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Spin diffusion in one-dimensional antiferromagnets

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We study the problem of spin diffusion in magnetic systems without long-range order. We discuss the example of the one-dimensional spin chain. For the system described by the Heisenberg Hamiltonian we show that there are no diffusive excitations. However, the addition of an arbitrarily small dissipation term, such as the spin-phonon interaction, leads to diffusive excitations in the long-time limit. For those excitations we estimate the spin-diffusion coefficient by means of the renormalization group analysis. [S0163-1829(96)01330-6]

I. INTRODUCTION

Spin dynamics in magnetic systems without long-range order is a longstanding problem. It has been assumed[1–4] that in the high temperature limit, where no long-range order is present, the microscopic spin fluctuations are governed by the classical diffusion equation, i.e., that at small frequencies and momenta the retarded spin-correlation function has a diffusive pole[1]

$$\int_0^\infty dt e^{i\omega t}\langle S^z_{-k}S^z_k(t)\rangle=\frac{Z}{i\omega-Dk^2}, \tag{1}$$

where D is the diffusion constant and Z is the residue.

Although the idea of spin diffusion is quite common, we are not aware of any theoretical approach, within which one has actually derived the correlation function Eq. (1), starting from the nearest-neighbor Heisenberg Hamiltonian

$$H_H=\sum_i J_i\vec{S}_i\cdot\vec{S}_{i+1}. \tag{2}$$

On the contrary, recent computer simulations have given us reason to question these assumptions. In particular, it was found that the usual hydrodynamic assumptions break down in one dimension, so that the asymptotic behavior of the spin-spin correlation function deviates from the predictions of the classical diffusion theory (see Ref. 5 and references therein).

In this paper we present a technique which allows for a direct calculation of the spin correlation function for a one-dimensional (1D) spin-1/2 Heisencberg chain of infinite length. At any nonzero temperature the chain is in a disordered state with exponentially decaying spin-spin correlations, and even at $T=0$ the correlations decay as a power law, so there is no true long-range order (where the correlation function would be constant in the long-time limit). We show that in one dimension it is impossible to derive the diffusive form of the spin-spin correlation function Eq. (1) from the Heisenberg Hamiltonian Eq. (2) without any kind of additional dissipation mechanism, in agreement with the prior expectations based on the results of the scaling theory.[6] We also present a general argument supporting this result. We show then that, if one takes into account an additional dissipation, for example due to spin-phonon interaction (which is always present in any real system at finite temperature), then the renormalization-group approach leads to the correlation function Eq. (1) in the long-time asymptotic.

Our results could be applied to materials like $KCuF_3$, $CuSO_4$ $5H_2O$, Sr_2CuO_3. In a broad temperature range they are nearly ideal 1D antiferromagnetic chains with the coupling constant J ranging from 1.45 K in $CuSO_4$ $5H_2O$ (Ref. 2) to 190 K in $KCuF_3$,[7,8] and 2600 K in Sr_2CuO_3.[9] The diffusion equation was successfully used in a number of experimental papers to fit the data and explain the results of the experiments.[2,3] This approach was also confirmed by computer simulations.[4]

The rest of this paper is organized as follows. In Sec. II we briefly review the mapping of the spin system onto 1D fermions. In Sec. III we map the fermions onto a boson system, justifying the bosonization procedure at finite temperatures. Section IV gives the calculation, which shows the absence of spin diffusion in the Heisenberg model Eq. (2) and presents some general qualitative arguments supporting this conclusion, based on the theory of the sine-Gordon equation. Section V discusses the spin-phonon interaction and its renormalization. Section VI contains conclusions and a discussion of results. The details of the calculations are presented in the Appendixes.

II. FERMION MODEL

In this section we review the well-known procedure of mapping the Heisenberg model Eq. (2) on a system of spinless fermions,[10] and establish the notations. The spin model Eq. (2) can be transformed into a model of spinless fermions, noting that operators S_i^+ and S_i^- anticommute. The Jordan-Wigner transformation then relates spin to fermion operators $(a_i,a_i^\dagger)$ via

$$S_i^+=a_i^\dagger\exp\left(i\pi\sum_{j=1}^{i-1}a_j^\dagger a_j\right),$$

$$S_i^z=a_i^\dagger a_i-\tfrac{1}{2}. \tag{3}$$

When transforming the Hamiltonian Eq. (2) the spin-flip terms give rise to the motion of the fermions (kinetic energy

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in the fermion Hamiltonian) and $S_i^z S_{i+1}^z$ interaction leads to a fermion-fermion interaction between adjacent cites. Since S^z is quadratic in fermion operators, the interaction between fermions is the four-particle interaction. Since the original spin model was formulated on a lattice, all possible types of four-fermion interaction are present, including the umklapp term.

The 1D fermion models are often treated using bosonization.[11,12] In the case of massless fermions with four-fermion interaction with small momentum transfer it allows an exact solution. It is shown in the thermodynamic limit that the system can be mapped onto a system of free bosons. The propagator of a free boson has a pole at $\omega = ck$ and therefore corresponds to the particle, propagating without dissipation, so that this interaction cannot lead to any significant change in the long-time dynamics of the system. Therefore the only source of dissipation might be interaction with large momentum transfer, such as the umklapp term.

