High-temperature expansion study of the Nishimori multicritical point in two and four dimensions

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We study the two- and four-dimensional Nishimori multicritical point via high-temperature expansions for the $\pm J$ distribution, random-bond, Ising model. In $2d$ we estimate the critical exponents along the Nishimori line to be $\gamma=2.37\pm0.05$ and $\nu=1.32\pm0.08$. These, and earlier $3d$ estimates $\gamma=1.80\pm0.15$ and $\nu=0.85\pm0.08$ are remarkably close to the critical exponents for percolation, which are known to be $\gamma=43/18$ and $\nu=4/3$ in $d=2$, and $\gamma=1.805\pm0.02$ and $\nu=0.875\pm0.008$ in $d=3$. However, the estimated $4d$ Nishimori exponents $\gamma=1.80\pm0.15$ and $\nu=0.8\pm0.1$, are quite distinct from the $4d$ percolation results $\gamma=1.435\pm0.015$ and $\nu=0.678\pm0.05$. [S0163-1829(96)02125-X]

In recent years there has been much interest in the study of critical phenomena in quenched-random, two-dimensional, thermodynamic systems. However, with the exception of percolation, for which various critical parameters are known exactly, other random fixed points are not fully understood. Such random critical phenomena are of interest from a theoretical point of view, and also from an experimental point of view. Among notable experimental systems showing two-dimensional random critical phenomena are plateau transitions in quantum Hall systems[1] and Bose-glass transitions in dirty superfluids and superconductors.[2]

Perhaps the simplest theoretical model with quenched randomness is the random-bond Ising model. In $d=2$ a lot is known about weak randomness.[3] The case where randomness has the most dramatic influence on thermodynamic properties is that of a symmetric distribution of bonds, that is one where there are roughly equal tendencies for ferromagnetic and antiferromagnetic ordering. In this case the system may only have a long-ranged spin-glass phase at low temperatures. There is considerable numerical evidence that in $d=2$ there is no finite temperature spin-glass phase.[4] The Nishimori manifold separates the region in parameter space where ferromagnetic (or antiferromagnetic) correlations are stronger from those where spin-glass correlations dominate. In certain random-bond Ising models many exact results can be obtained along this special manifold.[5] A Nishimori multicritical point can exist even in the absence of a finite-temperature spin-glass transition, and has been studied by renormalization-group[6] and various numerical methods.[7] By now the existence of the critical point and its location are reasonably well established,[7] although to our knowledge no reliable estimates of the critical exponents exist.

Here we study this model by high-temperature expansions, estimating the critical point and the various critical exponents. Our estimates for the critical temperatures are consistent with previous ones. Our interesting result is that the critical exponents γ and ν are remarkably close to that of percolation. This also turns out to be the case in three dimensions. To see if such a trend continues with dimensionality we study the four-dimensional Nishimori multicritical point. There the critical exponents are clearly distinct from percolation. Towards the end of this paper we speculate on the relevance of the percolation fixed point to the present problem.

We consider the Hamiltonian

$$\mathcal{H}=\sum_{\langle i,j\rangle} J_{i,j}S_iS_j \tag{1}$$

with the J_{ij} independent quenched random variable with distribution

$$P(J_{ij})=p\,\delta(J_{ij}-J)+(1-p)\,\delta(J_{ij}+J). \tag{2}$$

The Nishimori line in the parameter space of temperature T and ferromagnetic bond concentration p is given by

$$v=2p-1 \tag{3}$$

with $v=\tanh J/kT$. It is most convenient to directly develop the expansions along the Nishimori line.[8] In this case the expansion variable is $w=v^2$. We define susceptibilities $\chi_{m,n}$ by the relations

$$\chi_{m,n}=\frac{1}{N}\sum_{i,j}\,[\langle S_iS_j\rangle^m]^n. \tag{4}$$

Here angular brackets represent thermal averaging, and the square brackets an averaging with respect to the bond distribution. We carry out expansions to 19th order in $2d$ and 15th order in $4d$ for $\chi_{2,1}$ and $\chi_{2,2}$ using the star-graph method.[9] We note that along the Nishimori line the ferromagnetic susceptibility $\chi_{1,1}$ exactly equals the spin-glass susceptibility $\chi_{2,1}$. In order to study the crossover exponents we also calculate the series for $\chi_v=v\,\partial\chi_{2,1}/\partial v$. The expansion coefficients are given in Tables I and II.

