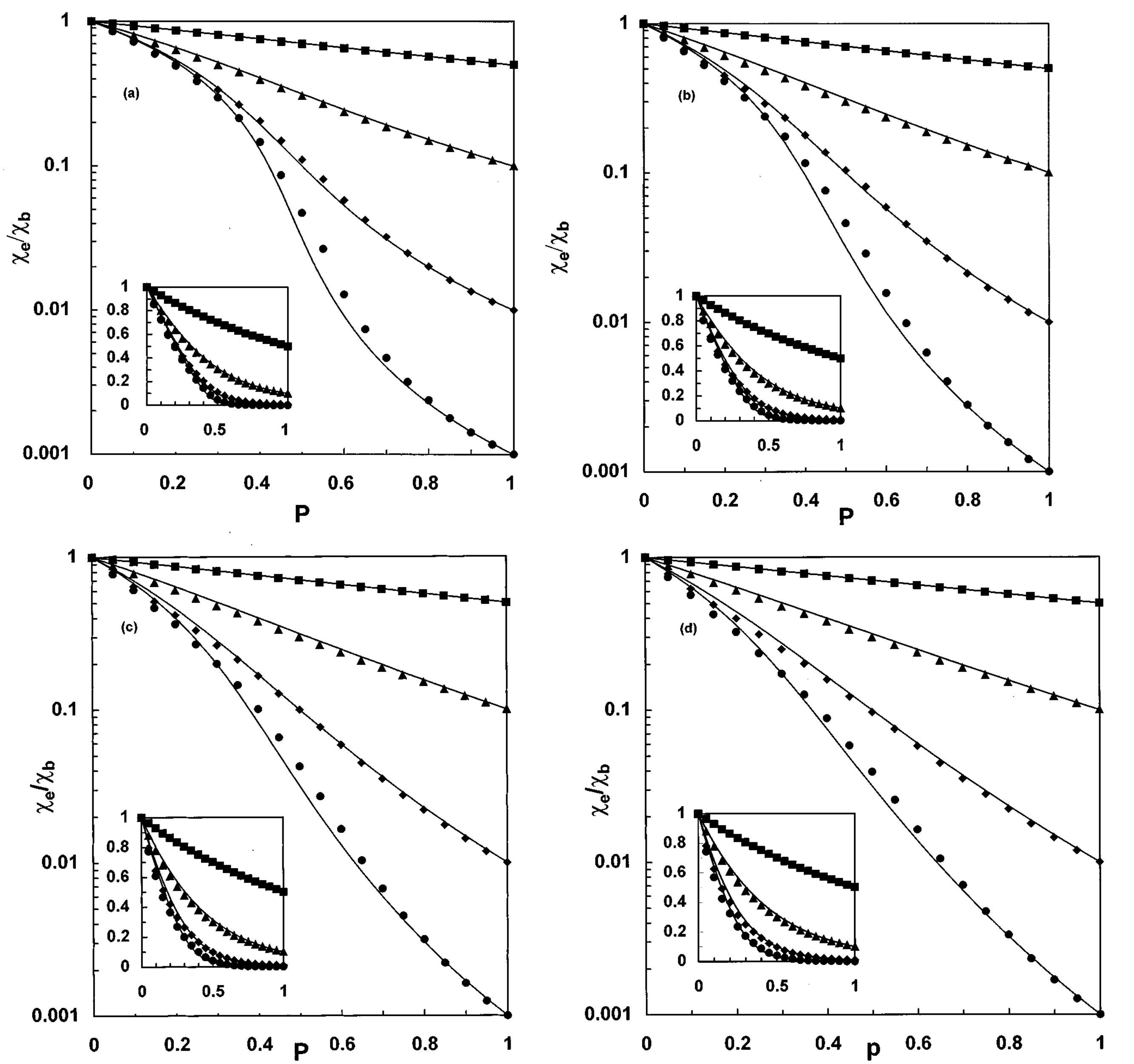

FIG. 1. The effective nonlinear response χ_e/χ_b is plotted on a semilogarithmic scale as a function of the concentration of the nonlinear component a for different values of χ_b/χ_a. The symbols represent simulation data for $\chi_b/\chi_a=2$ (squares), 10 (triangles), 100 (diamonds), and 1000 (dots). Each data point is an average over 200 different configurations. The solid lines are the corresponding results of the present mean-field theory. The insets shows the simulation data and results of the mean-field theory on linear scales. Four different values of β are investigated: (a) $\beta=1$, (b) $\beta=2$, (c) $\beta=3$, (d) $\beta=4$.