The treatment of the umklapp term is much more complicated, because in boson language it corresponds to a highly nonlinear term, namely, $\cos(2\beta\phi)$ [where $\phi(x,t)$ is the boson field and the number β is a parameter of the transformation, depending on the fermion interaction with small momentum transfer]. This term does not conserve momentum and is only marginally irrelevant, so at $T>0$ it might lead to some additional dynamics. Thus we should investigate the impact of that term on the long-time fermion (and thus spin) dynamics, disregarding all other possible four-fermion interaction terms. Sections III and IV present the results of this investigation.

Since we are interested only in the low-energy behavior of the system, we can linearize the fermion kinetic energy, thus allowing an exact bosonization. Therefore our model Hamiltonian is

$$H_m = v_F \sum_k \left[(k-k_F)\psi_R^\dagger \psi_R + (-k-k_F)\psi_L^\dagger \psi_L \right] + V \sum (\psi_R \psi_R \psi_L^\dagger \psi_L^\dagger + \text{H.c.}), \tag{4}$$

where $\psi_{R(L)}$ is the operator of a ''right'' (''left'') mover. The quantity we are interested in here is the density-density correlation function, which corresponds to the $\langle S_i^z S_j^z \rangle$ correlator in the spin problem. We shall now investigate whether the long-time asymptotic of this correlator has the diffusive form, Eq. (1).

III. BOSONIZATION

As mentioned above our main technical tool will be bosonization. The procedure is well established in 1D at zero temperature. We consider finite temperatures, and we look for the long-time asymptotic of the spin-spin correlator. It is quite difficult to obtain the results in that limit in the Matsubara technique, since the analytic continuation from the Matsubara frequencies ω_n to real frequencies much less that the inverse temperature would require a precise knowledge of the Green's functions on the infinite range of ω_n, which is usually not the case in perturbation theory. Therefore we have to resort to the Keldysh technique,[13] which incorporates finite temperatures and a real-time representation. In this section we construct the bosonisation procedure for the Keldysh technique and then confirm its correctness by comparing the results of the perturbation theory for bosons and fermions. The fermion and boson Green's functions in a space-time representation, which we are using in our calculations, are presented in Appendix A.

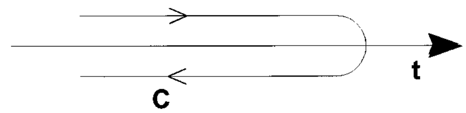

FIG. 1. Keldysh time contour.

Thirty years ago Keldysh has presented a field-theoretical technique to calculate the real-time correlation functions of a quantum system. To allow the treatment of advanced and retarded correlators, Keldysh introduced the time contour C (Fig. 1) with the upper branch going in positive direction from $-\infty$ to $+\infty$ and the backward lower branch. All the operator products are now time ordered along the contour C. To distinguish particles on the upper and lower branches of the contour, the fermion field operator ψ_γ is given an index γ, which equals 1 on the upper branch and 2 on the lower. Green's functions become 2×2 matrices with respect that index.

In a one-dimensional fermion problem we have four different operators $\psi_\gamma^{L(R)}$ — left and right movers on both branches of the contour. Since operators on each separate branch are completely analoges to the zero-temperature operators, we can proceed with the bosonization separately on each branch in exactly the same way as at $T=0$. Thus we introduce two boson fields ϕ_γ (one for each branch), which we shall treat as two components of the Keldysh field. The resulting bosonized Hamiltonian will thus be formulated in the Keldysh technique also.

The left- (ϕ_γ^L) and right- (ϕ_γ^R) moving bose fields expressed via ϕ_γ and its canonically conjugate P_γ,

$$\phi_\gamma^{L(R)}(x) = \frac{1}{2}\left[\phi_\gamma(x) \mp \int_{-\infty}^{x} P_\gamma(x')dx' \right] = \pm \int_0^\infty \frac{dp}{2\pi\sqrt{2|p|}} e^{-\alpha|p|}[\phi_\gamma(p)e^{ipx} + \text{H.c.}]. \tag{5}$$

As in the usual procedure, $\phi_\gamma^{L(R)}$ are functions of only $(x \mp t)$.

The fermion operators are constructed in analogy with the zero-temperature case,

$$\psi_\gamma^{L(R)} \sim \frac{1}{\sqrt{\alpha}} \exp(\pm i\beta\phi_\gamma^{L(R)}), \tag{6}$$

where $\beta^2 = 4\pi$, and the upper sign corresponds to the left mover.

The commutation relations between fermion fields hold for exactly the same reason[11] as at $T=0$. The fact that we have a different time contour (the Keldysh contour C as opposed to the usual time axis) does not change the calculation, for the integrals involved in Eq. (5) are over space coordinates, and the Bose fields commute no matter which branch of the time contour they are on. Another way of saying this is that the Keldysh operators on different branches still correspond to the same particle. Dividing the time contour into two parts is a matter of mathematical convenience rather than physical distinction.