We shall assume that the quantities $\chi\equiv\chi_{2,1}$, $\chi'\equiv\chi_{2,2}$, and χ_v become singular at a critical point w_c with exponents γ, γ', and γ'' respectively. Assuming standard scaling, the

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TABLE I. Expansion coefficients for the susceptibilities in $d=2$ [see Eq. (4)].

n	$\chi_{2,1}$	$\chi_{2,2}$	χ_v
0	1	1	0
1	4	0	8
2	12	4	48
3	36	0	216
4	76	36	512
5	196	−32	1640
6	316	236	1856
7	884	−464	9208
8	780	1988	−5824
9	3684	−5072	57 576
10	396	17 076	−109 264
11	22 740	−50 432	680 376
12	−22 596	164 108	−15 47 376
13	188 420	−500 496	8 163 624
14	−331 108	1 604 572	−21 618 432
15	1517 396	−5 042 160	84 085 560
16	−4 509 268	16 221 028	−311 253 632
17	15 654 148	−52 336 864	1 071 437 960
18	−56 714 548	170 687 620	−4 263 416 944
19	183 041 524	−561 493 296	14 787 979 576

exponents for the divergence of the different series can be related to the critical exponents ν, η, and ϕ as[8]

$$\gamma=(2-\eta)\nu, \quad \gamma'=(4-d-2\eta)\nu, \quad \gamma''=\gamma+\phi.$$

Here d is the dimensionality of the system.

We analyze the series based on the expectation that near the critical point the susceptibilities have the form

$$\chi\propto(w_c-w)^{-\gamma}[1+a(w_c-w)^{\Delta_1}+b(w-w_c)+\cdots].$$

We estimate the location of the critical point w_c, the dominant exponent γ, and the correction-to-scaling exponent Δ_1. The value of Δ_1 may be biased by the presence of even higher-order correction terms, though fitting to this form should ensure reliable evaluation of the critical point and the dominant exponent. Our analysis is carried out with no prior assumptions regarding exponent values, and was done by one of us without knowing prior literature values for the critical parameters or scaling relations between the exponents. We have studied the series with two methods, commonly known as $M1$ and $M2$.[10] They are based on suitable transformations of the series, and Padé approximants for the transformed series.

TABLE II. Expansion coefficients for the susceptibilities in $d=4$ [see Eq. 4)].

n	$\chi_{2,1}$	$\chi_{2,2}$	χ_v
0	1	1	0
1	8	0	16
2	56	8	224
3	392	0	2352
4	2552	200	19 840
5	16 904	−192	162 512
6	105 944	6584	1 179 840
7	679 784	−12 384	8 736 880
8	4 158 200	234 824	58 846 592
9	26 120 392	−649 056	412 368 720
10	157 020 984	8 748 712	2 651 405 536
11	974 362 408	−32 109 952	18 054 488 432
12	5 783 009 304	342 786 296	112 459 651 552
13	35 661 616 648	−1 523 180 000	755 621 878 608
14	209 506 120 728	14 008 147 224	4 590 427 798 720
15	1 289 118 273 320	−70 814 307 872	30 721 400 183 024

$M1$: In this method of analysis we study the logarithmic derivative of

$$B(w)=hH(w)-(w_c-w)\frac{dH(w)}{\mathrm{d}w}.$$

The dominant singularity is a pole at $w=w_c$ with a residue $(h-1)$.

We implement method $M1$ as follows: for a given value of w_c we obtain Δ_1 versus input h for many central and high Padé approximants, and we choose the triplet w_c, h, Δ_1, where all Padés yield as nearly as possible identical values of h.

$M2$: In the second method we first transform the series in w into a series in the variable y, where

$$y=1-(1-w/w_c)^{\Delta_1},$$

and then take Padé approximants to

$$G(y)=\Delta_1(y-1)\frac{d}{dy}\ln[H(w)],$$

which should converge to $-h$. Here we plot graphs of h versus the input Δ_1 for different values of w_c and again choose the triplet w_c, h, Δ_1, where all Padés converge to the same point.

