Critical behavior of weakly disordered anisotropic systems in two dimensions

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The critical behavior of two-dimensional (2D) anisotropic systems with weak quenched disorder described by the so-called generalized Ashkin-Teller model (GATM) is studied. In the critical region this model is shown to be described by a multifermion field theory similar to the Gross-Neveu model with a few independent quartic coupling constants. Renormalization group calculations are used to obtain the temperature dependence near the critical point of some thermodynamic quantities and the large-distance behavior of the two-spin correlation function. The equation of state at criticality is also obtained in this framework. We find that random models described by the GATM belong to the same universality class as that of the two-dimensional Ising model. The critical exponent ν of the correlation length for the three- and four-state random-bond Potts models is also calculated in a three-loop approximation. We show that this exponent is given by an apparently convergent series in $\epsilon = c - \frac{1}{2}$ (with c the central charge of the Potts model) and that the numerical values of ν are very close to that of the 2D Ising model. This work therefore supports the conjecture (valid only approximately for the three- and four-state Potts models) of a superuniversality for the 2D disordered models with discrete symmetries. [S0163-1829(96)04826-6]

I. INTRODUCTION

The critical properties of two-dimensional random spin systems have been extensively studied in the last few years.[1–3] Two-dimensional (2D) systems are particularly interesting due to a variety of reasons. First, there are numerous examples of layered crystals undergoing continuous antiferromagnetic and structural phase transitions.[4,5] More recently, 2D and quasi-2D crystals have begun to be fabricated and studied thanks to advances in deposition techniques, with an enormous increase in the variety of physical phenomena to be investigated.[6] Perfect crystals, however, are the exception rather than the rule, with quenched disorder always existing in different degrees. Even weak disorder may drastically affect the critical behavior, according to the celebrated Harris criterion.[7] Second, the conventional field-theoretic renormalization group (RG) approach based on the standard ϕ^4 theory in $(4-\epsilon)$ dimensions, and as applied to study properties of disordered systems by Harris and Lubensky[8] and Khmelnitskii,[9] does not work in 2D due to the hard restriction $\epsilon \ll 1$. Similar considerations apply to the $(2+\epsilon)$ low-temperature RG approach. Third, from a theoretical point of view a most challenging problem is to establish the relationship between random models and the corresponding conformal field theory (CFT) describing these at criticality.

Some early exact results concerning the 2D random-bond Ising model (IM) with a special type of disorder (where only the vertical bonds are allowed to acquire random values, while the horizontal bond couplings are fixed) have been obtained by McCoy and Wu.[10] This type of 1D quenched disorder without frustration was shown to smooth out the logarithmic singularity of the specific heat; the frustrated case was considered by Shankar and Murthy.[11] Dotsenko and Dotsenko[1] initiated some considerable progress in the study of 2D random bond IM's by exploiting the remarkable equivalence between this problem and the $N=0$ Gross-Neveu model. For weak dilution the new temperature dependence of the specific heat was found to become $C \sim \ln\ln\tau$, $\tau = (T-T_c)/T_c$ being the reduced deviation from the critical temperature T_c. However, their results concerning the two-spin correlation function at the critical point were later reconsidered by Shalaev,[12] Shankar,[13] and Ludwig.[14] By using the RG approach as well as the bosonization technique these authors showed that the large-distance behavior of this function at criticality was the very same as in the pure case. Some convincing arguments in favor of the critical behavior of the 2D IM with impurities as governed by the pure IM fixed point had been given earlier by Jug.[15] Recently, a good number of papers devoted to Monte Carlo simulations of the critical behavior of the random Ising model have been published.[16] Most Monte Carlo data are in good agreement with analytical results obtained in Refs. 12–14. It should be mentioned, however, that these analytical results have been obtained by employing the replica method. This, on the one hand, is known to give reliable results only in the framework of perturbation theory. On the other hand, the mathematical legitimacy of the replica trick has not yet been established. Moreover, replicas (though being very useful and convenient) appear not to capture the essentials of nonperturbative effects in the close vicinity of the phase transition point (Griffiths phase) (see, for instance, Ref. 17). The study of nonperturbative effects in the critical properties of random systems is, however, beyond the scope of the present paper.

Here, we point out that there is some scope to extend the previous analysis for the 2D random IM to other discrete-symmetry systems. A very interesting problem consists in considering minimal CFT models with $c<1$ as perturbed by randomness. These models comprise the three- and four-state Potts systems as particular cases, and these have interesting applications to real 2D crystals.[6] Because the critical exponent α is positive for all these models, the critical behavior is governed by a random fixed point in agreement with the Harris criterion. Some years ago Ludwig[18,19] and Ludwig and Cardy[20] made an attempt to calculate perturbatively the critical exponents of the random three-state Potts model. Their approach was essentially based on the powerful CFT technique. More recently, Dotsenko, Pujol, and Picco[21] obtained the critical exponents for the dilute three-state Potts model in a two-loop approximation by exploiting the Coulomb gas representation for the correlation functions and a special kind of ϵ regularization, where ϵ stands here for the difference between the pure system's central charge value and the conformal anomaly for the pure 2D IM ($\frac{1}{2}$). Dotsenko *et al.*[22] have also found the new universality class of the critical behavior as corresponding to the broken replica symmetry proposed by Harris *et al.*[23]

Another interesting possibility is to study critical phenomena in 2D dilute anisotropic systems with many-component order parameters. The analysis of the critical behavior of such systems in $(4-\epsilon)$ dimensions was developed in great detail years ago,[24] but cannot be directly applied to the 2D case. Therefore it would be interesting and important to consider studying these 2D models. This is the main goal of our paper. The key ingredient of our treatment is a fermionization trick first suggested by Shankar[25] for the N-color Ashkin-Teller model (see also Refs. 26,27). This method is quite general and may be extended to other systems. The initial Landau Hamiltonian as written in terms of scalar fields can be shown to map onto a multifermion field theory of the Gross-Neveu type with a few independent quartic couplings. This transformation can be done for Hamiltonians containing only even powers of each order parameter component, the fourth-order term being an invariant of the hypercubic symmetry group [this is the so-called generalized Ashkin-Teller model (GATM)].

The work presented in this paper is organized as follows. In Sec. II we consider in brief the critical behavior of the weakly disordered 2D Ising model with random bonds, this being the central theme of this research field. The tranfer matrix formalism is set up and the corresponding equations are written down. The computation of the two-spin correlation function for pure and random models at criticality is also reviewed. In Sec. III we give a description of the fermionization trick allowing us to study the critical behavior of the pure N-color Ashkin-Teller model. In Sec. IV the critical properties of two interacting N- and M-color quenched disordered Ashkin-Teller models are studied. The RG method is used to obtain the exact temperature dependence of the correlation length, specific heat, susceptibility, and spontaneous magnetization near criticality, as well as the two-point spin-correlation function and the equation of state at the critical point. In Sec. V, exploiting the approach of Dotsenko, Picco, and Pujol and of Ludwig we compute the critical exponent of the correlation length in a three-loop approximation for the weakly disordered minimal models of CFT, in particular for the three- and four-state Potts models with random bonds. We find that while for the GATM the introduction of disorder leads to critical behavior as characterized by the random-bond IM fixed point, for the minimal models of CFT this Ising behavior, conjectured by a number of authors recently for the 2D Potts models,[28] is actually only approximate. The accuracy with which the Ising values of the exponents is observed, however, justifies the use of the term ''IM superuniversality'' for all these models, when disordered. Sec. V contains a discussion and some concluding remarks.

