Effect of van Hove singularities on a spin liquid

L. B. Ioffe
Department of Physics, Rutgers University, Piscataway, New Jersey 08855
and Landau Institute for Theoretical Physics, Kosygina 2, Moscow

A. J. Millis
Bell Laboratories, Lucent Technologies, 600 Mountain Avenue, Murray Hill, New Jersey 07974
(Received 13 November 1995)

We determine the properties and leading instabilities of a spin liquid with a Fermi surface passing near a van Hove singularity. Our study is motivated by recent photoemission experiments on high-T_c cuprates in which it is found that for the optimally doped material the experimental Fermi surface passes near a van Hove singularity, while for underdoped materials, a pseudogap in the electron spectral function is formed in the vicinity of the van Hove point. We show theoretically that proximity to the van Hove singularity suppresses the inelastic scattering due to the gauge field and permits the formation of a d-wave RVB state in which the gap exists only near the van Hove points while finite regions of the Fermi surface remain gapless. This d-wave pairing provides a natural explanation of the pseudogap observed in photoemission. We also discuss the relation of the pseudogap observed in the spectral function to the pseudogaps observed in the magnetic susceptibility. [S0163-1829(96)04729-7]

I. INTRODUCTION

In this paper we report results of a theoretical study of a ''spin liquid'' with a van Hove singularity near the Fermi surface. By the term ''spin liquid'' we mean a liquid of charge 0 spin-1/2 fermions filling a large Fermi sea and coupled by a singular gauge-field interaction[1,2] and by additional nonsingular interactions. We determine exactly the effect of the van Hove singularity on the fermion gauge-field physics and treat the additional interactions by a ''leading logarithm'' renormalization-group analysis.

The problem of a spin liquid with a van Hove singularity is of interest on experimental and theoretical grounds. The experimental motivation for describing the high-T_c superconductors as spin liquids has been discussed at length elsewhere.[3] Recent angle-resolved photoemission experiments[4] also suggest that van Hove points are important. The qualitative doping dependence expected theoretically for noninteracting electrons is sketched in Fig. 1. For heavily overdoped samples (dotted curve) the Fermi surface is closed and electron-like. In a noninteracting model, the Fermi surface would grow as electrons are added, until it reached the van Hove points. A Fermi surface passing through the van Hove points, M, is shown as the thick dashed line in Fig. 1. The experimental result[4] is that for optimally doped materials the Fermi surface is very close to the van Hove singularity. It is not clear exactly where the Fermi surface is near van Hove points because quasiparticle peak is very broad there; most likely it intersects zone boundary at a point slightly displaced from M in the X direction. For instance, the data on slighly overdoped Bi-Sr-Ca-Cu-O presented in Fig. 4 of Ref. 4 seem to show that the Fermi surface crosses the M-X line rather than the M-Γ line. However, the area enclosed by the Fermi surface in this figure corresponds to a doping of 7%, whereas optimal doping is approximately 20%. Because the states at $(\pi,0)$ point shown in Fig. 2 of Ref. 4 are so close to the Fermi level, we suspect that there are uncertainties in the experimental determination of the Fermi surface, and that the true Fermi surface is much closer to the $(\pi,0)$ point than shown in Fig. 4 of Ref. 4.

More significantly, as more electrons are added the Fermi surface does not continue to evolve in the manner expected

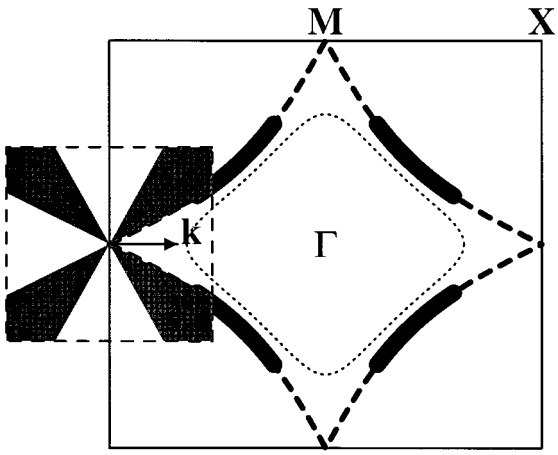

FIG. 1. Large square: Brillouin zone for fermions. Heavy dashed line: Fermi surface of noninteracting electrons with spectrum (2) and $t' = -0.3t$ passing through the van Hove point. Dotted line: Fermi surface of heavily overdoped material. Heavy arcs: region where Fermi surface was observed in photoemission experiment in underdoped Bi-Sr-Ca-Cu-O. Dashed square: inset showing phase space for gauge-field fluctuations. Gauge-field fluctuations with momentum **k** in one of the four shaded regions are described by the conventional overdamped propagator; fluctuations with **k** outside of these regions are much larger because there is no part of the Fermi-surface tangent to **k**.

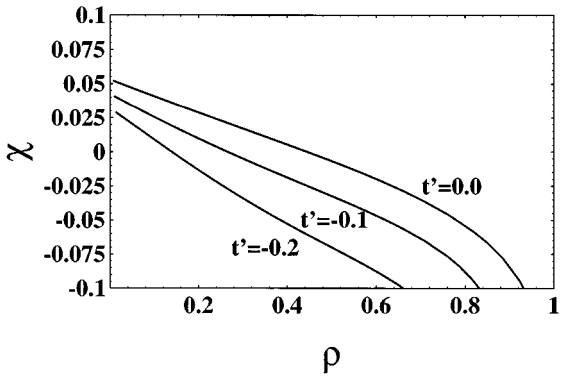

FIG. 2. Diamagnetic susceptibility of noninteracting Fermi gas with spectrum given by Eq. (2) as a function of filling, ρ, for $t=1$ and several values of the next-neighbor hopping, t'.

from the band theory. Instead, the material develops a gap near the van Hove points. States in the vicinity of the van Hove points are pushed away from the Fermi surface. For ''underdoped'' materials no states with energies near the chemical potential are observed near the zone edges. States are observed near the chemical potential only in the disconnected regions along the zone diagonal shown as solid arcs in Fig. 1. It is important to note that the experimental claim is not that the material develops a ''hole pocket'' Fermi surface centered around the point labeled X in Fig. 1 but that no states at all are observed near the chemical potential anywhere along the lines connecting M to Γ and M to X. The existence and consequences of this ''non-Luttinger'' Fermi surface require theoretical explanation.

The subject of non-Luttinger Fermi surfaces has attracted substantial theoretical attention. The general approach has been to start with fermions with a large (Luttinger) Fermi surface and then to invoke a physical mechanism to open a ''pseudogap'' which eliminates part or all of the Fermi surface. Three classes of mechanisms have been extensively considered: (i) quasi-long-ranged antiferromagnetic spin fluctuations, (ii) the ''d-wave RVB'' state, and (iii) the ''staggered flux phase.'' None has proven completely satisfactory; we discuss each in turn.

The logic behind the antiferromagnetic spin fluctuations approach is that static antiferromagnetic order at wave vector $\mathbf{Q}$ leads to Bragg scattering at $\mathbf{Q}$ which may open a gap over all or part of the Fermi surface. Schrieffer and co-workers have argued that sufficiently slowly varying antiferromagnetic fluctuations with a sufficiently long correlation length may also lead, if not to a gap, at least to a rather strong suppression of the Fermi surface density of states.[5] A theoretical difficulty with this picture is that magnetic instabilities of a wide variety of models have been investigated; pseudogaps have only been found in parameter regimes leading to long-ranged order at $T=0$.[6,7] The essential reason is that quasistatic (i.e., frequency much less than temperature) spin fluctuations are required for pseudogap formation.[5–7] In metallic and superconducting high-T_c materials, the spin fluctuations observed by NMR have characteristic frequency scale of order T or greater[8] (corresponding to no long-range order at $T=0$) and are not sufficient to open a pseudogap in the models which have been considered.