The cutoff α is a lattice spacing, and so should be the same for both bosons and fermions. The operator equality Eq. (6) means that any correlation function (in the limit $\alpha\to 0$), calculated in the Fermi theory with the cutoff α is reproduced in the bosonic theory with the same cutoff if the fermion operator on the left-hand side is replaced by the bosonic operator on the right-hand side of Eq. (6). Using this operator equivalence we can construct the boson Hamiltonian from the fermion Hamiltonian Eq. (4). The fermion kinetic energy corresponds to that of bosons. The umklapp interaction term gives rise to the cosine interaction of the boson field. The conjugate operators P_+ and P_- cancel out, so the interaction is a function only of the boson fields ϕ_+ and ϕ_- itself, as it is in the zero-temperature bosonization. The interaction constant V now acquires the factor $1/\alpha^2$ from the prefactor in Eq. (6). The boson Hamiltonian therefore is

$$H_B=(\partial_\mu\phi)^2+V'\cos 2\beta\phi. \tag{7}$$

where $V'=V/\alpha^2$.

Due to the special form of the interaction ($\cos 2\beta\phi$), it is most convenient to formulate the Keldysh technique in the path integral representation, developed by Schmid.[14] The boson action for our model is

$$A=\int dx\,dt[\phi^*_\gamma B_{\gamma\nu}\phi_\nu - V'(\cos 2\beta\phi_+ - \cos 2\beta\phi_-)], \tag{8}$$

where ϕ_+ and ϕ_- are the two components of the Keldysh boson field, and γ and ν denote Keldysh indices. $B_{\mu\nu}$ is the kinetic operator, so that the inverse $B^{-1}_{\mu\nu}$ is proportional to the boson Green's function in the Keldysh basis

$$B^{-1}_{\mu\nu}=-i\begin{pmatrix} D^{++} & D^{+-} \\ D^{-+} & D^{--}\end{pmatrix}. \tag{9}$$

We shall later use these functions in the space-time representation (see Appendix A).

The fermion density in the conventional bosonization is represented by the spatial derivative of the boson field φ,

$$\rho=\frac{1}{\sqrt{\pi}}\partial_1\varphi. \tag{10}$$

Here in the same manner we write the fermion density operators. The density-density correlators are represented by functional integrals in which the preexponential is some linear combination of the spatial derivatives of the Bose fields corresponding to that particular correlator. The advanced correlator is

$$\langle\rho\rho\rangle_A=\Pi(x_1-x_2)$$
$$=\frac{1}{Z}\int \mathcal{D}[\phi_1,\phi_2]\partial_1\phi^*_\alpha(x_1)\tau_{\alpha\beta}\partial_1\phi_\beta(x_2)\exp(-A), \tag{11}$$

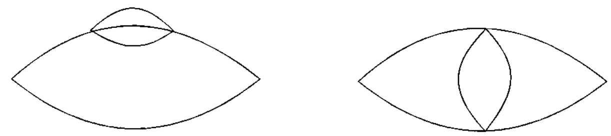

FIG. 2. First-order corrections to the fermion density-density correlator.

where the density vertex $\tau_{\alpha\beta}$ is

$$\tau_{\alpha\beta}=\frac{1}{\pi}\begin{pmatrix} 1 & -1 \\ 1 & -1\end{pmatrix}. \tag{12}$$

We now check, following the approach of Shankar,[12] that the above construction yields correct results in perturbation theory. The density-density correlation function for noninteracting fermions should be calculated separately for the ''left'' and ''right'' movers, and the results should be added. That gives, for the advanced correlator,

$$\Pi_0=\frac{2}{\pi}\frac{q^2}{\omega^2-q^2-2i\delta\omega}, \tag{13}$$

where δ is infinitesimately small. In the next order we have two topologically nontrivial diagrams given in Fig. 2 Again we have to repeat the calculation for the ''left'' movers and add the results with proper combinatorial factors. For simplicity here we give only the imaginary part of the first-order correction, which for us is most important:

$$\mathrm{Im}\Pi_1=\frac{V^2}{2\pi^2}\frac{q^2}{\omega^2-q^2}\frac{\omega T}{\omega^2-q^2}. \tag{14}$$

The next order of perturbation is discussed in Sec. IV.

We now turn to a calculation of the same perturbation series on the boson language. In the noninteracting case we have to calculate a simple Gaussian integral, which in momentum space gives

$$\Pi_0(\omega,q)=\frac{2}{\pi}q^2 D_A(\omega,q)=\frac{2}{\pi}\frac{q^2}{\omega^2-q^2-2i\delta\omega}, \tag{15}$$

which is the same as Eq. (13). $D_A(\omega,q)$ is the advanced boson Green's function, connected to the original Keldysh basis via

$$D_A=D^{--}-D^{+-}. \tag{16}$$

We now expand the exponent in the integral Eq. (11) in the series in V'. In the first nontrivial order we get

$$\Pi_1(x_1-x_2)=V'^2\int d^2y_1d^2y_2\frac{1}{Z}\int \mathcal{D}[\phi_1,\phi_2]\partial_1\phi^*_\alpha(x_1)\tau_{\alpha\beta}\partial_1\phi_\beta(x_2)\exp\left(-\int dxdt\phi^*_\mu B_{\mu\nu}\phi_\nu\right)[\cos 2\beta\phi_+(y_1)$$
$$-\cos 2\beta\phi_-(y_1)][\cos 2\beta\phi_+(y_2)-\cos 2\beta\phi_-(y_2)]. \tag{17}$$