II. TWO-DIMENSIONAL ISING MODEL WITH RANDOM BONDS

A. Transfer matrix, effective action, and RG for thermodynamic functions

We begin with the classical Hamiltonian of the 2D Ising model with random bonds defined on a square lattice with periodic boundary conditions:

$$H=-\sum_{i=1,j=1}^{N}[J_1(i,j)s_{ij}s_{ij+1}+J_2(i,j)s_{ij}s_{i+1j}], \quad (2.1)$$

where i,j label sites of the square lattice, $s_{ij}=\pm 1$ are spin variables, and $J_1(i,j)$ and $J_2(i,j)$ are horizontal and vertical independent random couplings having the same probability distribution, which reads

$$P(x)=(1-p)\delta(x-J)+p\,\delta(x-J'). \quad (2.2)$$

Also, p is the concentration of impurity bonds and both J and J' are assumed to be positive so that the Hamiltonian favors aligned spins. Notice that both antiferromagnetic couplings (creating frustration) and broken bonds $(J'=0)$ lead to ambiguities in the transfer matrix and must be excluded in the present treatment. Let us now consider the calculation of the partition function of the model under discussion:

$$Z=\sum \exp\left(-\frac{H}{T}\right), \quad (2.3)$$

where H is defined in Eq. (2.1) and the sum runs over all 2^{N^2} possible spin configurations. The partition function is known to be represented as the trace of the product of the row-to-row transfer matrices $\hat{T}_i$:[29–31]

$$Z=\mathrm{Tr}\prod_{i=1}^{N}\hat{T}_i. \quad (2.4)$$

The Hermitian $2^N\times 2^N$ matrix $\hat{T}_i$ rewritten in terms of spin variables reads:[29–31,2]

$$\hat{T}_i=\exp\left(\frac{1}{T}\sum_{j=1}^{N}J_1(i,j)\sigma_3(j)\sigma_3(j+1)\right)\times\exp\left(\frac{1}{T}\sum_{l=1}^{N}J_2^*(i,l)\sigma_1(l)\right), \quad (2.5)$$

where σ_α, $\alpha=1,2,3$ are Pauli spin matrices; here J_2 and J_2^* are related by the Kramers-Wannier duality relation:[29–31]

$$\tanh\left(\frac{J_2^*}{T}\right)=\exp\left(-\frac{2J_2}{T}\right). \tag{2.6}$$

In Eq. (2.5) we have set an irrelevant factor to unity. Since the nonaveraged operator $\hat{T}_i$ in Eq. (2.7) is random, the representation in Eq. (2.4) is in fact inappropriate for computing the partition function. In order to get a more convenient starting point for further calculations we apply the replica trick. We introduce n identical ''replicas'' of the original model labeled by the index α, $\alpha=1,\ldots,n$ and use the well-known identity for the averaged free energy:

$$\overline{F}=-T\overline{\ln Z}=-T\lim_{n\to 0}\frac{1}{n}\overline{(Z^n-1)}. \tag{2.7}$$

Substituting Eq. (2.4) into Eq. (2.7) one obtains

$$\overline{F}=-T\lim_{n\to 0}\left\{\overline{\mathrm{Tr}\prod_{\alpha=1}^{n}\prod_{i=1}^{N}\hat{T}_i^\alpha-1}\right\}\frac{1}{n}. \tag{2.8}$$

In contrast to the case of random-site disorder, for the random-bond problem the two matrices $\hat{T}_i^\alpha$ and $\hat{T}_j^\beta$ with different row indices $i\neq j$ depend on two different sets of random coupling constants and commute to each other for any α and β. This allows us to average these two operators independently. After some algebra one arrives at

$$\overline{Z^n}=\mathrm{Tr}\hat{T}^N, \tag{2.9}$$

where the tranfer matrix $\hat{T}$ of the 2D random-bond IM is given by[2]

$$\hat{T}=\overline{\prod_{\alpha=1}^{n}\hat{T}_i^\alpha}=\exp\left\{\sum_{j=1}^{N}\ln\left[(1-p)\exp\left(\frac{J}{T}\sum_{\alpha=1}^{n}\sigma_3^\alpha(j)\right)\sigma_3^\alpha(j+1)+p\exp\left(\frac{J'}{T}\sum_{\alpha=1}^{n}\sigma_3^\alpha(j)\sigma_3^\alpha(j+1)\right)\right]\right\}$$
$$\times\exp\left\{\sum_{j=1}^{N}\ln\left[(1-p)\exp\left(\frac{J^*}{T}\sum_{\alpha=1}^{n}\sigma_1^\alpha(j)\right)+p\exp\left(\frac{J^{*\prime}}{T}\sum_{\alpha=1}^{n}\sigma_1^\alpha(j)\right)\right]\right\}. \tag{2.10}$$

Setting p to zero (or $J=J'$) one is indeed led to the well-known expression for the T operator of the pure IM:[29]

$$\hat{T}_{\mathrm{PIM}}=\exp\left\{\frac{J}{T}\sum_{j=1}^{N}\sigma_3(j)\sigma_3(j+1)\right\}\exp\left\{\frac{J^*}{T}\sum_{j=1}^{N}\sigma_1(j)\right\}. \tag{2.11}$$

The T matrix is known to possess the Kramers-Wannier dual symmetry. In the language of spin variables this nonlocal mapping reads:[30,31]

$$\tau_1(k)=\sigma_3(k)\sigma_3(k+1)\qquad \tau_2(k)=i\sigma_1(k)\sigma_3(k)$$

$$\tau_3(k)=\prod_{m<k}\sigma_1(m), \tag{2.12}$$

where the operators $\tau_\alpha(k)$ satisfy the very same algebra as the Pauli spin matrices $\sigma_\alpha(n)$. It is easy to see that if $p=0,\frac{1}{2},1$ the T matrix given by Eq. (2.10) is invariant under the dual transformation. The plausible assumption that there is a single critical point yields the equation for the critical temperature T_c:

$$\exp\left(-\frac{2J'}{T_c}\right)=\tanh\left(\frac{J}{T_c}\right). \tag{2.13}$$

Notice that the point $p=\frac{1}{2}$ is not the percolation threshold, because the coupling constants J and J' are assumed to take nonzero values with the ferromagnetic sign. Writing $\hat{T}$ in the exponential form

$$\hat{T}=\exp(-\hat{H}), \tag{2.14}$$

one obtains the partition function in the following form:

$$Z=\mathrm{Tr}\,\exp(-N\hat{H}), \tag{2.15}$$

where by definition $\hat{H}$ is just the logarithm of the transfer matrix $\hat{T}$ (the ''quantum'' Hamiltonian). In the thermodynamic limit $N\to\infty$ the free energy is proportional to the lowest eigenvalue of the quantum Hamiltonian $\hat{H}$:[30,31]

$$F=-T\,\ln\mathrm{Tr}\,\exp(-N\hat{H})\to NTE_0,\quad \hat{H}|0\rangle=E_0|0\rangle \tag{2.16}$$

Here $|0\rangle$ is the ground state of $\hat{H}$ which is assumed to be nondegenerate. Actually this means that we assume $T>T_c$. From Eqs. (2.10) and (2.14) it follows that $\hat{H}$ is not a simple local operator. A crucial simplification occurs by taking the y-continuum limit with $a_y\to 0$ (the lattice spacing along the y-axis). In other words, after calculating the logarithmic derivative of $\hat{T}$ with respect to a_y and setting a_y to zero the quantum Hamiltonian takes on the following simple form (for details see Ref. 2):

$$\hat{H}=\left.\frac{d\ln\hat{T}}{da_y}\right|_{(a_y=0)}=-\sum_{j=1}^{N}\left\{K_1\sigma_3^\alpha(j)\sigma_3^\alpha(j+1)+K_2\sum_{\alpha=1}^{n}\sigma_1^\alpha(j)+K_4'[\sigma_3^\alpha(j)\sigma_3^\alpha(j+1)]^2+K_4''\left(\sum_{\alpha=1}^{n}\sigma_1^\alpha(j)\right)^2\right\}. \tag{2.17}$$

The higher-order terms in the spin operators are known to be irrelevant in the critical region, so that they can be dropped in Eq. (2.17). The replicated Hamiltonian, Eq. (2.17), may be converted into the fermionic one by means of the Jordan-Wigner transformation:[30,31]

$$c^{\alpha}(m)=\sigma_{-}^{\alpha}(m)\prod_{j=1}^{m-1}\sigma_{1}^{\alpha}(j)Q^{\alpha},$$

$$c^{\alpha^{\dagger}}(m)=\sigma_{+}^{\alpha}(m)\prod_{j=1}^{m-1}\sigma_{1}^{\alpha}(j)Q^{\alpha},\quad \sigma_{\pm}=\tfrac{1}{2}(\sigma_3\pm i\sigma_2),$$

$$Q^{\alpha}=\prod_{\beta=1}^{\alpha-1}\prod_{j=1}^{N}\sigma_{1}^{\beta}(j),\quad \alpha=1,\ldots,n, \tag{2.18}$$

where $c^{\alpha}(m)$ and $c^{\alpha^{\dagger}}(m)$ are the standard annihilation and creation fermionic operators which satisfy the canonical anticommutation relations:

$$\{c^{\alpha}(m),c^{\beta^{\dagger}}(n)\}=\delta^{\alpha\beta}\delta_{mn},\quad \{c^{\alpha}(m),c^{\beta}(n)\}=0. \tag{2.19}$$