We found good convergence for all three $d=2$ series, with $M2$ graphs being better converged everywhere, and $M1$ giving consistent results. The χ and χ' series behaved better than the χ_v series. For brevity, we only show some representative plots from which we have deduced our estimates of critical parameters. In Fig. 1 we present two three-dimensional graphs from the $M2$ analysis for the χ series. In Fig. 1(a) we show the three-dimensional version on a fairly coarse temperature scale, in Fig. 1(b) we show a finer scale. On the coarse scale we can see that at the trial w_c values of 0.525 and 0.550, convergence to a region of clear intersections is much poorer than at 0.575. Similarly, although the different approximants come together at the background plane with $w_c=0.625$, this type of almost flat graph with the asymptotic convergence at very high-Δ_1 values is indicative of behavior that does not give correct critical behavior in test systems or exactly solved models. (It is often seen near trial critical points that give exponent values that violate hyperscaling.) The fine scale shown in the enlargement in Fig. 1(b) shows us a set of graphs which mostly satisfy the considerations required of an intersection region, with the best of all being the central plane. Thus from the *M2* analysis, the best w_c estimate is 0.596 ± 0.008. This implies $p_c=0.886\pm0.003$, $T_c/J=0.975\pm0.006$, which are consistent with

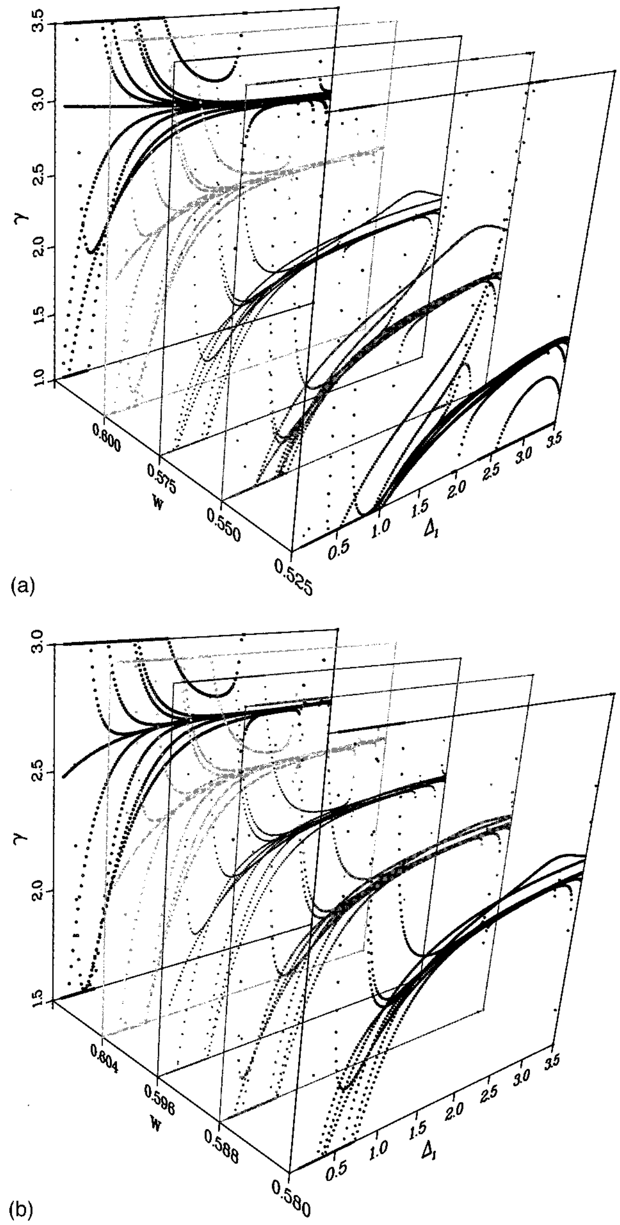

FIG. 1. Plots from the $M2$ analysis of the $d=2$ χ series (a) on a coarse scale and (b) on a fine scale.

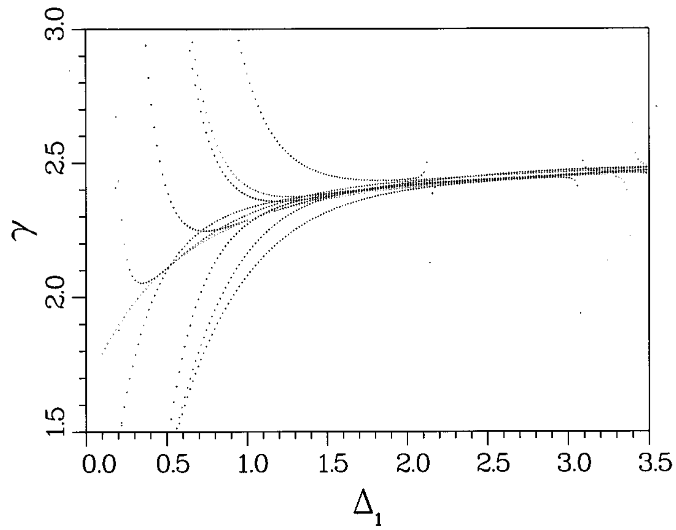

FIG. 2. The central slice for the $M2$ analysis in Fig. 1 with $w_c=0.596$.