Since the cosines contribute the linear terms in the exponent, the integral is still Gaussian with the same prefactor of the exponent. This feature will remain in all higher orders of the perturbation series. We shall discuss the calculation of that integral in more detail due to its importance for the later arguments. In the first-order integral Eq. (17) we have four different terms of the same type, arising from the cosines. In the exponent they have combinations like $2\beta[\phi_1(y_2)-\phi_2(y_1)]$ with all possible permutations of indices. To calculate the functional integral we make the Fourier transform of the Bose fields. We perform the transform in the most general way, since the same expressions will appear later. As we show below, in any order of the perturbation theory we shall need to calculate averages of the form

$$\left\langle \partial_1\phi^*_\alpha(x_1)\tau_{\alpha\beta}\partial_1\phi_\beta(x_2)\right.$$
$$\left.\times\exp\left(\int\frac{d^2k}{(2\pi)^2}{}^{(\mathbf{r})}I^*_\mu(k,y_i)\phi_\mu(k)+\text{H. c.}\right)\right\rangle. \tag{18}$$

For the first-order integral Eq. (17), we find four different factors ${}^{(\mathbf{r})}I^*_\mu(k)$:

$$\begin{pmatrix} {}^{(1)}I^*_1(k,y_i) \\ {}^{(1)}I^*_2(k,y_i \end{pmatrix}=\begin{pmatrix} i\beta e^{iky_1} \\ -i\beta e^{iky_2}\end{pmatrix},\quad \begin{pmatrix} {}^{(2)}I^*_1(k,y_i) \\ {}^{(2)}I^*_2(k,y_i) \end{pmatrix}=\begin{pmatrix} -i\beta e^{iky_2} \\ i\beta e^{iky_1}\end{pmatrix}$$

$$\begin{pmatrix} {}^{(3)}I^*_1(k,y_i) \\ {}^{(3)}I^*_2(k,y_i) \end{pmatrix}=\begin{pmatrix} i\beta(e^{iky_1}-e^{iky_2}) \\ 0\end{pmatrix},$$
$$\begin{pmatrix} {}^{(4)}I^*_1(k,y_i) \\ {}^{(4)}I^*_2(k,y_i) \end{pmatrix}=\begin{pmatrix} 0 \\ i\beta(e^{iky_1}-e^{iky_2})\end{pmatrix}. \tag{19}$$

Completing the square in the exponent, we calculate the functional integral in Eq. (17) and apart from the numerical factor, in momentum space we get

$$\Pi_1(\omega,q)\sim\int d^2kkq\int\prod d^2y_i\sum_{\{\mathbf{r}\}}{}^{(\mathbf{r})}I^*_\mu(q,x_i)B^{-1}_{\mu\alpha}(q)$$
$$\times\tau_{\alpha\beta}B^{-1}_{\beta\nu}(k){}^{(\mathbf{r})}I_\nu(k,x_i)K_1(y_i), \tag{20}$$

where

$$K_1(y_i)=\exp\left(\int\frac{d^2k'}{(2\pi)^2}{}^{(\mathbf{r})}I^*_\gamma(k',y_i)B^{-1}_{\gamma\delta}(k'){}^{(\mathbf{r})}I^*_\delta(k',y_i)\right), \tag{21}$$

and the sum is over four different factors ${}^{(\mathbf{r})}I_\nu(k,y_i)$ corresponding to the four different terms of type (18) in the first order integral Eq. (17).

Here the integration measure Πd^2y_i is equal to $dy_1dt_1dy_2dt_2$. All dependence on these variables is contained in the factors ${}^{(\mathbf{r})}I^*_\mu(k,y_i)$. These factors are nothing but exponentials βe^{iky} with different signs. Therefore the integral in the exponent in Eq. (21) is just the Fourier transform of the boson Green's function back from momentum space to the real space. The boson functions in real space are logarithms, so all four exponents $K_1(x_i)$ can be evaluated. All four have the same structure,

$$K_1(y=y_1-y_2)=\frac{f(y,t)}{S(y,t)}, \tag{22}$$

where

$$S^{-1}(y,t)=\frac{(\pi T\alpha)^4}{\sinh^2\pi T(y-t)\sinh^2\pi T(y+t)}, \tag{23}$$

while $f(y,t)$ is some algebraic function which has no poles. In the long-time asymptotic $K_1(y,t)$ acts like a derivative of a δ function. It has [due to the denominator Eq. (23)] a sharp singularity at $y,t=0$, and decays very fast as the variables go away from that point. Therefore the integral over y,t is dominated by the small region of size α around the origin. The remaining spatial integration over the sum y_1+y_2 involves only functions ${}^{(\mathbf{r})}I^*_\mu(k,y_i)$ from the prefactor, since $K_1(y_i)$ depends only on the difference of the variables y_1 and y_2. Since ${}^{(\mathbf{r})}I^*_\mu(k,y_i)$ are exponentials, the integral is easily evaluated to yield $\delta(k-q)$, which solves the momentum integration. So after the summation over all four terms we get the result Eq. (14), as expected.