After making different species anticommute, the Klein factors Q^{α} drop out of $\hat{H}$. For each species it is convenient to introduce a two-component Hermitean Majorana spinor field:[32,33]

$$\psi_1^{\alpha}(n)=\frac{1}{\sqrt{2a_x}}\left[c^{\alpha}(n)\exp\left(-i\frac{\pi}{4}\right)+c^{\alpha^{\dagger}}(n)\exp\left(i\frac{\pi}{4}\right)\right],$$

$$\psi_2^{\alpha}(n)=\frac{1}{\sqrt{2a_x}}\left[c^{\alpha}(n)\exp\left(i\frac{\pi}{4}\right)+c^{\alpha^{\dagger}}(n)\exp\left(-i\frac{\pi}{4}\right)\right], \tag{2.20}$$

with standard anticommutation rules

$$\{\psi_c^{\alpha}(n),\psi_b^{\beta}(m)\}=\frac{1}{a_x}\delta^{\alpha\beta}\delta_{bc}\delta_{mn},\quad c,b=1,2, \tag{2.21}$$

where a_x is the lattice spacing along the x axis. Using Eq. (2.21) and the relations

$$\sigma_1^{\alpha}(n)=2c^{\alpha^{\dagger}}(n)c^{\alpha}(n)-1,$$

$$\sigma_3^{\alpha}(n)\sigma_3^{\alpha}(n+1)=[c^{\alpha^{\dagger}}(n)-c^{\alpha}(n)]\times[c^{\alpha^{\dagger}}(n+1)+c^{\alpha}(n+1)], \tag{2.22}$$

one can easily rewrite the Hamiltonian, Eq. (2.17), in terms of Majorana fermionic fields. Now let us notice that in the vicinity of T_c the correlation length ξ goes to infinity and the system ''forgets'' the discrete nature of the lattice. For that reason we can simplify the Hamiltonian by taking the continuum limit $a_x\to 0$. Perfoming simple but cumbersome calculations we arrive at the O(n)-symmetric Lagrangian of the Gross-Neveu model,[1]

$$L=\int d^2x[i\bar{\psi}_a\hat{\partial}\psi_a+m_0\bar{\psi}_a\psi_a+u_0(\bar{\psi}_a\psi_a)^2], \tag{2.23}$$

where $\gamma_\mu=\sigma_\mu$, $\hat{\partial}=\gamma_\mu\partial_\mu$, $\mu=1,2$, $\bar{\psi}=\psi^T\gamma_0$, and

$$m_0\sim K_1-K_2\sim\tau=\frac{T-T_c}{T_c},\quad u_0\sim K_3+K_4. \tag{2.24}$$

Here m_0,u_0 are the bare mass of the fermions and their quartic coupling constant, respectively. Notice that if $p\ll 1$, $u_0\sim p$. Provided $p=\frac{1}{2}$ and $T=T_c$ we have $u_0\sim(J-J')^2$.

The RG calculations in the one-loop approximation are very simple. In fact, the O(n)-symmetric Gross-Neveu model being infrared free in the replica limit $n\to 0$, the one-loop approximation truly captures the essentials of the critical behavior of the model under consideration. The one-loop RG equations and initial conditions are given by

$$\frac{du}{dt}=\beta(u)=-\frac{(n-2)u^2}{\pi},\quad \frac{d\ln F}{dt}=-\gamma_{\bar{\psi}\psi}(u)=\frac{(1-n)u}{\pi},$$

$$u(t=0)=u_0,\quad F(t=0)=1, \tag{2.25}$$

where u is the dimensionless quartic coupling constant, $\beta(u)$ is the Gell-Mann-Low function, $\gamma_{\bar{\psi}\psi}(u)$ is the anomalous dimension of the composite operator $\bar{\psi}\psi=\epsilon(x)$ (in fact, the energy density operator), $t=\ln(\Lambda/m)$, $\Lambda=a^{-1}$ is an ultraviolet cutoff, and a and m are the lattice spacing and renormalized mass, respectively. Here F is the following Green's function at zero external momenta:

$$F=\frac{dm}{d\tau}=\int d^2xd^2y\langle\overline{\psi(x)}\psi(y)\overline{\psi(0)}\psi(0)\rangle. \tag{2.26}$$

The solution of these equations gives the temperature dependence of the correlation length ξ and specific heat C in the asymptotic region $t\to\infty,n=0$:[1]

$$u=\frac{\pi}{2t},\quad F\sim\tau^{-\frac{1}{2}},\quad \xi=m^{-1}\sim\tau^{-1}\left[\ln\frac{1}{\tau}\right]^{1/2},$$

$$C\sim\int dtF(t)^2\sim\ln\ln\frac{1}{\tau}. \tag{2.27}$$

These results follow from the solution of the one-loop RG equations, Eq. (2.25), but in fact it is worth noticing that they are a direct consequence of a renormalization statement valid to all orders in perturbation theory. Consider a version of the field theory, Eq. (2.23), in which the quartic term is decoupled by the introduction of a scalar Hubbard-Stratonovich field ϕ:

$$L=\int d^2x[\bar{\psi}_a(i\hat{\partial}+m_0)\psi_a+\tfrac{1}{2}\phi^2+\tfrac{1}{2}g_0\phi\bar{\psi}_a\psi_a], \tag{2.28}$$

with $g_0\propto\sqrt{u_0}$. As a consequence of the functional version of the classical equation of motion,[34]

$$\frac{\delta L}{\delta\phi}=\phi+\frac{1}{2}g_0\bar{\psi}_a\psi_a=0, \tag{2.29}$$

the vertex parts Γ of the correlation functions $G_{ab}^{(2,0;1)}=\langle\psi_a(x)\psi_b(y)\tfrac{1}{2}\bar{\psi}_c(z)\psi_c(z)\rangle$ and $G_{ab}^{(2,1)}=\langle\psi_a(x)\psi_b(y)\phi(z)\rangle$ are linked by the relationship

$$\Gamma^{(2,1)}=-g_0\Gamma^{(2,0;1)}_{\bar{\psi}\psi}, \tag{2.30}$$

where it has been indicated explicitly that the quadratic insertion refers to the $O_1=\bar{\psi}_a\psi_a$ operator. Imposing the renormalization conditions, Eq. (2.30) leads to[34]

$$g_0/g=\sqrt{u_0/u}=Z_\phi^{-1/2}Z_{11}, \tag{2.31}$$

where g is the renormalized coupling constant, Z_ϕ the ϕ-field renormalization constant, and Z_{ij} is the quadratic-insertion renormalization matrix for the operators $\{O_i\}=\{\bar{\psi}\psi,\phi^2\}$. Since for $n=0$ we have $Z_\phi=1$, Eq. (2.31) leads to the exact result $\beta^{(0)}(u)=-2u\gamma^{(0)}_{\bar{\psi}\psi}(u)$ between the Gell-Mann-Low and the anomalous dimension functions, implying $2\gamma_1/\beta_2=-1$ for the coefficients of the lowest-order nonzero terms in the expansion of these functions in u [that is, $\beta(u)=\beta_2u^2+\cdots$, $\gamma_{\bar{\psi}\psi}(u)=\gamma_1u+\cdots$]. Solving, for instance, the RG equation for the specific heat function leads to the remarkable Dotsenko-Dotsenko result for the leading asymptotic behavior, when $\tau\to 0$,

$$C\sim\int^{\tau}\frac{dx}{x}|\ln x|^{2\gamma_1/\beta_2}\sim\ln\ln\tau, \tag{2.32}$$

by virtue of the above $n=0$ exact results. Similar considerations lead to the announced behavior of the correlation length, ξ.