An alternative model for the origin of the pseudogap is the ''d-wave RVB'' model. This is a mathematical expression of Anderson's original insight[9] that in models involving a strong on-site repulsion and a density near one electron per site, singlet pairing and antiferromagnetism do not differ much in their short-ranged correlations and energies. To implement Anderson's idea one assumes that ''spin-charge separation'' occurs, and that the spin degrees of freedom are described by charge 0 fermionic ''spinons.'' These fill a Fermi sea with a large (Luttinger) Fermi surface. This Fermi sea of spinons may be derived in a mean-field theory, ''the uniform RVB phase,''[2,10] of the t-J model. The t-J model is believed by many[3] but not all[11,12] authors to contain the essential physics of high-T_c superconductors, and the uniform RVB phase is believed to be the most physically appropriate starting point, at least for materials near the optimal doping. To understand pseudogap formation one then considers instabilities of the uniform RVB state, which are due to residual interactions neglected in the mean-field theory. Before now calculations have been based on ''generic'' Fermi surfaces without van Hove points and have considered instabilities to antiferromagnetism, to d-wave pairing, and to a staggered flux phase. The antiferromagnetic instability has been considered and found not to lead to a pseudogap for essentially the same reason as in the Fermi-liquid case.[13,14]

The staggered flux phase involves the appearance of circulating spin currents. This entails spontaneous breaking of time-reversal symmetry which has been observed not to occur in cuprates.[15] For this phenomenological reason the staggered flux phase has been discarded.

The d-wave RVB state naturally leads to the formation of a pseudogap, and the resulting phenomenology provides an attractive scenario for the cuprates.[16] The d-wave RVB state may be viewed as arising from a pairing instability of the uniform RVB state; the resulting theory is very similar to that arising from conventional superconducting pairing. One important difference is that because the pairing involves chargeless ''spinons'' it does not lead to superfluidity or indeed any other observable which could serve as an order parameter. For this reason fluctuation corrections convert the pairing transition to a smooth crossover.

A difficulty with this scenario has been pointed out by Ubbens and Lee.[17] Their results, we believe, are most simply interpreted as saying that the spinon-gauge-field interaction produces a very short inelastic lifetime for the spinons. This inelastic scattering is so strongly pairbreaking that it completely suppresses the d-wave pairing instability. Of course, a first-order transition to a paired state would be possible,[17] but is not observed.

Very recently, a model with an SU(2)$\times$ SU(2) symmetry has been considered.[18] This symmetry is, in principle, broken down to SU(2) only at any nonzero doping, but the symmetry breaking is argued not to be important. In this model a staggered flux instability occurs which does not break time-reversal symmetry and can be transformed using an operation in the SU(2)$\times$ SU(2) group to a d-wave pairing state. The effects of gauge fluctuations on this state have not yet been determined. On the mean-field level it leads to a phenomenology very similar to the one we shall derive in this paper.

All of the previously discussed theoretical calculations

were based on ''generic'' models which did not contain van Hove singularities. In this paper we show that when the effects of proximity to van Hove singularities are properly included, the theoretical predictions are very substantially altered. Most notably, near Van Hove points the inelastic scattering is suppressed, permitting the formation of a d-wave paired state with a gap which is nonzero only near van Hove points. This d-wave state should be reconsidered as a possible explanation for the pseudogap.

Near a van Hove singularity the fermion density of states diverges, so that even arbitrarily weak interactions can produce large effects. This line of thinking has led to a large literature devoted to analyzing the implications of van Hove singularities in Fermi-liquid models.[19–24] We have already mentioned the strong phenomenological evidence for regarding high-T_c materials as spin liquids.[3] An additional difficulty with Fermi-liquid-based treatments is that the singularities due to the van Hove points are cut off by coherent hopping in third dimension. The bare value of this hopping may be determined from band-structure calculations and is not small.[25] Evidently in the actual materials the interplane coupling is renormalized down to a very small value in Bi-Sr-Ca-Cu-O and underdoped $YBa_2Cu_3O_{7-x}$ as seen in c-axis penetration depth[26] and resistivity[27] measurements. These measurements have been shown by Anderson and Ren[28] to imply that there is no coherent interplane hopping. This absence of coherent c-axis hopping cannot be understood in a Fermi-liquid picture but follows naturally in several non-Fermi-liquid pictures[11] including the spin-liquid model studied here. Although the absence of coherent hopping in the spin liquid is known to many workers a derivation has apparently not been published. For completeness we give it in the Appendix. Finally, we mention an alternative explanation of the pseudogap[29] based on the k-diagonal interplane pairing mechanism of Anderson and Chakravarty.[11]

The remainder of this paper is organized as follows. In Sec. II we formulate and solve a model of a spin liquid with a van Hove singularity coupled to a gauge field. In Sec. III we employ a leading logarithm approximation to determine the effects of residual nonsingular interactions. Section IV is a conclusion which summarizes the approximations employed, results obtained and consequences for photoemission and other physical properties. The Appendix gives derivations of some results used in the body of the paper.

II. SPINON-GAUGE MODEL

The spin-liquid state we shall start from is the ''uniform RVB'' state. The Hamiltonian is

$$H=\sum_{p\sigma}\epsilon_p c^\dagger_{p\sigma}c_{p\sigma}+\frac{1}{2}\sum_{pk\sigma}\vec{a}_k c^\dagger_{p+k/2,\sigma}[\vec{v}_{p+k/2}+\vec{v}_{p-k/2}]c_{p-k/2,\sigma}$$
$$+W\sum_p c^\dagger_{p_1\sigma}c_{p_2\sigma}c^\dagger_{p_3\sigma}c_{p_4\sigma}\delta(\Sigma p_i)+\frac{1}{4g_0^2}f^2_{\mu\nu}. \quad (1)$$

This Hamiltonian is the usual one (see, e.g., Refs. 1 and 30), however we have written the fermion-gauge-field coupling in its general form. Here $\vec{v}(p)=\partial\epsilon_p/\partial\vec{p}$ and $f_{\mu\nu}=\epsilon_{\mu\nu}\partial_\mu a_\nu$. It represents the low-energy spin degrees of freedom of the t-J model. Low energy in this context means energies less than J, which in high-T_c materials is about 0.15 eV. The coefficient g_0^{-2} contains the contributions to the gauge-field stiffness from the higher-energy spin degrees of freedom which were integrated out in the derivation of Eq. (1). It may be expressed in terms of the high-energy part of a six-spin-correlation function and is of the order of J in magnitude; it is discussed further in the Appendix.

We assume the fermion spectrum is

$$\epsilon_p=-2t[\cos(p_x a)+\cos(p_y a)]$$
$$-4t'\cos(p_x a)\cos(p_y a)-\mu-4t'. \quad (2)$$

Here a is the lattice constant, $\mu+4t'$ is the chemical potential with $\mu=0$ corresponding to the van Hove point, t is a first-neighbor hopping due mostly to the superexchange J with an additional contribution coming from a band structure $t_{\rm band}$ renormalized by the hole density and the parameter t' is derived from further neighbor hopping in the underlying band structure also renormalized by the hole density; roughly, we expect as $\delta\to 0$

$$t=\Theta J+t_{\rm band}\delta, \quad (3)$$

$$t'=t'_{\rm band}\delta. \quad (4)$$

Here Θ is a number of the order of unity, in the large-N limit $\Theta=2/\pi^2$. It is believed that $t'<0$ in high-T_c superconductors.[25] Note that there is no coherent hopping of spinons in the third dimension; in the model such hopping cannot occur unless the band-structure $t_\perp \gtrsim t_{\rm band}$. The derivation of the spinon ϵ_p from the t-J model is outlined in the Appendix. In Eq. (1) interplane hopping is entirely neglected. Even if there is no coherent hopping, interplane coupling may be important. We return to this issue in the conclusion.