$$\chi_e=1+(1-p)\delta-\frac{(2+\beta)p(1-p)\delta^2}{2(\beta+d)}-\frac{(2+\beta)(2\beta-\beta^2+3d-3\beta d-3d^2-4\beta p-\beta^2 p-6dp+3d^2 p)p(1-p)\delta^3}{6(\beta+d)^3}+\mathscr{O}[\delta]^4. \tag{15}$$

From Eq. (24) in Ref. 15, we obtain the series expansion of the HS lower bound as

$$\chi_e=1+(1-p)\delta-\frac{(2+\beta)p(1-p)\delta^2}{2(\beta+d)}-\frac{(2+\beta)(2\beta-\beta^2-3\beta d-4\beta p-\beta^2 p-3dp)p(1-p)\delta^3}{6(\beta+d)^3}+\mathscr{O}[\delta]^4. \tag{16}$$

Thus, to second order in δ, the present theory gives results which coincide with the HS lower bound. If the term of order δ^3 is considered, χ_e obtained within our theory is above the HS lower bound in the small contrast limit. As the contrast increases, the HS bound becomes progressively weak.[18] Hence, our predictions should also be well above the HS bound as the contrast increases. The agreement is also based on the fact that the present theory determines $\langle E^2\rangle$ more accurately and the treatment by Ponte Castaneda, de Botton, and Li is also based on the determination of $\langle E^2\rangle$ variationally.

V. SCALING BEHAVIOR FOR CUBIC NONLINEARITY

A. Cubic nonlinearity

It is interesting to study the scaling behavior of the effective response near the percolation threshold p_c within the mean-field theory. Consider the nonlinear conductors satisfying the $\mathbf{J}-\mathbf{E}$ relation of the form

$$\mathbf{J}=\chi_{a(b)}\langle|E|^2\rangle_{a(b)}\mathbf{E}. \tag{17}$$

To simplify notations, we define $A=\langle|E|^2\rangle_a/|E_0|^2$ and $B=\langle|E|^2\rangle_b/|E_0|^2$. Then the self-consistency equations read

$$A=\frac{1}{p}\left\{-\frac{1}{2}(1-2p)+\frac{1}{2}\frac{2\chi_bB-(1-2p)^2(\chi_bB-\chi_aA)}{\sqrt{(1-2p)^2(\chi_bB-\chi_aA)^2+4\chi_a\chi_bAB}}\right\}, \tag{18}$$

and

$$B=\frac{1}{1-p}\left\{\frac{1}{2}(1-2p)+\frac{1}{2}\frac{2\chi_aA+(1-2p)^2(\chi_bB-\chi_aA)}{\sqrt{(1-2p)^2(\chi_bB-\chi_aA)^2+4\chi_a\chi_bAB}}\right\}. \tag{19}$$

From Eq. (10), we have the following expression for χ_e:

$$\chi_e=\tfrac{1}{2}(1-2p)(\chi_bB-\chi_aA)+\tfrac{1}{2}\sqrt{(1-2p)^2(\chi_bB-\chi_aA)^2+4\chi_a\chi_bAB}. \tag{20}$$

Consider the normal-conductor–insulator (N/I) case in which the nonlinear susceptibility χ_b of component b is small compared with χ_a. We define $h=\chi_b/\chi_a$ and $\Delta p=p-p_c$, then from Eqs. (10), (11), and (12), we obtain

$$(1+2\Delta p)A=2\Delta p+\frac{Bh+2(A-Bh)\Delta p^2}{\sqrt{ABh+(A-Bh)^2\Delta p^2}}, \tag{21}$$

$$(1-2\Delta p)B=-2\Delta p+\frac{A-2(A-Bh)\Delta p^2}{\sqrt{ABh+(A-Bh)^2\Delta p^2}}, \tag{22}$$