The main conclusion of this section is that the critical behavior of the 2D random bond IM is governed by the pure Ising fixed point. It implies that all critical exponents of the weakly disordered system are the very same as for the pure model. Randomness gives rise to the self-interaction of the spinor field which leads to logarithmic corrections to power laws. In the special case $p=\frac{1}{2}$ duality imposes strong restrictions; in particular it gives the exact value of the critical temperature T_c which is believed to be unique. At the critical point the original lattice model and its continuum version described by the Gross-Neveu model Lagrangian become massless, irrespective of the value of n. We conjecture that there are only two phases divided by the single critical point given by the self-duality equation, Eq. (2.13). It implies that under this assumption the Griffiths phase shrinks to zero.

B. Two-spin correlation function at criticality

In order to complete the calculation of the temperature dependence of other thermodynamic quantities we have to compute the susceptibility and spontaneous magnetization near T_c. For these calculations we need to find the large-distance asymptotic behavior of the two-spin correlation function at criticality. The most effective way for calculating different correlation functions for the 2D IM is to use bosonization. Below we shall give a brief description of this procedure, exploiting simple physical arguments.

Before recalling the principles of the bosonization method, however, let us show how a straight formulation of the problem in terms of pure fermionic fields leads to some difficulties even in the case of the calculation of the pure Ising model correlation function exponent η $(=\frac{1}{4})$ at criticality. As shown, e.g., by Samuel,[35] the two-spin correlation function can be expressed in the lattice formulation as the partition function of a defective lattice where along the line T_{0R} of bonds joining the two sites (0,0) and (0,R) the "bond strengths" $\lambda_y\equiv\tanh(J_2/T)$ must be replaced by λ_y^{-1}. Namely,

$$G_y(R)=\langle s_{00}s_{0R}\rangle =\lambda_y^R\left\langle\exp\left\{-(\lambda_y-\lambda_y^{-1})\sum_{ij\in T_{0R}}y^\dagger_{ij}y_{ij+1}\right\}\right\rangle, \tag{2.33}$$

where the lattice (y) Grassmann variables $\{y^\dagger_{ij},y_{ij}\}$ have been introduced.[35,36] After suitable transformations, leading to the quadratic term of the effective Grassmann action in Eq. (2.23) without replicas, and in the continuum limit, the Eq. (2.33) reads

$$G_y(R,T_c)=\lambda_{yc}^R\left\langle\exp iT_0\int_0^R dy\,\bar{\psi}(0,y)\,\psi(0,y)\right\rangle, \tag{2.34}$$

with $T_0=(\lambda_c^{-1}-\lambda_c)/2\lambda_c=\sqrt{2}+1$ at criticality ($\lambda_c=\sqrt{2}-1$ for the isotropic model). The two-component Grassmann (or Majorana) field is the same as in Eq. (2.23) and is given by $\psi=a^{-1}(y^\dagger y)$. A possible strategy[1] is now to evaluate the $R\to\infty$ behavior of $\ln G_y(R,T_c)$ through an expansion in powers of T_0. Use must be made of the propagator ($\hat{x}=x_\mu\gamma_\mu$)

$$S_0(x-x')=\langle\bar{\psi}(x)\,\psi(x')\rangle_0=\frac{i}{2\pi}[\hat{x}-\hat{x}']^{-1}f_\Lambda(x-x'), \tag{2.35}$$

where f_Λ is some cutoff function. The typical term in the expansion for $\ln G(R)$ involves the multiple integral

$$\mathcal{I}_{2n}(R)=\int_0^R dy_1dy_2\cdots dy_{2n}\mathrm{Tr}[S_0(y_1-y_2)S_0(y_2-y_3)\cdots S_0(y_{2n}-y_1)]=a_nR+b_n\ln R+\cdots, \tag{2.36}$$

from which the $R\to\infty$ critical correlator could be evaluated through

$$\ln G(R)=-\sum_{n=1}^{\infty}\frac{(2T_0)^{2n}}{4n}\mathcal{I}_{2n}(R)+R\ln\lambda_c \tag{2.37}$$

(the odd-valued power terms vanishing). Taking the (conjectural) point of view that all terms in R must cancel exactly, the evaluation of the $\ln R$ terms can proceed[1] by taking the choice (natural, but leading to some ambiguities) $f_\Lambda=1$ and evaluating every other y integral exactly,

$$I_{2n}(R)=\int_0^R \frac{dy_1 dy_2\cdots dy_{2n}}{(y_1-y_2)(y_2-y_3)\cdots(y_{2n}-y_1)}$$
$$=\int_0^R dy_1 dy_2\cdots dy_n \frac{\ln[(1-R/y_2)/(1-R/y_1)]\ln[(1-R/y_3)/(1-R/y_2)]\cdots\ln[(1-R/y_1)/(1-R/y_n)]}{(y_1-y_2)(y_2-y_3)\cdots(y_n-y_1)}. \quad (2.38)$$

After a straightforward but laborious reparametrization of the integral,[1] we arrive at

$$\ln G(R)=R\ln\lambda_c-\sum_{n=1}^{\infty}\frac{(-T_0^2/\pi^2)^n}{2n}I_{2n}(R),\quad I_{2n}(R)=\int_{-\infty}^{+\infty}dz_1 dz_2\cdots dz_{n-1}\frac{\Sigma_i z_i/2}{\sinh\Sigma_i z_i/2}\prod_i \frac{z_i/2}{\sinh z_i/2}\int_{\Lambda^{-1}}^{R}\frac{R\,dx}{x(R-x)}=2\theta_n\ln R\Lambda, \quad (2.39)$$

with the R dependence now neatly factorized out and the cutoff $\Lambda\sim a^{-1}$ conveniently reinstated. The coefficient θ_n is evaluated through the Fourier representation

$$\frac{z/2}{\sinh(z/2)}=\int_{-\infty}^{+\infty}\frac{dp}{2\pi}F(p)e^{-ipz},\quad F(p)=\frac{\pi^2}{\cosh^2\pi p}, \quad (2.40)$$

leading to $\theta_n=(1/2\pi)\int_{-\infty}^{+\infty}dp[F(p)]^n$. Finally, we get (dropping the R terms)

$$\ln G(R)=-\sum_{n=1}^{\infty}\frac{\theta_n}{n}\left(\frac{T_0}{\pi}\right)^{2n}\ln R\Lambda=-\eta\,\ln R\Lambda, \quad (2.41)$$

where

$$\eta=\sum_{n=1}^{\infty}\frac{\theta_n}{n}\left(\frac{T_0}{n}\right)^2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dp\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{T_0^2}{\cosh^2\pi p}\right]^n. \quad (2.42)$$

The last sum converges to a logarithm and the p integral can be evaluated, provided $|T_0|<1$. For $T_0=1$, Eq. (2.42) leads to $\eta=1/4$;[1] however, the standard prescription[35] calls for $T_0=\sqrt{2}+1$ and this leads to a divergence in the summation. Clearly, this is associated with the use of a uniform cutoff function $f_\Lambda=1$, but it must be stressed that to date no further progress in evaluating the spin-spin correlator at criticality, using solely the fermionic formalism, can be reported. The situation is even more delicate when disorder is introduced; thus the method of the fermionic tail T_{0R} must be abandoned.

Let us now begin discussing bosonization, with the action

$$L=\int d^2x\{i\bar{\psi}\hat{\partial}\psi+[m_0+\tau(x)]\bar{\psi}\psi\}, \quad (2.43)$$

where ψ is a Majorana spinor and $\tau(x)$ is a random Gaussian field with the following probability distribution:

$$P[\tau(x)]\sim\exp\left\{-\frac{1}{2u_0}\int d^2x[\tau(x)]^2\right\},$$
$$\langle\tau(x)\tau(y)\rangle=u_0\delta(x-y). \quad (2.44)$$

In fact, the action, Eq. (2.43), describes free fermions moving in the random potential $\tau(x)$, which in our case is responsible for local fluctuations of the critical temperature T_c in the dilute ferromagnet. After applying the replica trick and averaging over ''all'' possible configurations of $\tau(x)$ one gets the very same Gross-Neveu Lagrangian as given by Eq. (2.23). The representation of the square of the spin-spin correlation function of the pure 2D IM, that is,

$$G(x-y)=\langle\sigma(x)\sigma(y)\rangle, \quad (2.45)$$

in terms of the path integral over the real bosonic field ϕ of quantum sine-Gordon model was found by Zuber and Itzykson[33] (see also Ref. 37) and reads

$$G(x-y)^2=Z^{-1}\frac{1}{2\pi^2a^2}\int D\phi\sin[\sqrt{4\pi}\phi(x)]\times\sin[\sqrt{4\pi}\phi(y)]\exp\{-S\},$$
$$S=\frac{1}{2}\int d^2x\left\{(\partial_\mu\phi)^2+\frac{2m_0}{\pi a}\cos(\sqrt{4\pi}\phi)\right\},$$
$$Z=\int D\phi\exp\{-S\}. \quad (2.46)$$