Near a van Hove point (e.g., $p_x=\pi, p_y=0$) we have

$$\epsilon_p=-u_0(p_x^2-\alpha^2 p_y^2) \quad (5)$$

with $u_0=(t+2t')a^2$ and $\alpha^2=(t-2t')/(t+2t')$. If $t'\neq 0$ then $\alpha^2\neq 1$ and the energy contour which passes through the van Hove points is not nested. An example of such a contour is shown as the solid line in Fig. 1. At the van Hove points the velocity vanishes, implying a diverging density of states and a vanishing of the fermion-gauge-field coupling.

We now consider the fermion-gauge-field interaction. Previous work[30] has shown that it is correct to analyze this interaction in two steps using a loop expansion controlled by the parameter N, the fermion spin degeneracy. In the physical problem $N=2$. It has been shown that results obtained at leading order in a $1/N$ expansion are not significantly changed at higher orders. First, one constructs the renormalized gauge-field propagator D by dressing the term $f^2_{\mu\nu}/g_0^2$ by the fermion transverse current-current polarizability $\Pi=\int vvGG$. Second, one uses this to compute the fermion self-energy. For a closed Fermi surface far from van Hove points, $\pi(\omega,\vec{k})=p_0|\omega|/\pi|k|+\chi k^2$. For a given direction of $\vec{k}$, the dissipative term $|\omega|/|k|$ comes from fermions near the points on the Fermi surface which are tangent to $\vec{k}$; p_0 is the curvature of the Fermi surface at these points and χ is the diamagnetic susceptibility of the fermions.

The presence of van Hove singularities leads to two effects. First, as can be seen from Fig. 1, there is a range of directions of $\vec{k}$ which are not tangent to any point on the Fermi surface. For these directions, the dissipative term in $\Pi(\omega,k)$, $\Pi_{\rm diss}$, is much smaller. To display the behavior of the dissipative term it is convenient to define coordinates $k_{\pm}=k_x\pm\alpha k_y$.

If

$$\frac{1-\alpha}{1+\alpha}<|k_+/k_-|<\frac{1+\alpha}{1-\alpha},$$

then

$$\Pi_{\rm diss}(\omega,k)=\frac{\alpha N|\omega|}{4\pi}\left(\frac{k^2}{k_x^2-\alpha^2k_y^2}+\frac{k^2}{k_y^2-\alpha^2k_x^2}\right), \tag{6}$$

while if $|k_+/k_-|$ is outside this range then

$$\Pi_{\rm diss}(\omega,k)=\frac{Np_0|\omega|}{2\pi|k|}. \tag{7}$$

Second, states in the vicinity of the van Hove point produce a negative, divergent contribution to χ, so

$$\chi=-\frac{Nu_0}{6\pi^2}\ln\epsilon_F/\Lambda+\chi_{\rm reg}. \tag{8}$$

Here the logarithm cutoff $\Lambda=\max(T,\mu,u_0k^2)$ and $\chi_{\rm reg}$ is the contribution from fermions far from the van Hove points. The total χ calculated for noninteracting fermions with $N=2$ moving in the band structure defined by Eq. (2) is shown in Fig. 2; as seen in this figure χ near half filling is negative and of small magnitude with a weak divergence near van Hove points as expected from Eq. (8). The full gauge propagator is thus

$$D(\omega,k)=\frac{1}{\Pi_{\rm diss}+(g_0^{-2}+\chi)k^2}. \tag{9}$$

The divergence of χ implies that in the vicinity of the van Hove singularity the uniform RVB state is unstable to a state of nonzero flux, at a temperature

$$T_{\rm flux}\sim\epsilon_F\exp-\left[\frac{6\pi^2(\chi_{\rm reg}+g_0^{-2})}{Nu_0}\right], \tag{10}$$

unless a different instability occurs first. States of nonzero flux break time-reversal invariance; as there is no evidence that this occurs in high-T_c materials we shall assume that $\chi_{\rm reg}+g_0^{-2}$ is sufficiently large and that $T_{\rm flux}$ is negligibly small. This assumption is consistent with the previously mentioned theoretical estimate $g_0^{-2}\sim J$ which is greater than the typical values of $\chi\sim0.1t\sim0.1J$ shown in Fig. 2. Also, a previous analysis of the T dependence of the resistivity at $T>100$ K predicted by Eq. (1) found $g_0^{-2}+\chi_{\rm reg}\approx500a^2K\sim Ja^2/3$. These estimates combined with the large numerical factor in Eq. (10) imply that the instability will occur only very close to the van Hove point, and only at very low temperature.

At scales of interest we may then neglect the logarithm and write

$$D(\omega,k)=\frac{1}{\Pi_{\rm diss}+k^2/g^2}, \tag{11}$$

where $g^{-2}=g_0^{-2}+\chi\approx500a^2K$. We now use this $D(\omega,k)$ to calculate the fermion self-energy. For fermions far from the van Hove points the calculation is identical to that given in previous work.[30] The kinematics of a scattering event imply that the fermion is scattered parallel to the Fermi surface. For these momenta one must use the $|\omega|/|k|$ form of $\Pi_{\rm diss}$, leading to

$$\Sigma_{\rm far}(\epsilon,\vec{p})=[\omega_0(\vec{p})]^{1/3}\epsilon^{2/3}. \tag{12}$$

Here

$$\omega_0(\vec{p})=\frac{v_F^3(\vec{p})g^4}{\pi^2p_0(2\sqrt{3})^3} \tag{13}$$

at leading order in N. The Fermi velocity $v_F(p)$ vanishes linearly as one approaches the van Hove point, implying $\Sigma_{\rm far}$ does also. Note also that $1/p_0$ vanishes for a nested Fermi surface; from Eq. (4) we conclude that $1/p_0\propto\delta$, so $\omega_0\propto\delta$.

For fermions near the van Hove point, scattering processes involving the other form of $\Pi_{\rm diss}$ become allowed. These lead to a contribution

$$\Sigma_{\rm near}(\epsilon,p)=-\frac{1}{2\pi^2}\left(\ln\frac{\epsilon_F}{\epsilon}\ln\frac{u}{\pi\chi}\right)[iA\epsilon-Bu(p_x^2-\alpha^2p_y^2)], \tag{14}$$

where the numerical coefficients $A=\ln(1+\alpha)/(1-\alpha)-1$, $B=3\ln(1+\alpha)/(1-\alpha)-1$. $\Sigma_{\rm near}(\epsilon,p)$ becomes important for $\epsilon<\epsilon^*$ where $(1/2\pi^2)\ln(\epsilon_F/\epsilon^*)\ln(u/\pi\chi)=1$; $\ln(u/\pi\chi)\approx1$, so $\ln(\epsilon_F/\epsilon^*)$ is within a factor of 2 of $\ln(\epsilon_F/T_{\rm flux})$, thus, for consistency, we must assume $\Sigma_{\rm near}(\epsilon,p)$ is negligible. The instability in $\Sigma_{\rm near}(\epsilon,p)$ reinforces that in χ. To study this instability more carefully we have written and solved coupled renormalization-group (RG) equations for u and g; we find that the more careful considerations do not change the estimate of the scale $T_{\rm flux}$ where these effects become important. Thus we assume

$$\Sigma(\epsilon,\vec{p})\cong\Sigma_{\rm far}(\epsilon,\vec{p}); \tag{15}$$

in other words the self-energy due to fermion-gauge field scattering is important far from the van Hove points and unimportant near them, provided that the instability towards the flux phase may be neglected.

III. EFFECT OF NONSINGULAR INTERACTIONS

We now turn to the residual, nonsingular interactions. It has already been shown that the short-range interaction between fermions far from van Hove points is renormalized to zero by the gauge field.[30,31] We thus need only to consider processes involving fermions near van Hove points. For these processes, vertex renormalizations due to the gauge field lead to the same logarithms we have already agreed to neglect. The theory we must consider is therefore of fermions near the van Hove points, with self-energy $\Sigma_{\rm far}(\epsilon,p)$, coupled by short-range interactions which may be parametrized by four constants: particle-particle and particle-hole

interactions with momentum transfer near zero or $G=(\pi,\pi)$.