and the effective nonlinear response as

$$\chi_e/\chi_a=\Delta p(A-Bh)+\sqrt{ABh+(A-Bh)^2\Delta p^2}, \tag{23}$$

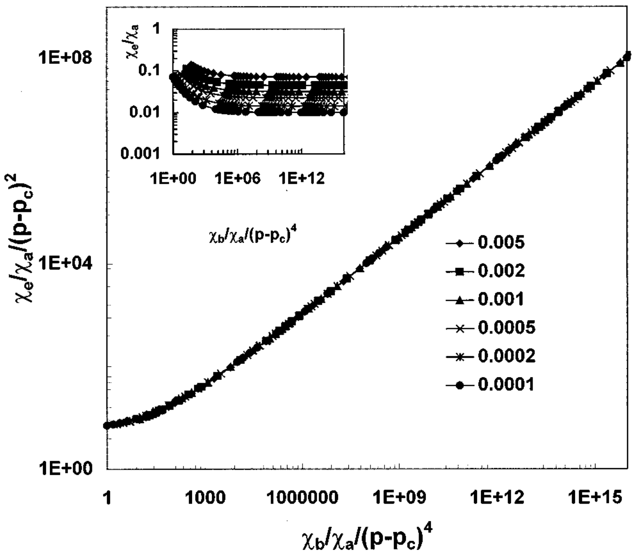

FIG. 2. The rescaled effective response $\chi_e/(\chi_a\Delta p^2)$ is plotted on a log-log scale as a function of the scaling variable, $z=\chi_b/(\chi_a\Delta p^4)$, for various values of χ_b/χ_a for the case of cubic nonlinearity. The inset shows the same data obtained by the present mean-field theory on a log-log plot of χ_e/χ_a as a function of z for different values of χ_b/χ_a.

where $p_c=1/2$ is the percolation threshold in 2D within EMA. As $h\to 0$, for $p\geqslant p_c$ we found that the local fields squared behave as

$$A\sim\Delta p, \tag{24}$$

and

$$B\sim\Delta p^{-1}. \tag{25}$$

Hence we propose that the local fields satisfy the following scaling form:

$$A=\Delta p\Phi_a\left(\frac{h}{\Delta p^{\phi}}\right), \tag{26}$$

and

$$B=\Delta p^{-1}\Phi_b\left(\frac{h}{\Delta p^{\phi}}\right), \tag{27}$$

where ϕ is a crossover exponent, and Φ_a and Φ_b are some scaling functions that can be obtained from the self-consistency equations. For small arguments, the leading terms in these scaling functions are constants. Trying $\phi=4$ and writing $z=h/\Delta p^4$, then by substituting Eqs. (26) and (27) into Eqs. (23) we have for $p\geqslant p_c$

$$\frac{\chi_e/\chi_a}{\Delta p^2}=\Phi_e(z), \tag{28}$$

where $\Phi_e=\Phi_a+\sqrt{\Phi_a^2+z\Phi_a\Phi_b}$.

Equation (28) suggests that, within the mean-field theory, proper rescaling of the variables χ_e, h, and Δp will give a universal curve. The inset in Fig. 2 gives results of the present mean-field theory calculated for different values of h and Δp. To demenstrate the scaling behavior, the same re-