previous estimates.[7] In Fig. 2(a) the central slice at $w_c=0.596$ is shown. From this we conclude an exponent estimate of $\gamma=2.37\pm0.05$. The $M1$ analysis is consistent with these values. In the $M2$ analysis for the χ' series, convergence was again optimal at $w_c=0.596$. The exponent γ' is deduced to be 2.11 ± 0.07. Via scaling this gives $\nu=1.32\pm0.08$. We analyze the χ_v series in two ways, first by considering χ_v/w and second by studying the series for $d\chi_v/dw$. From these analyses we conclude $\gamma+\phi=3.0\pm0.3$.

In four dimensions, the $M2$ analysis of the χ series gives best convergence at $w_c=0.1764$, with $\gamma=1.9$. From $M1$ a slightly lower $w_c=0.176$ value and a correspondingly lower $\gamma=1.8$ seems optimal. For the χ' series optimal convergence is at $w_c=0.176$ with the central values of the exponent γ' of 0.41 from $M1$ and 0.40 from $M2$. Since three of the four analysis support the lower value of $w_c=0.176$ we use this for our final estimates. We quote overall values of $w_c=0.176\pm0.001$ and $\gamma=1.80\pm0.15$, $\gamma'=0.40\pm0.03$. By scaling this gives us $\nu=0.8\pm0.1$.

As stated in the Introduction, these estimated critical parameters are remarkably close to percolation[11] in $d=2$. The numbers for percolation are $\gamma=43/18$ and $\nu=4/3$. Furthermore, in $d=3$ the Nishimori exponents were found to be $\gamma=1.80\pm0.15$ and $\nu=0.85\pm0.08$.[8] These numbers are also confirmed by our present analysis. The $d=3$ percolation exponents are $\gamma=1.805\pm0.02$ and $\nu=0.875\pm0.008$.[11] Thus in $d=3$ the Nishimori exponents are also very close to percolation. However, in $d=4$ the percolation exponents are $\gamma=1.435\pm0.015$ and $\nu=0.678\pm0.05$, which are clearly distinct from those found here.

One can conclude that in $d=2$ and 3 the Nishimori critical behavior is consistent with the universality class for percolation. However, this is not so in $d=4$. The possibility that the closeness of the Nishimori exponents to percolation in $d=2$ and 3 is purely accidental cannot be ruled out. However, the following considerations suggest a possible connection. If we consider a bond distribution of the form

$$P(J_{ij})=p\,\delta(J_{ij}-J)+(1-p)\,\delta(J_{ij}+J)+c\,\delta(J_{ij}),$$

that is, if the bonds are allowed to take values $\pm J$ as well as zero, then in the generalized parameter space of p, c and temperature T, the Nishimori manifold is a two-dimensional plane. For $p=0$ (or unity), this plane reduces to the $T=0$ dilution axis and thus contains the percolation fixed point.[12] However, it is generally believed that T is always an unstable direction for percolation and, hence, finite temperature Nishimori criticality should have different exponents.

Secondly, Nishimori has argued[13] that the spin-glass ferromagnetic transition is a *geometry-induced* phase transition (as opposed to a thermal transition), which is also true for percolation. However, if this led to the identification of the Nishimori fixed point with percolation it should be true independent of dimensions. However, our results in $d=4$ contradict this. Furthermore, the epsillon expansions (around

$d=6$) for the exponents at the Nishimori multicritical point are different from percolation.[6]

Finally, a very different way in which this model is of significant interest, is through the mapping between the $2d$ Ising model and free fermions in $1+1$ dimension, and the connection between the latter and the pleateau transitions in the quantum Hall effect. The Chalker model for the pleateau transitions in the quantum Hall effect[14,15] can thus be mapped onto random-bond Ising models. However, the $\pm J$ model studied here does not have the correct symmetries for the quantum Hall problem.[16] It is well known that percolation occurs in one limit of the quantum Hall systems, when the disorder potential is slowly varying in space.[17] However, numerical studies of the Chalker models lead to exponents clearly different from percolation.[14]

Thus there is no compelling theoretical reason why the Nishimori multicritical point should be in the universality class of percolation. One possible explanation for our findings could be that in low dimensions, where the multicritical point occurs at very low temperatures, there are crossover effects which produce effective exponents close to percolation. These issues deserve further attention.

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