The theory has some formal similarity to theories of fermions in one dimension. In one-dimensional models one considers only fermions near the Fermi surface of the left- and right-hand branches of the dispersion relation; here one considers only fermions near van Hove points. In one-dimensional (1D) models a weak-coupling leading logarithm approximation exists because some particle-hole and particle-particle propagators diverge logarithmically due to kinematics in one dimension. In the present model the divergent density of states at the van Hove singularity similarly leads to logarithmic divergences of susceptibilities.

The theory is more complicated than theories of fermions in one dimension because there are two small scales: $\omega_\alpha=[(1-\alpha)/(1+\alpha)]\epsilon_F$ which will be shown below to set the scale at which deviations from the perfect nesting become important, and ω_0 which sets the scale at which gauge-field effects become important. Both ω_α and ω_0 are proportional to δ as previously discussed; because they have the same doping dependence and ω_0 is small even at large doping due to the large value of the gauge stiffness g^{-2} in Eq. (13), we believe that $\omega_0<\omega_\alpha$ is the only relevant case.

The charges in the antiferromagnetic and d-wave pairing channels diverge. At scales larger than ω_α and ω_0 both channels have a $\ln^2(1/\epsilon)$ divergence. At scales less than ω_α the antiferromagnetic divergence becomes $\ln(1/\epsilon)$. At scales less than ω_0 the coefficient of the $\ln^2(1/\epsilon)$ in the superconductivity channel (and, if $\omega_0>\omega_\alpha$, in the antiferromagnetic channel) is reduced, because the gauge field produces a strong inelastic scattering in some regions of momentum space.

In the remainder of this section we present the results of the leading logarithm calculations. We assume throughout that at most one coupling becomes large. Especially in the regime of $\ln^2(1/\epsilon)$ renormalizations the problem of coupled charges is very involved and has been treated elsewhere.[20]

A. Antiferromagnetic instability

We proceed to construct leading logarithm RG equations. There is no logarithm in the small momentum transfer particle-hole susceptibility, so the corresponding charge is not renormalized. Moreover, in contrast to 1D, the small momentum transfer particle hole susceptibility has only a weak singularity as $\omega=vq$, so these processes can be neglected. The **G** momentum transfer particle-hole processes lead to logarithmic divergences. For a nested Fermi surface ($\alpha^2=1$) the two momentum integrals are *each* logarithmically divergent leading to a $\ln^2(1/\epsilon)$ renormalization of the charge. For a non-nested Fermi surface ($\alpha^2\neq1$) one logarithm is cut off by $(1-\alpha^2)$ if it has not already been cut off by temperature. The crossover occurs at the scale $\omega\approx\omega_\alpha$. The existence of a logarithmic divergence in the particle-hole bubble means that the charge, $g_{\rm AF}$, associated with **G** momentum transfer particle-hole processes grows. In the leading logarithm approximation we find that antiferromagnetic charge is renormalized by

$$\frac{\delta g_{\rm AF}}{g_{\rm AF}}=\begin{cases}\dfrac{g_{\rm AF}}{8\pi^2 u}\ln^2\dfrac{\epsilon_F}{T}, & T>\omega_0,\omega_\alpha\\[2ex] \dfrac{g_{\rm AF}}{8\pi^2 u}\ln\dfrac{1+\alpha}{1-\alpha}\ln\dfrac{\epsilon_F\omega_\alpha}{T^2}, & T<\omega_\alpha.\end{cases}\tag{16}$$

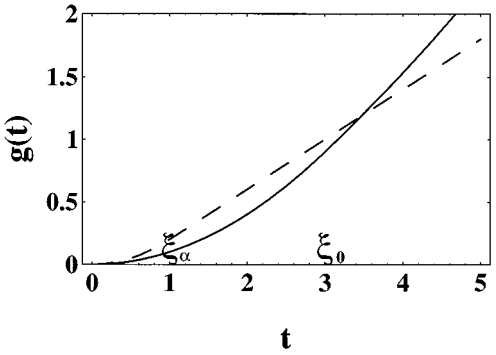

FIG. 3. Renormalization of the interaction constants $g=\delta g_{\rm AF}/g_{\rm AF}$, $\delta g_p/g_p$ plotted against $t=\ln(\epsilon_F/T)$. Solid line: $\delta g_p/g_p$ calculated from Eq. (22) with initial condition $g_p/8\pi^2u=0.1$. Dashed line: $\delta g_{\rm AF}/g_{\rm AF}$ calculated from Eq. (16) with initial condition $g_{\rm AF}/8\pi^2u=0.2$.

The second of these formulas was derived on the assumption that $\omega_\alpha>\omega_0$; if not, an additional crossover occurs. The formulas relevant to this case will be presented and discussed in the next subsection treating pairing. The renormalization of $g_{\rm AF}$ calculated from Eq. (16) is shown as the dashed line in Fig. 3. The existence of the crossover scale ω_α was noted by Schultz in Ref. 21 and an expression similar to our Eq. (16) was obtained by Markiewicz[22] in the context of a Fermi-liquid model.

We define $T_{\rm AF}$ as the scale at which $\delta g_{\rm AF}/g_{\rm AF}$ becomes of the order of unity. At scales larger than $T_{\rm AF}$, the effects of $g_{\rm AF}$ are negligible within the weak-coupling approximation. At scales of order $T_{\rm AF}$, the antiferromagnetic susceptibility becomes large, the renormalization of the fermion propagator becomes large, and perturbation theory breaks down. It is difficult to make definite predictions for temperatures lower than $T_{\rm AF}$ because one is then dealing with a strongly interacting model. We can imagine two scenarios — a crossover to the d-wave RVB regime to be discussed below or a crossover to an antiferromagnetic regime involving critical fluctuations. We discuss the antiferromagnetic regime here, focusing on whether the important fluctuations are quantum or classical and whether they can open a gap in the fermion spectrum.

It is helpful to compare the present calculation to the well-known BCS theory of superconductivity which also has a logarithmic divergence of the coupling $g_{\rm sc}$. In superconductivity the important fluctuations are classical (i.e., involve modes with energies less than k_BT) and are generally weak but grow as T approaches T_c. These fluctuations are described by the classical Landau theory

$$F_{\rm sc}=\nu\int d^dk(\tau+\xi_0^2k^2)\Delta_k^2+\beta_{\rm sc}\int d^dr\Delta_r^4,\tag{17}$$

where $\nu\sim1/\epsilon_F$ is the density of states, the coherence length $\xi_0=v_F/T_c$ and $\beta_{\rm sc}\sim\nu/T_c^2$. The renormalization of $\beta_{\rm sc}$ may be estimated to be[32]

$$\frac{\delta\beta_{\rm sc}}{\beta_{\rm sc}}\sim\left(\frac{T_c}{\epsilon_F}\right)^{d-1}\tau^{(d-4)/2}\tag{18}$$

where $\tau=(T-T_c)/T_c$. This estimate shows that at the scale $\tau\approx 1$ where the coupling g_{sc} has scaled to the order of unity, thermal fluctuations are negligible in dimension $d>1$. By continuity quantum fluctuations must be also negligible at this scale. This may be seen directly: at scales larger than T_c, the ϕ^4 vertex of the quantum Landau theory $\beta_{sc}(\epsilon,k)\sim 1/[\epsilon^2+(vk)^2]$ and the Cooper propagator has negligible momentum dependence so the leading correction $\delta\beta_{sc}/\beta_{sc}\sim(T_c/\epsilon_F)^{d-1}$. Thus we see from the quantum calculation that $d=1$ is the marginal dimensionality for the quantum fluctuations; this fact is revealed in the classical calculation because in $d=1$ the Ginzburg parameter $(T_c/\epsilon_F)^{d-1}$ ceases to be small.