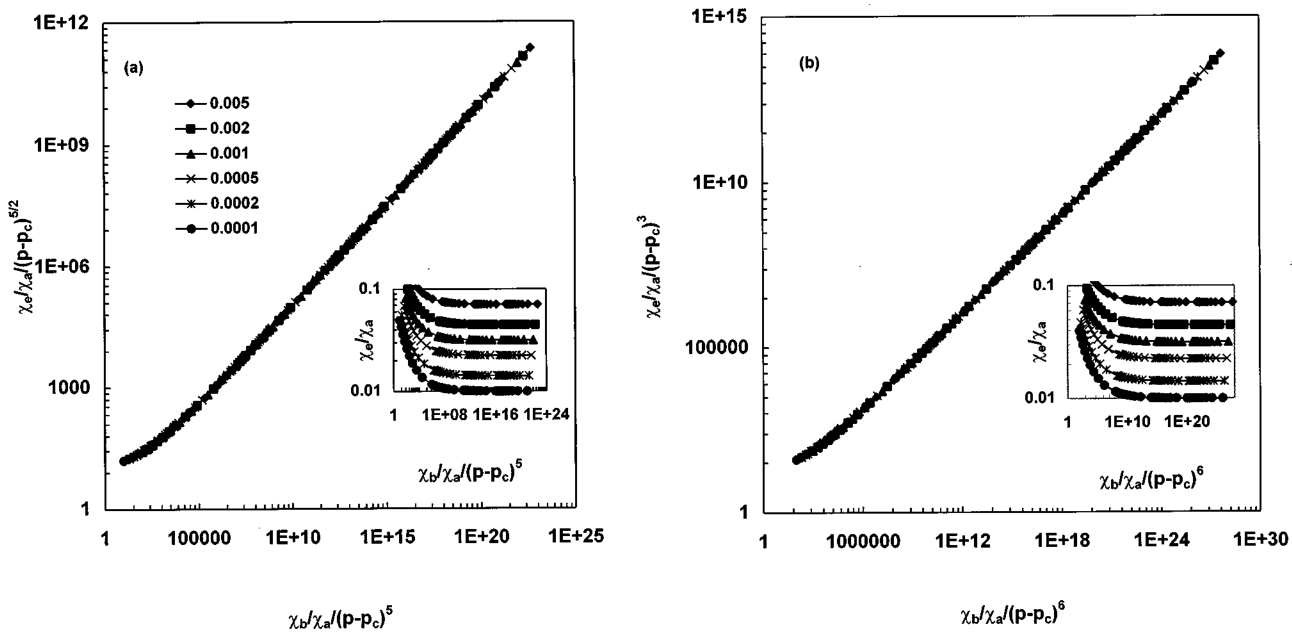

FIG. 3. The rescaled effective response $\chi_e/(\chi_a \Delta p^{\beta/2+1})$ is plotted on a log-log scale as a function of the scaling variable, $z=\chi_b/(\chi_a \Delta p^{\beta+2})$ for various values of χ_b/χ_a. The inset shows the log-log plot of χ_e/χ_a as obtained by the present mean-field theory as a function of z for different values of χ_b/χ_a. Two values of β are investigated: (a) $\beta=3$, (b) $\beta=4$.

sults are plotted with $\chi_e/(\chi_a \Delta p^2)$ against the scaling variable $z=\chi_b/(\chi_a \Delta p^4)$, and the data fall onto the same curve. Similar scaling arguments have been given within the context of an effective-medium approximation based on the variational method,[19] which however suffers from the unphysical result of a β-dependent percolation threshold. It should be pointed out that similar scaling variable has been proposed previously in Ref. 16.

B. Arbitrary nonlinear exponents

In order to generalize the above discussion to arbitrary nonlinear exponent, we define $A=\langle |E|^2\rangle_a^{\beta/2}/|E_0|^\beta$ and $B=\langle |E|^2\rangle_b^{\beta/2}/|E_0|^\beta$. Then Eqs. (11) and (12) become

$$(1+2\Delta p)^{\beta/2}A=\left(2\Delta p+\frac{Bh+2(A-Bh)\Delta p^2}{\sqrt{ABh+(A-Bh)^2\Delta p^2}}\right)^{\beta/2}, \tag{29}$$

$$(1-2\Delta p)^{\beta/2}B=\left(-2\Delta p+\frac{A-2(A-Bh)\Delta p^2}{\sqrt{ABh+(A-Bh)^2\Delta p^2}}\right)^{\beta/2}. \tag{30}$$

For $h\to 0$, we have

$$A\sim \Delta p^{\beta/2} \tag{31}$$

and

$$B\sim \Delta p^{-\beta/2}. \tag{32}$$

For cubic nonlinearity, $\beta=2$ and Eqs. (29) and (30) reduce back to Eqs. (21) and (22). Deducing from the case of cubic nonlinearity, we propose that the local fields satisfy the following scaling form with crossover exponent $\phi=\beta+2$:

$$A=(\Delta p)^{\beta/2}\Phi_a\left(\frac{h}{\Delta p^{\beta+2}}\right), \tag{33}$$

and

$$B=\Delta p^{-\beta/2}\Phi_b\left(\frac{h}{\Delta p^{\beta+2}}\right), \tag{34}$$