At criticality, $m_0=0$, the path integral being Gaussian, the result of its evaluation is easily seen to be

$$G(x-y)\sim|x-y|^{-(1/4)}. \quad (2.47)$$

The representation for the two-spin correlation function may be extended to the dilute system by replacing the bare mass $m_0\sim\tau$ with the random one $m_0+\tau(x)$ into Eq. (2.46). Of course, in the inhomogeneous case the nonaveraged $G(x,y)$, being sample dependent, depends on x and y separately. The averaged correlation function $\overline{G(x-y)}$ at the critical point may be computed (even without using the replica trick) in two stages: (i) First, the square root of $G(x,y)^2$ is formally evaluated by means of expanding it in a power series in $\tau(x)$; (ii) second, the resulting expression is integrated with respect to $\tau(x)$ (for technical details of the calculations see Refs. 2,38). The conventional RG equation for the renormalized averaged correlation function reads

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(u)\frac{\partial}{\partial u}+\eta(u)\right\}\overline{G_R(p,u,\mu)}=0, \quad (2.48)$$

where μ is a renormalization momentum, $\beta(u)$ is the beta function, and $\eta(u)$ is defined as

$$\eta(u)=\beta(u)\frac{d\ \ln Z_\sigma(u)}{du}. \quad (2.49)$$

The spin renormalization constant $Z_\sigma(u)$ and the renormalized correlation function are defined in the standard way:

$$\overline{G(p,u_0,\Lambda)}=Z_\sigma(u)\overline{G_R(p,u,\mu)}. \quad (2.50)$$

The Kramers-Wannier symmetry was shown to apply in some vanishing terms linear in u in the expansions for $\eta(u)$ and $Z_\sigma(u)$,[2] that is,

$$Z_\sigma(u)=1+O(u^2), \quad \eta(u)=\frac{7}{4}+O(u^2). \quad (2.51)$$

Given $\beta(u)$ and $\eta(u)$ in the one-loop approximation, the solution of the Ovsyannikov-Callan-Symanzik equation for the correlation function is quite simple:

$$G(p)\sim p^{-7/4}, \quad G(R)\sim R^{-1/4}. \quad (2.52)$$

So the Fisher critical exponent takes the very same value $\eta=\frac{1}{4}$ as in the pure model. Notice that in contrast to higher moments of the spin correlation function, the first one does not contain the logarithmic factor due to the above-mentioned dual symmetry.[18] From this remark it follows that the temperature dependence of the homogeneous susceptibility and spontaneous magnetization are described by power-law functions of the correlation length ξ (without logarithmic corrections like $\ln\xi$):

$$\chi\sim\xi^{2-\eta}\sim\tau^{-7/4}\left[\ln\frac{1}{\tau}\right]^{7/8}, \quad M\sim\xi^{\eta/2}\sim(-\tau)^{1/8}\left[\ln\frac{1}{(-\tau)}\right]^{1/16}. \quad (2.53)$$

The equation of state at the critical point may be obtained from the usual scaling relation

$$H\sim M^{(4+\eta/\eta)}\sim M^{15}. \quad (2.54)$$

As we predicted, all critical exponents of the quenched disordered system are identical to those of the pure model, apart from some logarithmic corrections.[15]

III. *N*-COLOR ASHKIN-TELLER MODEL

The N-color Ashkin-Teller model (ATM) was introduced by Grest and Widom[39] and consists of a system of N 2D Ising models coupled together like in the conventional two-color model. The lattice Hamiltonian of the isotropic N-color ATM reads

$$H=\sum_{a=1}^{N}H_I(s^a)+J_4\sum_{a\neq b=1}^{N}\sum_{\langle nn\rangle}\epsilon_a\epsilon_b$$
$$=-\sum_{\langle nn\rangle}\left\{J\sum_{a=1}^{N}s_i^a s_j^a+J_4\left[\sum_{a=1}^{N}s_i^a s_j^a\right]^2\right\}, \quad (3.1)$$

where $s^a=\pm1$, $a=1,\dots,N$, $\langle\rangle$ indicates that the summation is over all nearest-neighboring sites, $H_I(s^a)$ is the Hamiltonian of the pure 2D IM, $\epsilon_a=s_i^a s_j^a$ is the density energy operator, and J_4 is a coupling constant between the Ising planes.

This model was shown to be the lattice version of a model with hypercubic anisotropy, describing a set of magnetic and structural phase transitions in variety of solids.[24,40] The corresponding Landau Hamiltonian reads

$$H=\int d^2x\left\{\frac{1}{2}(\partial_\mu\Phi)^2+\frac{1}{2}m_0^2\Phi^2+\frac{1}{8}u_0(\Phi^2)^2+\frac{1}{8}v_0\sum_{a=1}^{N}\Phi_a^4\right\},$$

$$\Phi^2=\sum_{a=1}^{N}\Phi_a^2,(\partial_\mu\Phi)^2=\sum_{a=1}^{N}(\partial_\mu\Phi_a)^2, \quad (3.2)$$

where Φ is an N-component order parameter, $m_0^2\sim\tau$, $u_0\sim J_4$, and v_0 are some coupling constants. In particular, in the replica limit the Hamiltonian, Eq. (3.2), describes the random-bond IM (for $v_0>0,u_0<0$). If $v_0=0$, a phase transition in the O(N)-symmetric model with nonzero value of the spontaneous magnetization is known to be forbidden by the Mermin-Wagner theorem.[41] If $v_0\neq0$, the spontaneous breakdown of the discrete hypercubic symmetry occurs at $T_c>0$. Since the term with v_0 is strongly relevant, the perturbation theory expansion with respect to v_0 is actually hopeless near T_c.

By exploiting the operator product expansion (OPE) approach, Grest and Widom obtained the one-loop β function for the quartic coupling constant J_4. If $J_4<0$ and $N>2$, the phase transition was shown to be continuous and the critical behavior belonging to the 2D IM universality class.[39]

The exact solution of the multi color ATM in the large-N limit was found by Fradkin,[42] who developed a rather complicated formalism based on bosonic fields and showed that a second-order phase transition with IM critical exponents occurs if $J_4<0$. In fact, as was shown by Aharony,[43] the model with hypercubic anisotropy, Eq. (3.2), in the large-N limit is equivalent to the IM with equilibrium impurities. Moreover, for the 2D case he predicted the Ising-type critical behavior with logarithmic corrections. Such being the case, one expects that the critical behavior is identical to the IM critical behavior. Since $\alpha=0$, Fisher's renormalization of the critical exponents[44] is inessential and gives rise only to logarithmic factors. Notice also that in contrast to the pure case the specific heat C is finite at T_c. The exact solution of the 2D IM with equilibrium defects obtained by Lushnikov[45] many years ago confirms these conclusions.