In the antiferromagnetic problem of interest here, the marginal dimension is $d=2$ because the fermion spectrum is $\epsilon\propto p^2$ rather than $\epsilon\propto p$. The free energy for the classical fluctuations is

$$F_{cl}=\frac{\ln[(1+\alpha)/(1-\alpha)]}{u}\int (d^2k)\,\phi_k^2\left[F_1\tau+F_2\frac{uk^2}{T_{AF}}\right]+\beta_{AF}\int \phi_r^4 d^2r \tag{19}$$

with $\beta_{AF}=F_3/(uT_{AF}^2)$ and $\tau=(T-T_{AF})/T_{AF}$ (F_i are numerical coefficients of the order of unity). From the free energy (19) one finds

$$\frac{\delta\beta_{AF}}{\beta_{AF}}\sim\frac{G_i}{|\tau|},\quad G_i=\ln^{-2}\frac{1+\alpha}{1-\alpha}. \tag{20}$$

We see that $d=2$ is the critical dimension because the only parameter (apart from τ) controlling the fluctuation correction is $\ln[(1+\alpha)/(1-\alpha)]$ which does not depend on T_c/ϵ_F. The calculation of the leading quantum fluctuation corrections shows that these also are small only by a power of $\ln[(1+\alpha)/(1-\alpha)]$. We cannot proceed further in the general case, but in the limit $\alpha\to 1$ we can use the $\ln[(1+\alpha)/(1-\alpha)]$ to control the calculation assuming that the antiferromagnetic charge is the only relevant one. However, a theory of the general $\alpha\to 1$ case found more than one relevant charge and the resulting theory of the competing instabilities is very involved.[20]

In our $\alpha\to 1$ limit, the only important fluctuations are classical. There is no long-range order at any $T>0$; rather, for $T<T_{AF}$, Δ crosses over to $\Delta\sim\exp(G_i/\tau)$ implying an exponentially growing correlation length $\xi_{AF}\sim\Delta^{-1/2}$. These fluctuations are thus quasistatic and long ranged and in particular have energy much less than the typical fermion energy k_BT and momentum much less than the typical fermion momentum, $p\sim\sqrt{T/u}$, so as far as the fermions are concerned these fluctuations may be treated as static periodic scatterers, and lead to a gap $\Delta_F\approx T_{AF}\sqrt{-\tau}$. The magnetically induced fermion pseudogap found in this model is not relevant to the underdoped high-T_c materials because the fluctuations producing the pseudogap would also lead to a very rapid T dependence of the Cu NMR relaxation rates proportional to powers of $\xi_{AF}\sim\exp(G_iT_{AF}/T)$; such a rapid temperature dependence is not consistent with Cu relaxation rate measurements in high-T_c materials.[8]

B. Superconducting instability

We now consider particle-particle processes. As usual the leading divergence happens in the $q=0$ momentum channel. The Cooper propagator, $C(T)$, is

$$C(T)=\int\frac{d^2p\,d\epsilon}{(2\pi)^3}\frac{g_p}{\epsilon_p^2+[\epsilon+\omega_0(p)^{1/3}\epsilon^{2/3}]^2}. \tag{21}$$

Near the corners $\omega_0(p_\pm)\sim(p_+p_-)^3$ and $\epsilon_p=-up_+p_-$, so one finds

$$C(T)\approx\begin{cases}\dfrac{g_p}{8\pi^2u}\ln^2\dfrac{\epsilon_F}{T}, & T\geqslant\omega_0,\\[2ex] \dfrac{g_p}{24\pi^2u}\left(\ln^2\dfrac{\epsilon_F^3}{\omega_0^2T}-6\ln^2\dfrac{\epsilon_F}{\omega_0}\right), & T<\omega_0.\end{cases} \tag{22}$$

Here g_p is the pairing coupling constant.

The $\ln^2(\epsilon_F/T)$ divergence of $C(T)$ is due to the divergence of the density of states and the vanishing of the inelastic lifetime near the corners. The change in $C(T)$ as T is reduced below ω_0 reflects the pair-breaking effect of the gauge fluctuations in the regions away from van Hove points. One sees immediately from (21) that $C(T)$ is nondivergent in models where $\omega_0(p)\sim$const, as shown by Ubbens and Lee.[17] Ubbens and Lee derived their results by an energy argument which demonstrated that a second-order pairing transition is not possible in models with $\omega_0(p)\sim$const. We believe the argument using the gap equation is equivalent but more transparent, however we note that if we apply their energy arguments to the present model we discover that a second-order transition is possible.

The $\ln^2(\epsilon_F/T)$ divergence of $C(T)$ means that pairing is the instability which dominates the logarithmic antiferromagnetic and flux instabilities in weak coupling. However, because the interactions are repulsive the pairing instability occurs in the d-wave channel. Formally, one must consider the pairing amplitudes Δ_{11} and Δ_{22} near the two inequivalent corners 1 and 2 and the appropriate interaction amplitudes g_{11}, g_{12}. The gap equation is dominated by the corners, so becomes a 2×2 matrix equation

$$\begin{pmatrix}\Delta_{11}\\ \Delta_{22}\end{pmatrix}=-C(T)\begin{pmatrix}g_{11} & g_{12}\\ g_{12} & g_{11}\end{pmatrix}\begin{pmatrix}\Delta_{11}\\ \Delta_{22}\end{pmatrix}. \tag{23}$$

The growing eigenvalue is for $\Delta_{11}=-\Delta_{22}$ (corresponding to d-wave symmetry) and requires that $g_p\equiv g_{12}-g_{11}>0$. The pairing scale T_p at which the interaction becomes of order unity is given by

$$T_p\approx\begin{cases}\epsilon_F\exp-\sqrt{\dfrac{8\pi^2u}{g_p}}, & T_p>\omega_0,\\[2ex] \dfrac{\epsilon_F^3}{\omega_0^2}\exp-\sqrt{\dfrac{24\pi^2u}{g_p}+6\ln^2\dfrac{\epsilon_F}{\omega_0}}, & T_p<\omega_0.\end{cases} \tag{24}$$

In a model with pointlike interactions, $g_{12}=g_{11}$; it is then necessary to go to higher order. Contributions to the pairing instability of order $g_{12}^2\ln^3(\epsilon_F/T)$ exist and, for sufficiently small coupling dominate over the AF instability. The renormalization of g_p is shown in Fig. 3.

By repeating the arguments of the previous section we see that the Ginsburg parameter is $G_i \approx 1/\ln(\epsilon_F/T_p)$, so mean-field theory provides a reasonable description at temperatures less than T_p in weak coupling. There is, however, one important caveat: mean-field theory predicts a phase transition to a paired state T_p; in reality this is not a true transition. The interaction between vortices of the pairing field is not logarithmic because of screening by the gauge field, so as long as spin-charge separation occurs vortices exist and prevent long-range order. However, the small value of the Ginzburg parameter implies that the number of vortices present is very small, so the magnitude of the pairing field and the implications of the pairing for the physical properties are well described by the mean-field theory. In particular, in regions of momentum space where the pairing amplitude is appreciable there will be a strong suppression of the electron spectral function.

If the Fermi surface does not pass through a van Hove point the $\ln^2$ divergence in Eq. (22) is cut off when $T \sim \mu$ (recall that we measure the chemical potential relative to the van Hove point); however because $\omega_0(p)$ is small near the van Hove point there is an additional contribution of

$$\Delta C(T) = \frac{g_p}{24\pi^2 u} \ln\frac{\epsilon_F^3}{\omega_0^2 \mu} \ln\frac{\mu}{T} \tag{25}$$

from the part of the Fermi surface with $\omega_0(p) < T$. This contribution exists only for intermediate temperature range $T > \omega_0^{\min} \sim \mu^{3/2}$.