The next orders of the perturbation series, which we discuss in Sec. IV, could be calculated in the same manner, and are also the same for the boson and the fermion formulations of he theory. The bosonization procedure is thus justified.

IV. ABSENCE OF DIFFUSION FOR THE HEISENBERG MODEL

Although the boson and fermion versions of the theory are completely equivalent and should give the same results in the perturbation theory, the bosonized version allows an easier evaluation or the higher orders of the perturbation theory. Indeed, all higher-order corrections differ from the first order Eq. (17) only by the appearance of additional cosine brackets (and corresponding spatial integrations). This means that in any order of perturbation we have to calculate averages of the form Eq. (18). The functions ${}^{(\mathbf{r})}I_\mu(k,x_i)$ will be now sums of exponentials, for example, in the next order of perturbation we will have terms like

$${}^{(\mathbf{r})}I_\mu(k,x_i)=\beta[e^{ikx_1}-e^{ikx_2}+e^{ikx_3}-e^{ikx_4}], \tag{24}$$

similar to the first-order factors Eq. (19), but constructed from four different exponentials. We can still perform the functional integration and get the same general formula Eq. (20). Since functions ${}^{(\mathbf{r})}I_\mu(k,x_i)$ are exponentials, the integral in the exponent in Eq. (20) still yields the boson Green's functions in the space-time representation, but now instead of one such a function as in the first order we have a sum of them. In the second order we shall get

$$K_2(x_1,\dots x_4)=\frac{S(x_2-x_3)S(x_1-x_4)}{S(x_1-x_2)S(x_1-x_3)S(x_2-x_4)S(x_3-x_4)}\times f_2(x_1,\dots,x_4), \tag{25}$$

where $S(x)$ is the ''singular'' denominator, defined in Eq. (23).

In higher orders functions K have the same structure, but with more factors $S(x)$ in the numerator and denominator. These factors have the same singular behavior as described in Sec. III. Therefore the integration over multidimensional space will be dominated by the regions where the two pairs of variables are almost equal to each other, for example $x_1=x_2$ and $x_3=x_4$. The integration over one such pair effectively reduces the function Eq. (25) to the form of the previous order Eq. (22) (in the long-time asymptotic). One can see it by inspecting Eq. (25). Consider, for instance, the contribution from the region $x_1=x_2$. In this region $K_2\sim S^{-1}(x_1-x_2)S^{-1}(x_3-x_4)$, so the integral over x_2 leads to the same form $[\sim 1/S(x_3-x_4)]$, which is the main part of the first-order function K_1, Eq. (22). The integration over x_3 and x_4 is then the same as in the first order . Therefore the spatial integration in in the higher-order terms does not yield any additional singularity to that produced by the prefactor of the exponent, which is the same as in the first order Eq. (20) and contains just two boson Green's functions. So the structure of any term in the perturbation series is

$$\mathrm{Im}\Pi_1(\omega,q)\sim P\frac{\omega q^2}{(\omega^2-q^2)^2}g(\omega,q). \tag{26}$$

Here $g(\omega,q)$ is some function, which has no poles at small ω,q. We note that $\mathrm{Im}\Pi_1(\omega,q)$ differs from the first order Eq. (17) only by $g(\omega,q)$, and P denotes the principal value.

Combining our conclusions, we can write down the structure of the density-density correlation function in our model,

$$\Pi(\omega,q)=\frac{2}{\pi}\frac{q^2}{\omega^2-q^2+2i\delta\omega}+i\frac{\omega T q^2}{(\omega^2-q^2)^2}g(\omega,q,V). \tag{27}$$

The significance of this result for us is in the fact that the higher-order corrections do not acquire additional poles in the imaginary part, which could sum up in a diffusive pole in $\Pi(\omega,q)$.

The fermion perturbation theory gives the same results in the low orders of the perturbation. In the second nontrivial order we have six topologically different diagrams, presented in Fig. 3. Our purpose is to show that the correction to the density-density correlation function, which is the sum of these diagrams, is not qualitatively different from the first-order result. The only diagrams that produce the extra pole are the diagrams *a* and *b* in Fig. 3. These singular terms cancel out exactly, so that the second-order correction has a pole of the same order as the first-order one. This cancellation seems to be an accidental property of the problem in the fermion representation, but the bosonization approach shows that it happens in all orders of perturbation theory.