The effective method for solving the model under discussion, based on a mapping of the original model, Eq. (3.1), onto the O(N)-symmetric Gross-Neveu model, was suggested by Shankar[25] (see also Ref. 26,27). In order to show this equivalence let us transform the partition function Z by applying the Hubbard-Stratonovich identity:

$$Z=\int D\Phi\exp[-H(\Phi)]$$
$$=\int D\Phi D\lambda\exp\left\{-\int d^2x\left[\frac{1}{2}(\partial_\mu\Phi)^2+\frac{1}{2}m_0^2\Phi^2+\frac{1}{8}v_0\sum_{a=1}^{N}\Phi_a^4+i\lambda(x)\Phi^2+\frac{1}{2u_0}[\lambda(x)]^2\right]\right\}$$
$$=\int D\lambda\exp\left(-\frac{1}{2u_0}\int d^2x\lambda^2\right)\{Z_I[m_0^2+i\lambda(x)]\}^N, \quad (3.3)$$

where $\lambda(x)$ is an auxiliary field; Z_I is the exact partition function of the 2D IM which is known to correspond to a path integral over Grassmann variables (Sec. II):

$$Z_I=\int D\overline{\psi}D\psi\ \exp\left\{-\int d^2x[i\overline{\psi}\hat{\partial}\psi+\kappa_0\overline{\psi}\psi]\right\}. \quad (3.4)$$

Now let us replace $\kappa_0=m_0^2$ in Eq. (3.4) by $m_0^2+i\lambda(x)$ and substitute Eq. (3.4) into Eq. (3.3). This replacement is based on the fact that the energy-density operator in the ϕ^4 theory is ϕ^2 while in 2D fermionic models this is given by $\overline{\psi}\psi$. We have:[25,27]

$$Z=\int D\lambda\exp\left(-\frac{1}{2u_0}\int d^2x\lambda^2\right)\int\prod_{a=1}^{N}D\overline{\psi_a}D\psi_a\times\exp\left\{-\int d^2x[i\overline{\psi_a}\hat{\partial}\psi_a+(m_0^2+i\lambda(x))\overline{\psi_a}\psi_a]\right\}$$
$$=\int\prod_{a=1}^{n}D\overline{\psi_a}D\psi_a\exp(-S_{\rm GN}), \quad (3.5)$$

where $S_{\rm GN}$ is the Gross-Neveu action, given by Eq. (2.23). In going from Eq. (3.2) to Eq. (3.5) it is assumed that u_0 has been rescaled as $u_0\to u_0{}'=u_0a^{-2}$ so as to make u_0 dimensionless (the prime will be ignored hereafter). We see that the discrete hypercubic symmetry of the N-color ATM evolves into the continuous O(N) symmetry, hidden when the system approaches the critical point.

The one-loop RG equations for the N-color ATM have been already obtained in Sec. II, these being Eq. (2.25) where we must set $n=N$. Solving these equations gives the temperature dependence of the correlation length and specific heat in the vicinity of the critical point:[27]

$$\xi\sim\tau^{-1}\left[\ln\left(\frac{1}{\tau}\right)\right]^{(N-1/N-2)}, \quad C\sim\left[\ln\left(\frac{1}{\tau}\right)\right]^{(N/2-N)}. \quad (3.6)$$

As for the calculation of the correlation function, one can apply the procedure described in Sec. II. Like for the random IM case, the term linear in u for $\eta(u)$ and $Z_\sigma(u)$ vanishes due to the Kramers-Wannier symmetry. This implies the anomalous dimension of the spin s_α to be equal to $\frac{1}{4}$. We get

$$G(R)\sim R^{-1/4}, \quad \chi\sim\tau^{-7/4}\left[\ln\frac{1}{\tau}\right]^{7(N-1)/4(N-2)},$$
$$M\sim(-\tau)^{1/8}\left[\ln\frac{1}{(-\tau)}\right]^{(N-1)/8(N-2)}, \quad H\sim M^{15}. \quad (3.7)$$

Notice that these results are valid only for $N\geq 2$, $J_4\leq 0$. If $J_4>0$, the discrete γ_5 symmetry $\psi\to\gamma_5\psi,\overline{\psi}\psi\to-\overline{\psi}\psi$ is spontaneously broken. From the γ_5-symmetry breaking it follows that $\langle\overline{\psi}\psi\rangle\neq 0$. It means that we have a finite correlation length or, in other words, a first-order phase transition.[39,25] So Eqs. (3.7) reproduce the well-known results for some particular cases: $N=0,1,\infty$ corresponding to the random-bond IM problem, Onsager problem, and IM with equilibrium impurities, respectively.

The symmetric eight-vertex model (or Baxter model) is known to be isomorphic to the $N=2$ color ATM in the vicinity of the critical line. The phase diagram of the two-color ATM contains the ferromagnetic phase transition line beginning from the IM critical point and ending at the point corresponding to the four-state Potts model. Along this line the model exhibits weakly universal critical behavior, with the critical exponents continuously varying. For instance, the critical exponent α changes continuously from $\alpha=0$ (IM) to $\alpha=\frac{2}{3}$ (four-state Potts model[29]). The above results obviously show the special nature of the $N=2$ situation, due to the factor $1/(N-2)$. In this case the system under discussion is described by the O(2)-symmetric Gross-Neveu model or, equivalently, by the massive Thirring model with the β function being equal to zero identically and presenting nonuniversal critical exponents.[25] Since the $N=3$ color ATM is equivalent to the O(3)-symmetric Gross-Neveu model which is known to be supersymmetric,[46] this model should possess a hidden supersymmetry (see for details Refs. 25,26).

Notice that in contrast to the 2D case, the critical behavior of the N-color ATM in $4-\epsilon$ dimensions ($0<\epsilon\leqslant 2$) is governed by either the Gaussian or the cubic fixed point and never by the IM fixed point. The type of critical behavior crucially depends on the order parameter component number N. If $N>N_c(\epsilon)$, the RG flow arrives at the cubic fixed point; in the opposite case, $N<N_c(\epsilon)$, the Heisenberg (isotropic) fixed point is stable. Here $N_c(\epsilon)$ is the critical dimensionality of the order parameter, its expansion in powers of ϵ being:[47]

$$N_c(\epsilon)=4-2\epsilon-\left(\frac{5}{2}\zeta(3)-\frac{5}{12}\right)\epsilon^2+O(\epsilon^3), \quad (3.8)$$

where $\zeta(3)=1.202\,052\,8$ is the Riemann zeta function, and $N_c(1)\cong 2.9$.[48] If $\epsilon\to 2$, N_c decreases and all the cubic fixed points approach the IM fixed point, merging at $\epsilon=2$, irrespectively of the value of N.[27]

IV. GENERALIZED ASHKIN-TELLER MODEL WITH RANDOMNESS

Now we extend our study of the N-color ATM to two interacting M- and N-color quenched disordered Ashkin-Teller models, giving rise to a generalized Askin-Teller model (GATM). The Landau Hamiltonian is given by

$$H=\int d^2x\left\{\frac{1}{2}(\partial_\mu\Phi)^2+\frac{1}{2}[m_0^2+\tau_1(x)]\Phi_a^2+\frac{1}{2}[m_0^2+\tau_2(x)]\Phi_c^2+\frac{1}{8}u_1(\Phi_a^2)^2+\frac{1}{8}u_2(\Phi_c^2)^2+\frac{1}{8}w_0\Phi_a^2\Phi_c^2+\frac{1}{8}v_1\sum_{a=1}^{N}\Phi_a^4+\frac{1}{8}v_2\sum_{c=N+1}^{N+M}\Phi_c^4\right\}, \quad (4.1)$$

where $\Phi_k, k=1,\ldots,M+N$ is an $(M+N)$-component order parameter, $a=1,\ldots,N, c=N+1,\ldots,N+M$, $m_0^2\sim\tau$, $v_\mu, u_\nu>0$, and $\mu,\nu=1,2$. Summation over indices in the quadratic operators is understood. We may study two types of impurities: (i) uncorrelated impurities and (ii) correlated ones. In these cases the two-point correlators for the independent random Gaussian fields τ_μ read

$$\langle\tau_\mu(x)\tau_\nu(y)\rangle=z_\mu\delta_{\mu\nu}\delta(x-y),$$

$$\langle\tau_\mu(x)\tau_\nu(y)\rangle=z_0\delta(x-y). \quad (4.2)$$

In fact we study some multicritical point in the model under discussion, Eq. (4.1), since it has been assumed that $m_{10}=m_{20}=m_0\sim\tau$. This model in $4-\epsilon$ dimensions (without disorder) was initially studied by Bruce and Aharony[49] and by Lyuksyutov, Pokrovskii, and Khmelnitskii[50] (without cubic anisotropy) and then in numerous other papers.[24] By applying the replica trick and the ''fermionization'' method described in Sec. III, one arrives at the following effective fermionic action involving several types of quartic fermionic interactions:

$$H=\int d^2x\{i\overline{\Psi}_k^\alpha\hat{\partial}\Psi_k^\alpha+m_0\overline{\Psi}_k^\alpha\Psi_k^\alpha+u_1\overline{\Psi}_a^\alpha\Psi_a^\alpha\overline{\Psi}_b^\alpha\Psi_b^\alpha+u_2\overline{\Psi}_c^\alpha\Psi_c^\alpha\overline{\Psi}_d^\alpha\Psi_d^\alpha+w_0\overline{\Psi}_a^\alpha\Psi_a^\alpha\overline{\Psi}_c^\alpha\Psi_c^\alpha+z_1\overline{\Psi}_a^\alpha\Psi_a^\alpha\overline{\Psi}_b^\beta\Psi_b^\beta+z_2\overline{\Psi}_c^\alpha\Psi_c^\alpha\overline{\Psi}_d^\beta\Psi_d^\beta+r_0\overline{\Psi}_a^\alpha\Psi_a^\alpha\overline{\Psi}_c^\beta\Psi_c^\beta\}, \quad (4.3)$$

where Ψ_k^α is a (real) Majorana fermionic field, $\alpha,\beta=1,\ldots,n\to0$ are replica indices, $a,b=1,\ldots,N$, and $c,d=N+1,\ldots,N+M$. Naively one expects the appearance of two impurity quartic fermionic couplings in the replicated Hamiltonian, Eq. (4.3). Instead, we have one additional four-fermion vertex r_0 (absent in the bare action). This counterterm arises in the course of the renormalization procedure and provides the closedness of the operator algebra. In some sense the appearance of this term means violating the Harris criterion. The latter is indeed essentially based on the assumption of the existence of only one operator responsible for the impurity-induced interaction of the order parameter fluctuations.

The one-loop RG equations for the six coupling constants u_μ, v_ν, r, and w are given by (for $n=0$)

$$\frac{du_1}{dt}=-(N-2)u_1^2-2z_1u_1-Mw^2, \quad (4.4)$$

$$\frac{du_2}{dt}=-(M-2)u_2^2-2z_2u_2-Nw^2,$$

$$\frac{dw}{dt}=-w[(N-1)u_1+(M-1)u_2+z_1+z_2],$$

$$\frac{dz_1}{dt}=-2z_1[z_1+(N-1)u_1+2Mr],$$

$$\frac{dz_2}{dt}=-2z_2[z_2+(M-1)u_2+2Nr],$$

$$\frac{dr}{dt}=-r[(N-1)u_1+(M-1)u_2+z_1+z_2]-w[Nz_1+Mz_2].$$

The initial conditions for both (i) uncorrelated and (ii) correlated impurities are as follows:

$$\text{(i)}\quad z_1(0)=z_2(0)=z_0, r(0)=0,$$

$$\text{(ii)}\quad z_1(0)=z_2(0)=2r(0)=z_0. \quad (4.5)$$

It is easy to see that if one sets $M=N=1$ (random-bond Baxter or, equivalently, two-color ATM) one arrives at the RG equations obtained by Dotsenko and Dotsenko.[51] In this case the coupling constants u_μ decouple from the others; moreover, instead of two couplings we have only one coupling constant $z=z_1=z_2$. It was shown that in almost all cases even weak disorder would drastically change the critical behavior of the two-color ATM from a nonuniversal behavior to the Ising-type one, modified by some logarithmic corrections (see also Ref. 2). Even though the critical exponent α of the pure model is negative for $w_0<0$, for uncorrelated defects we find that the critical behavior of the model under discussion is changed by the emergence of the new scaling field r. In the case of correlated defects with $w_0<0$ the critical behavior of the random model was shown[51] to be still nonuniversal with critical exponents α and ν depending on both w_0 and on the concentration of impurities (r_0). This is the only exceptional case in which we would have nonuniversal critical behavior for a disordered system. In all cases the two-spin correlation function was shown to have, however, the same large distance behavior as for the 2D IM.[2]

Now let us consider two interacting N- and M-color ATM without randomness $(z_\mu=r_0=0)$. There are several different types of asymptotic behavior of the coupling constants $u_\mu(t), w(t)$, but there is only one stable solution exhibiting infrared-free behavior. That is given by

$$u_1(t)=\frac{1}{(N-2)t},\quad u_2(t)=\frac{1}{(M-2)t},$$

$$w(t)=O\left(\frac{1}{t\ln t}\right),\quad t\to\infty. \quad (4.6)$$

As a result, the original model decouples into two independent N- and M-color models as described in Sec. III. Thus, the hidden symmetry of the model near the critical point is the continuous $O(N)\times O(M)$ group. There also exists a solution of the RG equations given by

$$u_1(t)=u_2(t)=\pm\frac{1}{2}w(t)=\frac{1}{(N+M-2)t},\quad t\to\infty, \quad (4.7)$$

corresponding to the higher symmetry $O(M+N)$ being explicitly broken in the original Landau Hamiltonian, Eq. (4.1). This is shown to be unstable. For instance, provided $N=2$, $M=1$ (or vice versa), and $u_1(0)=u_2(0)=\frac{1}{2}w(0)$, we would obtain the supersymmetric asymptotic solution of Eq. (4.7). Were these conditions to be broken, i.e., were the supersymmetry explicitly broken, this would not be restored in the infrared limit.[26] Notice that our model without cubic anisotropy and randomness was shown to exhibit this enhanced asymptotic symmetry in $(4-\epsilon)$ dimensional space, provided $M+N<4$.[50]

One may expect that, due to the critical decoupling of two multicolor ATM's into two independent models, the quenched disorder does not affect the critical behavior of the system, Eq. (4.1). This is because if $N,M>2$ the specific heat is finite at criticality [Eq. (3.5)] and randomness is irrelevant in accordance with the Harris criterion. As was explained above this reasonable assumption should be checked in view of the obvious breakdown of the Harris criterion due to the appearance of the additional scaling field r. The answer is that this is indeed the case. In fact, it is easy to check that the solution given by Eq. (4.6) and describing pure models is stable despite the presence of three disorder couplings.

Thus, from our RG calculations it follows that, in contrast to a 2D IM with random bonds, weak quenched disorder here is irrelevant near T_c. Moreover, in the critical region the decoupling of two interacting multicolor ATM's was found to occur even in the presence of quenched disorder. The temperature dependence of the main thermodynamic quantities near the critical point, the two-spin correlation function and equation of state at criticality of the model under consideration are given by Eqs. (3.5) and (3.7).

V. WEAKLY DISORDERED MINIMAL CONFORMAL FIELD THEORY MODELS

The critical behavior of the minimal models of conformal field theory with $c<1$ and as perturbed by a small amount of impurities is far from being solved and therefore is of considerable interest. In accordance with the Harris criterion, weak quenched disorder is expected to be strongly relevant near criticality since the critical exponent α of these models is always positive and given by $\alpha=2(m-3)/3(m-1)$, with $m=3,4,\ldots$. In particular, for the three- and four-state Potts model we have $\alpha=\frac{1}{3}$ $(m=5)$ and $\alpha=\frac{2}{3}$ $(m=\infty)$, respectively.

Results in this field were obtained in some papers by Ludwig[19] and by Dotsenko, Picco, and Pujol.[21] They succeeded in developing a powerful approach closely connected with the formalism exploited in the previous sections for describing the multicolor ATM. These authors suggested a special kind of ϵ expansion for computing the critical exponents, where now $\epsilon=c-\frac{1}{2}$. Here c is the central charge of the minimal models without randomness and $\frac{1}{2}$ is the conformal anomaly of the 2D IM. The main result of their considerations is that the $\beta(u)$ and $\gamma_{\bar{\psi}\psi}(u)$ functions coincide with the corresponding functions for the $O(N)$-symmetric Gross-Neveu model obtained in the framework of the minimal substraction scheme combined with dimensional regularization. The distinguishing feature of this scheme is that these functions do not depend on ϵ except for the first term in the β function. Thus, there is a clear possibility to apply the results of multiloop RG calculations for the Gross-Neveu model in order to compute the critical exponents of random minimal models. At the present time we have the five-loop expressions for the $\beta(u)$ function and anomalous dimension functions $\gamma_\psi(u)$ and $\gamma_{\bar{\psi}\psi}(u)$ of the fermionic field ψ and composite operator $\bar{\psi}\psi$, respectively, and as obtained in Ref. 52 Unfortunately, these expressions contain a few unknown coefficients in the four- and five-loop terms. As for the anomalous dimension of the spin variable $\eta(u)$, this function was obtained in Ref. 21 in a three-loop approximation. Notice that according to the conformal field theory classification, the spin variable corresponds to the operator $\Phi_{m,m-1}$, whilst $\Phi_{1,2}=\epsilon(x)$ is the energy-density operator. Thus, one may use the RG functions obtained only in the three-loop approximation for the calculation of the critical exponents.