We now discuss the momentum space structure of the gap function focusing on the physically relevant $T_p < \omega_0$ case. The gap equation is

$$\Delta_{\epsilon,p} = \int \frac{g_{\rm pair}(\epsilon,p)\Delta_{\epsilon',p'}(dp')}{[\epsilon' + \omega_0^{1/3}(p')\epsilon^{2/3}]^2 + (\epsilon_{p'} + \Delta_{\epsilon',p'})^2}. \tag{26}$$

The integral in (26) is dominated by the van Hove point which allows us to ignore the p', ϵ' dependence of $g_{\rm pair}$ in (26). For p away from the van Hove points $g_p(\epsilon,p,p')$ is suppressed at low scales by the gauge-field fluctuations,[30] roughly $g_{\rm pair}(\epsilon,p) = g_p[\epsilon/\omega_0(p)]^\kappa$ for $\epsilon < \omega_0(p)$ with $\kappa > 2/3$. Equation (26) implies

$$\Delta_{\epsilon,p} = \begin{cases} \Delta, & \omega_0(p) < \Delta, \\ \left(\dfrac{\epsilon}{\omega_0(p)}\right)^\kappa \Delta, & \omega_0(p) > \Delta. \end{cases} \tag{27}$$

Here $\Delta \sim T_p$ is the pairing amplitude at the van Hove points. Clearly, as one moves far enough from the van Hove points so that $\omega_0(p) > \Delta$, the pairing amplitude drops rapidly and becomes less than the scattering rate $\omega_0^{1/3}(p')\epsilon^{2/3}$ due to the gauge field; these portions of the Fermi surface may be regarded as gapless.

IV. CONCLUSION

We have studied theoretically a model of a spin liquid with a Fermi surface which passes near a van Hove singularity. We considered two sorts of interactions: the singular gauge interaction arising from the spin-charge separation which established the spin liquid in the first place, and residual short-range interactions between spinons. The gauge-field interactions lead to two effects. One is an instability at $T = T_{\rm flux}$ to a ''flux phase'' in which time-reversal symmetry is spontaneously broken. An expression for $T_{\rm flux}$ is given in Eq. (10). The numerical factors are such that $T_{\rm flux}$ is negligible in the weak-coupling limit. As there is no experimental evidence for time-reversal symmetry breaking in high-T_c materials, we assume that parameters in the physical model are such that $T_{\rm flux}$ is negligibly low. This assumption allows us to neglect also the logarithmic renormalization of the fermion dispersion shown in Eq. (15). The second effect is a suppression, for spinons near the van Hove point, of inelastic scattering due to the gauge field. The physics is simple: the gauge field couples to the fermions via the velocity; this vanishes at the van Hove point and the vanishing coupling overcomes the diverging density of states.

We then turned to the nonsingular interactions. Processes involving spinons far from the van Hove points are known to be renormalized to zero by the gauge field, but near the van Hove points the vanishing of the coupling implies that the renormalization is ineffective. We therefore argued that we could specialize to a model involving fermions near van Hove points coupled by nonsingular interactions. We treated this theory via weak-coupling leading logarithm methods similar to those used to study one-dimensional models.

We found diverging interactions in the d-wave pairing and antiferromagnetic channels. The coupling constant flows are shown in Fig. 3. For sufficiently weak couplings, d-wave pairing dominates and a controlled expansion based on the parameter $\ln(\epsilon_F/T)$ is possible. Below the pairing temperature T_p given in Eq. (24) the d-wave pairing leads to a gap in the fermion spectrum near the van Hove points but leaves a finite region of gapless Fermi surface near the zone diagonal. The resulting fermion spectrum is similar to that observed in recent photoemission experiments.[4] This pairing which eliminates some but not all of the Fermi surface will have implications for other physical properties. For example, the uniform spin susceptibility will decrease as T is decreased through the pairing scale, but will tend to a nonzero limit as $T \to 0$ because some of the Fermi surface remains ungapped. Precisely this behavior occurs in $La_{2-x}Sr_xCuO_4$ at an x-dependent pairing scale $T^*(x)$ varying from $T^*(x=0.15) \approx 300$ K to $T^*(x=0.04) \approx 700$ K. Other properties of $La_{2-x}Sr_xCuO_4$ also exhibit crossover at $T^*(x)$.[33] Similar behavior occurs in $YBaCu_3O_{6+x}$, although the situation is complicated by the occurence of a second, lower scale $T_{\rm SG}(x)$ below which all susceptibilities and relaxation rates drop rapidly. A detailed phenomenological discussion of the data was given by Barzykin and Pines.[34] The identification of the RVB pairing scale T_p with the empirical $T^*(x)$ was suggested by Tanamoto *et al.*;[16] the features of the present paper are a theoretical justification for the existence of the gap despite strong inelastic scattering and the result that the gap opens only over a small portion of the Fermi surface.

The present theory does not explain the *rapid* drop of $\chi_s(T)$ and NMR relaxation rates observed in bilayer and trilayer materials below a spin-gap temperature $T_{\rm SG} \sim 200$ K. Explaining these observations requires opening a gap over the whole Fermi surface; one possibile mechanism has been discussed elsewhere.[35]

The theoretical situation is less clear if the antiferromagnetic channel is dominant. There is no generic small parameter to control fluctuations in this regime. However, if the antiferromagnetic channel is dominant and the Fermi surface is nearly nested, a controlled calculation turned out to be possible. In this case we showed that the physics is controlled by classical spin fluctuations with correlations which grow exponentially as T decreases below a mean-field scale T_{AF}. These quasistatic fluctuations produce a gap, Δ_F, in the fermion spectrum near the van Hove points which varies as $\Delta_F = T_{\mathrm{AF}}\sqrt{T_{\mathrm{AF}} - T/T_{\mathrm{AF}}}$ for $T < T_{\mathrm{AF}}$. This mechanism for producing a pseudogap predicts a very rapid T dependence of NMR rates which is not observed in experiment on superconducting materials.[8]

We now discuss the doping dependence of our results. Our calculations are based on a spin-liquid saddle point. At low T or large doping it is believed[3] that the behavior of the model is better described by a Fermi-liquid fixed point. In the Fermi-liquid regime our results are not directly relevant and the pairing scale T_p, if it exists, corresponds to the true superconducting transition temperature, T_c. In a Fermi-liquid model one would expect $T_p \equiv T_c$ to rise and then fall as the chemical potential is tuned up to the van Hove point and then beyond. This expectation does not agree with experiment: as the chemical potential is increased, T_p becomes larger than T_c and continues to increase as the chemical potential is increased beyond the van Hove point. This is natural in the spin-liquid model because the particle-hole symmetry breaking term t' itself depends on doping as seen in Eq. (4). As the doping is decreased the Fermi surface gets flatter. This tends to pin the Fermi level to the corners. Note that the van Hove point may be slightly above or *below* the chemical potential, depending on the magnitude of t' when the spin-liquid regime is entered. The important point is that the van Hove singularity gets closer to the chemical potential as half-filling is approached. Further, as the Fermi surface flattens, p_0 increases, so that ω_0 in Eq. (13) decreases and T_p increases. However, ω_α also decreases so the importance of antiferromagnetism grows. As the doping is reduced towards zero, all instability scales become greater than both ω_0 and ω_α. In this regime the pairing and antiferromagnetic charges scale in the same way, as shown in Fig. 3. Because we expect the bare value of the antiferromagnetic charge to be larger than the bare value of the pairing charge we expect antiferromagnetism to be dominant, as was also found by Dzyaloshinshkii and Yakovenko.[20]

Within our leading logarithm approximation we have shown that the observed pseudogap cannot be due to long-ranged antiferromagnetic fluctuations, and must be due to RVB pairing. We now offer qualitative arguments that this conclusion survives even when the leading logarithm approximation is not reasonable. First, it seems that antiferromagnetic fluctuations can produce a pseudogap only if they are long ranged and quasistatic; such fluctuations are ruled out by NMR so we believe that a pairing origin of a pseudogap is more likely. Second, increases in the antiferromagnetic charge apparently feed back into the pairing equation in a way that increases T_c, so it seems natural to expect that the theory with antiferromagnetic charge ~ 1 is unstable to d-wave pairing *if* the Fermi surface is not too flat. However, when the Fermi surface becomes flat, antiferromagnetism becomes favored. On the basis of these considerations we propose the phase diagram shown in Fig. 4.