Our results clearly show the absence of spin diffusion in the Heisenberg model. Instead of the diffusive pole in the density-density correlation function we found some kind of a propagating behavior. Note that the second-order pole in Eq. (27) should be regarded as a principal value, so the imaginary part of the general susceptibilities does not contain unphysical contributions proportional to $\delta^2(\omega^2-q^2)$. This result should have been expected. In the boson representation our problem is essentially the sine-Gordon model. It is known in the theory of the sine-Gordon equation that due to the infinite number of conserved charges the excitations of the model are propagating and *not* diffusive. So by showing the mapping of the Heisenberg model onto the sine-Gordon model we have showed the absence of spin diffusion in the model Eq. (2).

V. SPIN-PHONON INTERACTION

The Heisenberg model Eq. (2) does not exactly describe the physics of a real material. We now try to make it a bit more realistic. Even in the absence of disorder, at finite temperatures there always are phonons in the system. The exchange integral depends in general on the instantaneous separation of magnetic ions. We consider the linear approximation for spin-phonon coupling. That is, we expand the separation-dependent Heisenberg coupling constant to the first order in ionic displacement $u(R_i)$,

$$J(r_i-r_j)=J_0(R_i-R_j)+\chi[u(R_i)-u(R_j)], \tag{28}$$

where R_i is the equilibrium position of the ion and $r_i=R_i+u(R_i)$. That leads to the simplest form of the spin-phonon interaction Hamiltonian,

$$\mathcal{H}_{\text{sp-ph}}=J'\sum_{\langle ij\rangle}\vec{S}_i\vec{S}_j(b_i^\dagger+b_i). \tag{29}$$

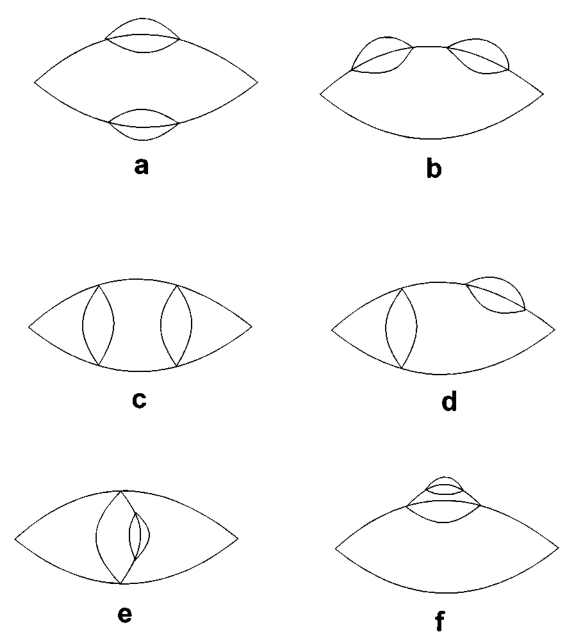

FIG. 3. Second-order corrections to the fermion density-density correlator. The terms with the third-order pole exactly cancel.

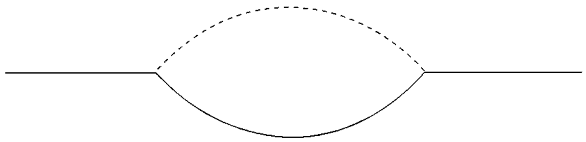

FIG. 4. Boson self-energy eiagram. The solid line denotes the boson propagator, and the dashed line denotes the phonon propagator.

We shall now try, treating interaction Eq. (29) as a small perturbation of the original Hamiltonian Eq. (2), to examine its impact on the long-time spin dynamics. To do that we first bosonize Eq. (29). We get

$$\mathcal{H}_{b\text{-ph}}=J'\sum_k |\phi_k|^2(b_k^\dagger+b_k). \tag{30}$$

The boson self-energy in the first nontrivial order in $\mathcal{H}_{b\text{-ph}}$ is presented by the diagram on Fig. 4. The solid line represents a boson, and the dashed a phonon. For the imaginary part we are interested in we get

$$\mathrm{Im}\Sigma(\omega,q)=\begin{cases}\eta\omega q v_F & \text{if } q>T\\ \eta\omega T & \text{if } q<T,\end{cases} \tag{31}$$

where restoring the units $\eta\sim(J'/c)^2$ and c is the speed of sound. The momentum dependence of $\mathrm{Im}\Sigma(\omega,q)$ arises due to the momentum in the numerator of the phonon Green's function

$$\mathcal{D}_{\mathrm{ph}}(\omega,q)=\frac{c^2q^2}{\omega^2-c^2q^2+i\delta\omega}, \tag{32}$$

where δ is infinitesimally small. Note that although we consider a 1D spin chain, the phonons in a real material are three dimensional, so when evaluating the self-energy Eq. (31) one must integrate out the two other components of the phonon momentum.

We shall now investigate how the umklapp term renormalizes this self-energy. We now divide the boson field into ''slow'' and ''fast'' parts, separated by some cutoff k_0. We integrate out all ''fast'' degrees of freedom (those with momentum larger than the cutoff) and see how the imaginary part of the boson self-energy (namely, the coefficient η) changes with the cutoff. The result is that when the cutoff is larger than the temperature T, η rises as some power of the cutoff. But after the cutoff becomes smaller than the temperature, the imaginary part grows exponentially. This means that at large distances the motion becomes diffusive. We notice that this should happen in any experimental situation, because it is just the existence of the spin-phonon interaction (no matter how small) that is responsible for the diffusion.