For small arguments, the leading terms in these scaling functions are constants. Writing $z=h/\Delta p^{\beta+2}$ and substituting Eqs. (33) and (34) into Eq. (23), we have for $p\geqslant p_c$:

$$\frac{\chi_e/\chi_a}{\Delta p^{\beta/2+1}}=\Phi_e(z), \tag{35}$$

where $\Phi_e=\Phi_a+\sqrt{\Phi_a^2+z\Phi_a\Phi_b}$. To demonstrate data collapse, we carry out similar calculations as in the case of cubic nonlinearity. Figure 3 shows data collapse for $\beta=3$ and 4, respectively. Again, data collapse is evident.

Similar consideration can be carried out for the superconductor–normal-conductor (S/N) case in which $\chi_a\to\infty$. Analytic asymptotic expressions for the local fields and effective nonlinearity can also be extracted. Define $\chi_b/\chi_a=h^{-1}$, and write $p_c-p=\Delta p$, then from Eqs. (11) and (12), we obtain

$$(1-2\Delta p)^{\beta/2}A=\left(-2\Delta p+\frac{B+2(Ah-B)\Delta p^2}{\sqrt{ABh+(Ah-B)^2\Delta p^2}}\right)^{\beta/2}, \tag{36}$$

$$(1+2\Delta p)^{\beta/2}B=\left(2\Delta p+\frac{Ah+2(B-Ah)\Delta p^2}{\sqrt{ABh+(B-Ah)^2\Delta p^2}}\right)^{\beta/2}. \tag{37}$$

As $h\to 0$, we have

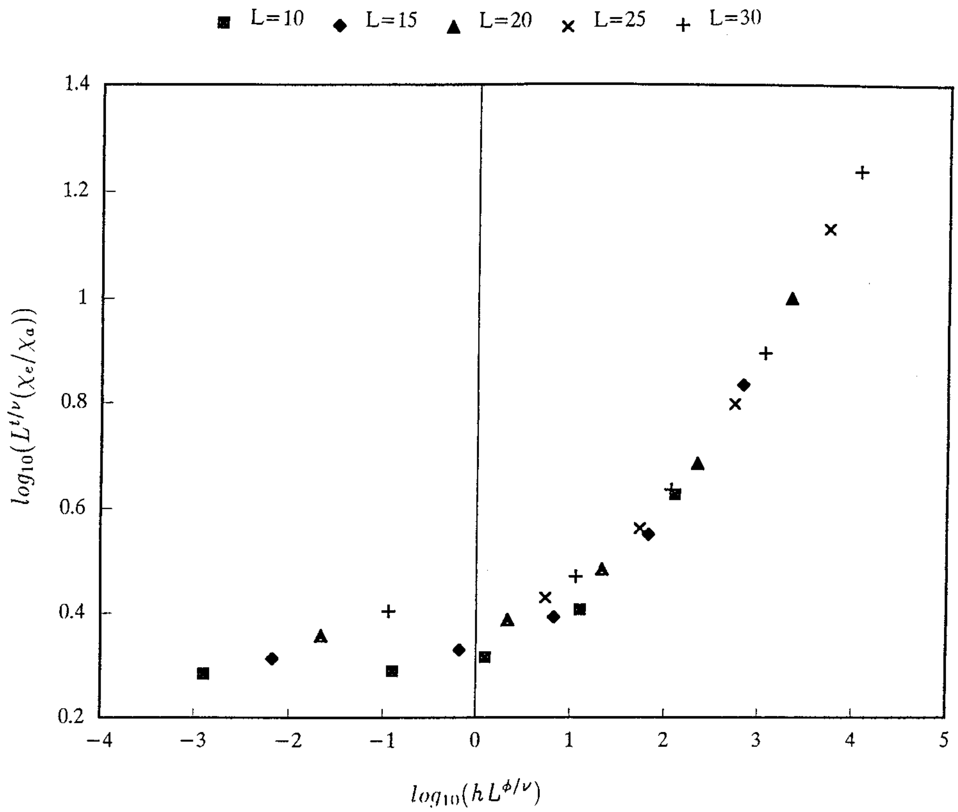

FIG. 4. Log-log plot of $L^{t/\nu}(\chi_e/\chi_a)$ versus $hL^{\phi/\nu}$ using data obtained from numerical simulations on systems with different sets of values of conductance ratio h and system size L. Data points corresponding to the same L are represented by the same symbol.