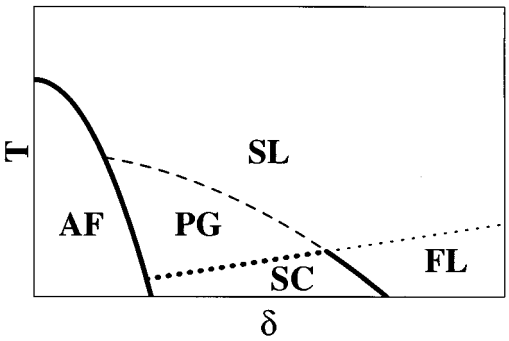

FIG. 4. Qualitative phase diagram of the spin liquid near van Hove point. AF denotes antiferromagnetic phase, SL denotes spin liquid with ungapped Fermi surface, PG denotes the pseudogap regime of the spin liquid, FL denotes the Fermi-liquid regime, and SC denotes the superconducting phase. The dotted line indicates the Bose condensation which leads to the SL-FL crossover at high doping and the superconducting transition at low doping.

The results of this paper concern the interplay of the spin-liquid (SL), pseudogap (PG), and antiferromagnetic regimes. For completeness we have also added to the figure a dotted line representing the ''Bose condensation'' considered by previous workers.[3] Below this line our calculations are not relevant. At large doping this line represents a crossover from spin-liquid (SL) to Fermi-liquid (FL) behavior; at small doping it represents the transition to the true superconducting state. Finally, we note that the key assumption underlying our work is that as the doping is varied the Fermi surface reaches the van Hove point somewhere near the maximum superconducting T_c, but the precise location of the van Hove point is not crucial.

Finally, we discuss the spin fluctuations expected in a different regime of this phase diagram. In the large δ, spin liquid, regime and in the pseudogap regime near the SL-PG boundary we expect spin fluctuations dominated by the ''$2p_F$'' effects discussed elsewhere.[13] As δ is decreased we expect enhancement of the spin fluctuations near the (π,π) point due to the flattening of the Fermi surface and the increase of the renormalized interaction. Moreover, inside the PG regime the density of states will be very much suppressed on those parts of the Fermi surface which could damp a (π,π) fluctuation. Therefore these fluctuations would be undamped as is apparently required by analyses of Cu NMR T_1 and T_2 experiments.[36]

Last, we note that in all of the theoretical development of this paper the effects of the between-planes hopping were neglected. One such effect is a between-planes coupling $J_\perp$, which has been shown to lead to spinon pairing.[35] However, other effects may occur as emphasized by Anderson.[11,29] A complete theory of the effects of $t_\perp$ is an important open problem.

In summary, we have studied the effect of van Hove points on the ''uniform RVB'' state with weak residual interactions. We have identified three instabilities: to a flux phase with spontaneous breaking of time-reversal symmetry, magnetic phase, and a d-wave RVB phase. We argued that

the d-wave state is likely to happen at large doping while antiferromagnetism dominates at small doping. Our proposed phase diagram is shown in Fig. 4. We showed that the inelastic scattering due to the gauge field is negligible for fermions near the corners and that this in combination with the divergent density of states permits a continuous crossover to a d-wave RVB state, which is not allowed in models without corners.

ACKNOWLEDGMENT

We thank Z. X. Shen for very helpful discussions on the photoemission data.

APPENDIX: PARAMETERS OF THE SPINON HAMILTONIAN

The spinon Hamiltonian is obtained expanding about a mean-field approximation to the t-J model which may be written

$$H_{tJ}=-\sum_{i,j} t_{ij}^{(\mathrm{band})} b_i^\dagger b_j c_{j\alpha}^\dagger c_{i\alpha}+\frac{J}{2}\sum \vec{S}_i\vec{S}_j. \tag{A1}$$

Here $c^\dagger$ is the spinon creation operator discussed in the text, $\vec{S}_i=(1/2)\Sigma_{\alpha\beta}c_{i\alpha}^\dagger\vec{\sigma}_{\alpha\beta}c_{i\alpha}$, the Bose operator $b_i^\dagger$ creates a spinless charge 1 ''holon'' and the constraint $b_i^\dagger b_i+\Sigma_\alpha c_{i\alpha}^\dagger c_{i\alpha}=1$ is assumed. To obtain the fermion dispersion one writes the second term in H_{tJ} as a product of four fermion operators[2]

$$\vec{S}_i\vec{S}_j=-\frac{1}{2}c_{j\alpha}^\dagger c_{i\alpha}c_{i\beta}^\dagger c_{j\beta}+\frac{1}{4} \tag{A2}$$

and approximates this by

$$\vec{S}_iS_j\approx-\frac{1}{2}(c_{j\alpha}^\dagger c_{i\alpha}+\mathrm{H.c.})\langle c_{i\beta}^\dagger c_{j\beta}\rangle. \tag{A3}$$

If $\langle c_{i\beta}^\dagger c_{j\beta}\rangle\neq 0$ then the bosons acquire a dispersion $H_{\mathrm{Bose}}=-\langle c_{i\beta}^\dagger c_{j\beta}\rangle t_{ij}^{\mathrm{band}} b_i^\dagger b_j$; because the density of bosons is low the bosons are mostly in small k states so it follows that for nearby i and j $\langle b_i^\dagger b_j\rangle=\delta$. Inserting this in Eq. (A1) we get the fermionic part of Eq. (1) with

$$t_{ij}=\delta t_{ij}^{(\mathrm{band})}+\frac{1}{2}J\langle c_{i\beta}^\dagger c_{j\beta}\rangle. \tag{A4}$$

The expectation value $\langle c_{i\beta}^\dagger c_{j\beta}\rangle$ turns out to be $4/\pi^2+O(\delta)$.

We now estimate the bare gauge stiffness g_0. This represents the effect of short-wavelength fluctuations which are integrated out in the definition of Eq. (1); in the $N\to\infty$ limit g_0^{-2} vanishes,[2] but it is nonzero for the physical $N=2$. If the only term in H, Eq. (1), were $(1/4g_0^2)f_{\mu\nu}^2$, then one would find

$$\int_{-\infty}^{\infty} dt\langle h_i(t)h_i(0)\rangle=g_0^2. \tag{A5}$$

Here as usual $h=\partial_x a_y-\partial_y a_x$. To find g_0^2 we evaluate the $\langle h_i(t)h_i(0)\rangle$ correlator directly from $S=1/2$ Heisenberg model, including in our calculation only short-wavelength degrees of freedom. Formally, the gauge field is related to the phase of the operator Δ_{ij} which decouples the four fermion operator in Eq. (A2)[2]; the field h is equal to the gauge flux through the elementary placquette with vertices i,j,k,l; the latter is related to the Δ operators by

$$Re^{ih}=\Delta_{ij}\Delta_{jk}\Delta_{kl}\Delta_{li}. \tag{A6}$$

Here R is a real operator. By undoing the decoupling one may express the product of four Δ around the placquette in terms of fermion operators; in the low doping limit these in turn may be expressed in terms of spin operators [with corrections $O(\delta)$], yielding

$$Re^{ih}=\frac{1}{8}+\frac{1}{2}\sum_{(ij)}\vec{S}_i\vec{S}_j+i\sum_{(ijk)}\vec{S}_i(\vec{S}_j\times\vec{S}_k)+2[(\vec{S}_i\vec{S}_j)(\vec{S}_k\vec{S}_l)+(\vec{S}_i\vec{S}_l)(\vec{S}_k\vec{S}_j)-(\vec{S}_i\vec{S}_k)(\vec{S}_j\vec{S}_l)], \tag{A7}$$

where (ij) denotes all distinct pairs of spins, (ijk) denotes all distinct triads with the clockwise order of i,j,k around the placquette. If there are strong short-range antiferromagnetic correlations then we may replace the real terms by c numbers, obtaining

$$Re^{ih}=\frac{1}{2}+i\sum_{(ijk)}\vec{S}_i(\vec{S}_j\times\vec{S}_k). \tag{A8}$$