After we introduce the ''slow'' and ''fast'' variables as

$$\phi_\mu=\phi_\mu^s+\phi_\mu^f, \tag{33}$$

the integral Eq. (11) is then

$$\Pi(x_1-x_2)=\frac{1}{Z}\int\mathcal{D}[\phi_1^s,\phi_2^s]\mathcal{D}[\phi_1^f,\phi_2^f]\times\partial_1\phi_\alpha^{s*}(x_1)\tau_{\alpha\beta}\partial_1\phi_\beta^s(x_2)\exp(-A), \tag{34}$$

where the prefactor contains only slow degrees of freedom since we are looking for the infrared asymptotic. We can separate the integral over fast variables, and the density integral now becomes

$$\Pi(x_1-x_2)=\frac{1}{Z'}\int\mathcal{D}[\phi_1^s,\phi_2^s]\partial_1\phi_\alpha^{s*}(x_1)\tau_{\alpha\beta}\partial_1\phi_\beta^s(x_2)\times\exp\left(\int\frac{d^2k'}{(2\pi)^2}[\phi_\mu^{s*}B_{\mu\nu}\phi_\nu^s]\right)\exp(\ln M), \tag{35}$$

where M is the integral over the fast variables, which in the first nontrivial order in perturbation is a Gaussian integral without any prefactor,

$$\begin{aligned}M=&\frac{V'^2}{2}\int\mathcal{D}[\phi_1^f,\phi_2^f]\exp\left(\int\frac{d^2k'}{(2\pi)^2}[\phi_\mu^{f*}B_{\mu\nu}\phi_\nu^f]\right)\\&\times\int d^2x_1d^2x_2\{\cos2\beta[\phi_1^s(x_1)+\phi_1^f(x_1)]\\&-\cos2\beta[\phi_2^s(x_1)+\phi_2^f(x_1)]\}\{\cos2\beta[\phi_1^s(x_2)+\phi_1^f(x_2)]\\&-\cos2\beta[\phi_2^s(x_2)+\phi_2^f(x_2)]\}.\end{aligned} \tag{36}$$

In the functional integral Eq. (35), $\ln M$ plays the role of renormalization of the imaginary part of the kinetic operator. Now we calculate the functional integral and expand the result in the boson fields ϕ_i^s to get their bilinear combination, which gives the renormalization of η. We get

$$M=\frac{(V'\beta)^2}{4}\int d^2x_1d^2x_2[\phi_\alpha^s(x_1)\mathcal{R}_{\alpha\beta}(x_{12})\phi_\beta^s(x_2)], \tag{37}$$

where as usual $x_{12}=x_1-x_2$. The elements of the matrix $\mathcal{R}_{\alpha\beta}(x)$ are the exponentials of the boson Green's functions in the space-time representation. They are presented in Appendix B, where we give the detailed renormalization calculation. The boson Green's functions entering the matrix $\mathcal{R}_{\alpha\beta}(x_{12})$ are now different from those in Appendix A, due to the self-energy Eq. (31). The retarded function in the momentum space is now

$$D_R(\omega,q)=\frac{1}{\omega^2-q^2+i\,\mathrm{Im}\Sigma(\omega,q)} \tag{38}$$

and the corresponding functions $D_A(\omega,q)$ and $D_F(\omega,q)$ acquire the same self-energy. Note that $\mathcal{R}_{\alpha\beta}(x_{12})$ is expressed via boson Green's functions in the unrotated Keldysh basis. However, the coefficient η is easier to extract from the Green's functions in the rotated Keldysh basis Eq. (52). The element $\mathcal{R}_R$ of the matrix $\mathcal{R}_{\alpha\beta}$ in the rotated basis which corresponds to the renormalization of the retarded function is

$$\mathcal{R}_R=\tfrac{1}{2}[\mathcal{R}_{11}(x)-\mathcal{R}_{22}(x)-\mathcal{R}_{12}(x)+\mathcal{R}_{21}(x)]. \tag{39}$$

If the cutoff is large, $v_F k_0 > T$, this yields, after the Fourier transform,

$$\mathcal{R}_R = i\omega \frac{\eta}{k_0^3} \exp\left(-\frac{\beta^2}{2\pi} \ln k_0 \alpha \right), \tag{40}$$

so that the change in η is

$$\Delta\eta = \eta \frac{V^2}{4(k_0\alpha)^6}, \tag{41}$$

since $\beta = \sqrt{4\pi}$. Here as always, α is the lattice spacing which cutsoff the large momentum integration. So here we have a power-law rise in η. When the cutoff is less that the temperature, we get an exponential renormalization:

$$\Delta\eta = \eta_T \exp\left(4\frac{T}{v_F k_0} \right) \frac{V^2}{4(k_0\alpha)^4}. \tag{42}$$

Here η_T is the effective damping at scales $v_F k_0 \sim T$. That means that, on scales of momenta less than the temperature, the imaginary part coefficient η grows very rapidly, and we immediately get the diffusive dynamics. The renormalization-group treatment breaks down when the correction Eq. (42) becomes of the order of unity. That determines the length scales, where the dynamics becomes diffusive. The mean free path is

$$l \sim \frac{v_F}{T} \ln(\eta_0 V^2), \tag{43}$$

where η_0 is the original value of the coefficient η, proportional to the spin-phonon coupling constant [Eq. (31)].