$$A \sim \Delta p^{-\beta/2}, \tag{38}$$

$$B \sim \Delta p^{\beta/2}. \tag{39}$$

Hence we propose

$$A = (\Delta p)^{-\beta/2} \Psi_a \left(\frac{h}{\Delta p^{\beta+2}} \right), \tag{40}$$

and

$$B = \Delta p^{\beta/2} \Psi_b \left(\frac{h}{\Delta p^{\beta+2}} \right), \tag{41}$$

where Ψ_a and Ψ_b are some scaling functions. The leading terms in these scaling functions are constants. Writing $z = h/\Delta p^{\beta+2}$, and substituting Eqs. (40) and (41) into Eqs. (23), we have for $p \leqslant p_c$

$$(\chi_e/\chi_a) \Delta p^{\beta/2+1} = \Psi_e(z), \tag{42}$$

where $\Psi_e = z^{-1}[\Psi_b + \sqrt{\Psi_b^2 + z\Psi_a\Psi_b}]$. In view of the symmetry between the N/I and S/N limits, data could be collapsed for the S/N limit when they are properly rescaled.

VI. DISCUSSION

While Figs. 2 and 3 represent data generated using the present mean-field theory, we can also test the scaling properties using numerical data from simulations on random resistor networks. In order to do so, we invoke the idea of finite-size scaling. For the case of a normal-conductor–insulator (N/I) mixture as discussed in the previous section, the correlation length ξ is related to $(p-p_c)$ by $(p-p_c) \sim \xi^{-1/\nu}$. For a small network or for p closed to p_c, its correlation length is well approximated by the size of the system L. Hence, for $\beta = 2$, we have

$$L^{t/\nu}(\chi_e/\chi_a) = \Phi_e(hL^{\phi/\nu}), \tag{43}$$

where within the present theory, $t = 2$ and $\phi = 4$. It is expected that this scaling form will be satisfied by numerical data, but probably with different values of the exponents. It is typical that mean-field exponents will not properly scale numerical data. To test the validity of the scaling form, we perform numerical calculations right at the percolation threshold for 2D square random nonlinear resistor networks with different sizes L and conductance ratios h. We consider samples in which the first component is spanning between busbars to which a potential difference is applied. An open boundary condition is applied in the other directions. We start with $h = 0.01$ and reduce its value down to 10^{-6}. For each value of h, L is varied from 10 to 30 in increments of 5. Averaging over 2000 configurations for each set of h and L gives results with reasonably good statistics. Since we expect that mean-field theory does not predict the exponents correctly, we use values of exponents, $t/\phi = 0.37$ and $\phi/\nu = 4.1$, previously reported in Ref. 18 to rescale our data. Figure 4 shows a plot of the variable $L^{t/\nu}(\chi_e/\chi_a)$ against $hL^{\phi/\nu}$ on a log-log scale. These data are calculated using numerical results from simulations based on different values of h and L. Data from different system sizes are plotted as different symbols. It is evident that data collapse on to a universal curve.

In summary, we have proposed a mean-field theory for the effective response in strongly nonlinear random composites with components of power-law **J-E** relations with arbitrary nonlinear exponent. Results are compared with simulation data and good agreements are found. Our results are consistent with the lower bound obtained in Ref. 15. The crossover and scaling behavior are discussed within the context of the mean-field theory together with EMA. Data collapse implied by the present theory are discussed and demonstrated. To our knowledge, the present theory gives the simplest, and yet reliable way for estimating the effective response in strong power-law nonlinear composites of arbitrary nonlinear exponent.

Although our discussion has been focused on media composed of two components with the same nonlinearity exponent β, the present theory can be readily generalized to treat composites made up of components with different power dependences on a local electric field. The effective nonlinear response of the composite will then have a power dependence on external electric field applied which varies with the strength of the electric field, as well as the composition of the mixture. The effective response, for example, in a random composites consisting of linear conductor and strongly nonlinear conductor[20] can be readily described by the present theory.[12]

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