This allows us to identify

$$h=2\sum_{(ijk)}\vec{S}_i(\vec{S}_j\times\vec{S}_k), \tag{A9}$$

and thus to relate g_0^2 to a six-spin correlator. Evaluating this in the spin-wave approximation leads to

$$g_0^2=B\sum_{k,p}\frac{1}{\omega_k+\omega_p}, \tag{A10}$$

where B is a number of the order of unity and ω_k and ω_p are spin-wave energies. Because the typical value for $\omega_k\sim 2J$ and the sums are dominated by short wavelengths, we see that $g_0^{-2}=B'J$ with B' a number of the order of unity. Of course we cannot expect this simple estimate to yield a reliable value for B', but it does show that it is reasonable to expect a large contribution to the gauge stiffness from short-range correlations.

Finally we consider the effective between-planes hopping $t_\perp$. If $t_\perp\neq 0$ then also the between-planes exchange $J_\perp\neq 0$ and we must add to Eq. (A1) the terms

$$H_\perp=\sum_i t_\perp b_i^{(1)\dagger}b_i^{(2)}c_{i\alpha}^{(1)\dagger}c_{i\alpha}^{(2)}+\frac{J_\perp}{2}\sum\vec{S}_i^{(1)}\vec{S}_j^{(2)}. \tag{A11}$$

One may factorize the four fermion term as in Eqs. (A2,A3); however the equation determining the amplitude is

$$\Delta_\perp\equiv(J_\perp+\delta t_\perp)\langle c_{i\beta}^{(1)\dagger}c_{i\beta}^{(2)}\rangle=\sum_{\epsilon,p}\frac{(J_\perp+\delta t_\perp)}{(i\epsilon-\epsilon_p)^2-\Delta_\perp^2}. \tag{A12}$$

A solution only becomes possible for $J_\perp+\delta t_\perp\approx t$, so for physically relevant parameters there is no coherent between-planes hopping in the model.

[1] G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. **63**, 973 (1987).
[2] L. B. Ioffe and A. I. Larkin, Phys. Rev. B **39**, 8988 (1989).
[3] P. A. Lee, in *High Temperature Superconductivity: Proceedings*, edited by K. S. Bedell, D. Coffey, D. E. Meltzer, D. Pines, and J. R. Schreiffer (Addison-Wesley, Reading, MA, 1990), p. 96; L. B. Ioffe and B. G. Kotliar, Phys. Rev. B **42**, 10 348 (1990); L. B. Ioffe, V. Kalmeyer, and P. B. Wiegmann, *ibid.* **43**, 1219 (1991).
[4] D. M. King, D. S. Dessau, A. G. Loeser, Z.-X. Shen, and B. O. Wells, J. Phys. Chem. Solids **56**, 1865 (1995).
[5] J. R. Schreiffer, X. G. Wen, and S. C. Zhang, Phys. Rev. Lett. **60**, 944 (1980); A. Kampf and J. R. Schreiffer, Phys. Rev. B **41**, 6399 (1990); **42**, 7967 (1990).
[6] P. A. Lee, T. M. Rice, and P. W. Anderson, Phys. Rev. Lett. **31**, 462 (1973).
[7] M. V. Sadovskii, Zh. Éksp. Teor. Fiz. **66**, 1720 (1974).
[8] See, e.g., A. J. Millis, in *High Temperature Superconductivity: Proceedings* (Ref. 3), p. 198; or C. P. Slichter, in *Strongly Correlated Electronic Materials*, edited by K. S. Bedell, Z. Wang, D. Meltzer, A. V. Balatsky, and E. Abrahams (Addison-Wesley, Redwood City, CA, 1990), p. 427.
[9] P. W. Anderson, Science **235**, 1196 (1987).
[10] I. A. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988).
[11] S. Chakravarty, A. Sudbo, P. W. Anderson, and S. P. Strong, Science **261**, 337 (1993); A. Sudbo, S. Chakravarty, S. Strong, and P. W. Anderson, Phys. Rev. B **49**, 12 245 (1994).
[12] C. M. Varma, S. Schmitt-Rink, and E. Abrahams, Solid State Commun. **62**, 681 (1987).
[13] B. Altshuler, L. B. Ioffe, A. I. Larkin, and A. J. Millis, Phys. Rev. B **52**, 4607 (1995).
[14] B. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B **52**, 5563 (1995).
[15] S. Spielman, K. Fester, C. B. Eom, T. H. Geballe, M. M. Fejer, and A. Kapitulnik, Phys. Rev. Lett. **65**, 123 (1990).
[16] T. Tanamoto, K. Kohno, and H. Fukuyama, J. Phys. Soc. Jpn. **61**, 1886 (1992).
[17] M. Ubbens and P. A. Lee, Phys. B **50**, 438 (1994).
[18] X.-G. Wen and P. A. Lee, Phys. Rev. Lett. **76**, 503 (1996).
[19] P. A. Lee and N. Read, Phys. Rev. Lett. **50**, 2691 (1987).
[20] I. E. Dzyaloshinshkii, Sov. Phys. JETP **66**, 848 (1987); I. E. Dzyaloshinshkii and V. Yakovenko, *ibid.* **67**, 844 (1988); Int. J. Mod. Phys. B **2**, 667 (1988).
[21] H. J. Schultz, Europhys. Lett. **4**, 609 (1987).
[22] R. S. Markiewicz, Physica C **169**, 63 (1990); J. Phys. Chem. Solids **56**, 1637 (1995).
[23] D. C. Newns, C. C. Tseui, P. C. Pattnaik, and C. L. Kane, Commun. Condens. Matter Phys. **15**, 273 (1992); D. M. Newns and P. C. Paittnaik, in *Strong Correlations and Superconductivity*, edited by H. Fukuyama, S. Maekawa, and A. Malozemoff (Springer-Verlag, Heidelberg, 1989), p. 146.
[24] E. Dagotto, A. Nazarenko, and A. Moreo, Phys. Rev. Lett. **74**, 310 (1995).
[25] See, e.g., Table I of W. E. Pickett, Rev. Mod. Phys. **61**, 463 (1989).
[26] See, e.g., the tabulation of observed penetration depths in C. C. Homes, T. Timusk, D. A. Bonn, R. Liang, and W. N. Hardy, Physica C **254**, 265 (1995).
[27] See, e.g., S. Martin *et al.*, Phys. Rev. B **41**, 846 (1990).
[28] P. W. Anderson and Y. Ren, in *High temperature Superconductivity: Proceedings* (Ref. 3), p. 3.
[29] P. W. Anderson (private communication).
[30] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B **50**, 14 048 (1994).
[31] J. Polchinski, Nucl. Phys. B **422**, 617 (1994).
[32] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1988), Vol. II, Sec. 45.
[33] H. Y. Hwang, B. Batlogg, H. Takagi, H. L. Kao, J. Kwo, R. J. Cava, J. J. Krajewski, and W. F. Peck, Jr., Phys. Rev. Lett. **72**, 2636 (1994).
[34] V. Barzykin, D. Pines, A. V. Sokol, and D. Thelen, Phys. Rev. B **49**, 1544 (1994).
[35] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B **53**, 423 (1996); B. L. Altshuler, L. B. Ioffe, A. I. Larkin, and A. J. Millis, JETP Lett. **59**, 67 (1994); B. L. Altshuler and L. B. Ioffe, Solid State Commun. **82**, 253 (1992).
[36] A. V. Sokol and D. Pines, Phys. Rev. Lett. **71**, 2813 (1993).