Let us now compute the critical exponents of the correlation length and specific heat of the random minimal models in the three-loop approximation. The corresponding expressions for the β function and temperature critical exponent function are given by:[52]

$$\beta(u)=2\epsilon u-2(N-2)u^2+4(N-2)u^3+2(N-2)(N-7)u^4,$$

$$\gamma_{\bar{\psi}\psi}(u)=2(N-1)u-2(N-1)u^2-2(N-1)(2N-3)u^3,$$

$$\epsilon=\frac{3-m}{2m},\quad m=3,4,\ldots. \tag{5.1}$$

Here N is the number of planes (colors), coupled to each other in the usual way like in the N-color ATM [Eq. (3.1)]. The critical behavior of the multicolor minimal models is governed by the nontrivial fixed point of Eq. (5.1). From this equation it follows that

$$\frac{1}{\nu}=\frac{1}{\nu_0}+\gamma_{\bar{\psi}\psi}(u^*)=\frac{1}{\nu_0}+2(N-1)\left[\frac{\epsilon}{N-2}+\frac{\epsilon^2}{(N-2)^2}-\frac{N\epsilon^3}{(N-2)^3}\right],\quad \nu_0=\frac{2m}{3(m-1)}, \tag{5.2}$$

where ϵ takes on the discrete values defined in Eq. (5.1) and ν_0 is the critical exponent of the correlation length of the pure model.

To check the self-consistency of Eq. (5.1) let us consider the limit $N\to\infty$, describing the system with equilibrium impurities. The result is easily seen to be

$$\nu_{\rm imp}=\frac{2m}{m+3}=\frac{\nu_0}{1-\alpha_0}. \tag{5.3}$$

From the expression for the anomalous dimension of the order parameter $\eta(u)$ obtained in Ref. 21 it follows that the Fisher critical exponent η_0 is unchanged in this limit, $\eta_{\rm imp}=\eta_0$, where for the q-state Potts η_0 is given by[53]

$$\eta_0=\frac{(m+3)(m-1)}{4m(m+1)},\quad q=4\cos^2\frac{\pi}{m+1},\quad m=2,3,5,\infty. \tag{5.4}$$

TABLE I. Critical correlation length exponent ν for random minimal models: TIM (tricritical Ising model), 3PM (three-state Potts model), TPM (tricritical Potts model), and 4PM (four-state Potts model). m denotes the minimal model, $\epsilon=(3-m)/2m$, ν_0 is the homogeneous exponent, $\nu_r=\nu_0+\nu_1+\nu_2+\nu_3$ the random one, and ν_n denotes the n-loop contribution to ν_r.

Model	m	ϵ	ν_o	ν_1	ν_2	ν_3	ν_r
TIM	4	-0.125	0.889	0.099	0.017	0.003	1.008
3PM	5	-0.2	0.833	0.139	0.038	0.008	1.018
TPM	6	-0.25	0.8	0.16	0.052	0.014	1.026
4PM	∞	-0.5	0.667	0.222	0.13	0.062	1.081

As was expected, we have obtained the duly renormalized critical exponent of the correlation length and an unchanged value of the order parameter anomalous dimension, in agreement with the predictions of the general theory.[44] As expected, in the trivial case $N=1$ one gets the critical exponents of pure systems.

The critical exponent values ν_r for random models can be easily obtained from Eq. (5.2) by expanding ν_r in powers of ϵ and setting $N=0$. The results of these calculations are presented in Table I. The most striking feature of the above expansions is that they look like rapidly convergent series, even in the case of the four-state Potts model. As a matter of fact, this is not so surprising, because in the Thirring model $(N=2)$ the anomalous dimensions $\gamma_{\bar{\psi}\psi}(u)$ and $\gamma_{\psi}(u)$ are known to be some geometric progressions in u. Notice also that if we set $N=2$ in Eq. (5.1) we do not obtain these progressions.[52] The reasons why the O(2)-symmetric Gross-Neveu model in the minimal substraction scheme is not completely equivalent to the Thirring model have not been completely understood as yet.[52] It is also important that all values of ν_r for any arbitrary integer m slightly exceed unity, so that all values of α_r are negative in full agreement with the Harris criterion.

As was mentioned above, the three-loop results concerning the critical exponent $\eta(u)$ were obtained in Ref. 21 It turns out that the numerical values of η are in the close vicinity of the Ising model value, $\eta=0.25$. The reasons for that are as follows. The one-loop approximation term which is expected to give rise to the main contribution to the deviation of η from η_0 vanishes due to the Kramers-Wannier dual symmetry. The two-loop correction was shown to be proportional to ϵu^2; therefore the deviations of η from the pure values are proportional to ϵ^3.[21]

Thus, the numerical values of critical exponents of the weakly disordered minimal models are very close to the critical exponents of the 2D IM. This has clear implications for the three- and four-state Potts models with random bonds, as shown by Table I. From a numerical point of view one might be tempted to conclude that all these models are described by the 2D IM fixed point.[28] We see here that this ''superuniversality'' is only approximate, though accidentally verified to a high degree of accuracy.

VI. CONCLUSIONS

It has been shown that the critical behavior of a good number of 2D anisotropic systems controlled by the IM fixed point is stable in the presence of weak quenched disorder. This statement was found to hold quite generally for the 2D IM, multicolor ATM, and some of its generalizations for which randomness is marginally relevant. In the case of the two-color ATM or Baxter model, disorder drastically changes the nonuniversal critical behavior inherent in this model over to the Ising-type critical behavior. Although some of these models exhibit a breakdown of the Harris criterion, this does not affect, in general, the stability of the IM fixed point. It is commonly believed that the type of randomness (random bond or site disorder) does not play a role near T_c, despite the fact that random-site disorder has not been studied in great detail as yet.

On the numerical side, Monte Carlo (MC) simulation results are in good agreement with the analytical results based on the RG calculations.[16] For instance, the high-accuracy MC simulation results for a 1024×1024 Ising lattice with ferromagnetic impurity bonds, recently obtained by Schur and Talapov,[16] show that the exponent of the two-spin correlation function at criticality is numerically very close to that for the pure model. On the other hand, numerical results obtained by Domany and Wiseman[28] do somewhat contradict the theoretical predictions. These results, for a 256×256 lattice, favor a logarithmic-type behavior of the specific heat near T_c for the disordered two-color ATM and four-state Potts models, and a double-logarithmic behavior of the specific heat for the random-bond IM. As was established by Dotsenko and Dotsenko,[51] the specific heat of the random two-color ATM should exhibit the double-logarithmic divergence at the critical point.

The critical behavior of the random four-state Potts model was shown to be described by a new fixed point which does not coincide with the IM one.[21,19] The conjecture made in Ref. 2 that the perturbation theory expansion around the free fermion theory appropriate for the Ising model is valid until the end point of the ferromagnetic phase transition line (describing the four-state Potts model) is actually incorrect. Exact results for the repulsion sector of the sine-Gordon theory obtained by means of the Bethe ansatz in Ref. 54 show that the perturbation theory around $\beta^2=4\pi$ (free fermions) diverges at $\beta^2=16/3\pi$ (see also Ref. 55); i.e., it has a finite radius of convergence, and cannot be continued to $\beta^2=8\pi$.

The critical exponents corresponding to the disordered four-state Potts fixed point slightly differ from the IM ones. The numerical results are believed to be sensitive to that difference. It is interesting to compare the estimate for the critical exponent ν of the four-state Potts model based on the three-loop approximation with known numerical results. From Table I it follows that $\nu=1.081$. Novotny and Landau[56] obtained for the Baxter-Wu model (equivalent to the four-state Potts model) the following value: $\nu=1.00(7)$. The result of Andelman and Berker[57] is given by $\nu=1.19$. Finally, the recent result obtained by Schwenger *et al.*[58] is as follows: $\nu=1.03(8)$.

We end this section by giving a remark that logarithmic corrections to the power-law dependence and corrections to scaling may give rise to a dependence of effective critical exponents on the concentration of defects as observed in some numerical experiments.[59]

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