The estimate of the mean free path allows us to estimate the diffusion coefficient as $D = l v_F$, so that

$$D \sim \frac{v_F^2}{T} \ln\left(\frac{J' V}{c} \right). \tag{44}$$

VI. CONCLUSIONS

We now briefly review our results. We started from a 1D Heisenberg Hamiltonian Eq. (2). Our goal was to calculate the spin-spin correlation function at long times and nonzero temperature. and check whether or not it had a diffusive pole Eq. (1) in some region in phase space. To perform such a calculation we mapped our original problem onto 1D fermion model Eq. (4) using the Jordan-Wigner transformation Eq. (3). We used the usual diagrammatic technique to calculate a few first orders of the perturbation theory (shown in Figs. 2 and Fig. 3). In order to go beyond that simple approximation we bosonized the fermion model. To get the long-time asymptotic of the spin-spin correlator, we had to combine bosonization with the Keldysh technique, which allows one to calculate real-time correlation functions at finite temperatures without any need of analytic continuation.

The bosonization procedure allowed us to find the general form of higher-order corrections, and to sum up the perturbation series. It turned out that in each order of perturbation the correction to the spin-spin correlator had the same form Eq. (26); therefore we concluded that the exact spin-spin correlation function does not acquire a diffusive pole from the summation of the perturbation series. This result is also known in the theory of the sine-Gordon model (which coincides with our boson model), where it has been found that due to the infinite number of conserved charges excitations are propagating and not diffusive.

We also check whether these results (the absence of spin diffusion in the perturbation theory) are robust with respect to small dissipation effects present in real physical systems. Specifically, we considered the effect of a weak spin-phonon interaction Eq. (29). We mapped the full problem (including the spin-phonon interaction) to the bosonic model. We found that the interaction with phonons leads to the boson self-energy Eq. (31), the imaginary part of which is proportional to the constant η at small momenta. Further, we applied the renormalization-group analysis; we integrated out ''fast'' degrees of freedom, and showed that this constant η grows moderately while the cutoff is larger than the temperature [Eq. (41)], but grows exponentially after the cutoff becomes smaller than the temperature [Eq. (42)]. At scales where the imaginary part of the spin-spin correlator becomes of the order of the real part the spin dynamics becomes purely diffusive. By associating the mean free path with the scale on which the renormalization procedure breaks down [namely, the renormalization Eq. (42) becomes of the order of unity] we estimate the diffusion coefficient Eq. (44). Restoring the original units and estimating the spin-phonon coupling constant J' from the expansion of the exchange integral Eq. (28), we have

$$D \sim \frac{\pi^2 (J\alpha)^2}{\hbar k_B T} \ln\left(\frac{J\alpha}{\hbar c} \right). \tag{45}$$

where α is the lattice spacing and c is the speed of sound. Thus we found that the presence of the spin-phonon interaction changes the long-time behavior of the spin-spin correlator, which becomes diffusive.

After this work was completed we learned about the dynamical NMR study on the 1D spin chain Sr_2CuO_3.[15] The data suggest the presence of weak spin diffusion in the chain, with the diffusion coefficient much larger than the classical expectation $(J/\hbar)\sqrt{2\pi S(S+1)/3}$, which is consistent with the estimate Eq. (45) (in the experiment $T << J$). Unfortunately the temperature dependence turns out to be very difficult to measure, although the data suggests that D tend to increase when temperature is decreasing.

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APPENDIX A

We present here the boson and fermion Green's functions, which we use in our calculations. It is the certain similarity between them that inspired the bosonization. As in the usual zero-temperature bosonization we need the Green's functions in the space-time representations. We perform the bosonization on the original Keldysh basis, but for simplicity here we calculate Green's functions on the rotated basis and then transform them back.

The retarded fermion Green's function in the momentum space is

$$G_R(\epsilon,k)=\frac{1}{\epsilon\mp k+i0}, \tag{A1}$$

where ''-'' is for the ''left'' and ''+'' for the ''right'' movers. For the simplification of the formulas the Fermi velocity is set equal to unity. The Fourier integral, which one has to calculate in order to transform the function Eq. (A1) real space, is formally divergent at large momenta. As usual in 1D calculations we introduce a cutoff α by adding the exponent $e^{-\alpha|k|}$ to the integral. Thus we have

$$G_R(x,t)=\int \frac{d\epsilon\, dk}{(2\pi)^2} e^{ikx-i\epsilon t}\frac{1}{\epsilon\mp k+i0}e^{-\alpha|k|}. \tag{A2}$$

Now we have a perfectly converging integral and get

$$G_R(x,t)=-\frac{\theta(t)}{2\pi}\frac{2i\alpha}{(x\mp t)^2+\alpha^2}, \tag{A3}$$

where

$$\theta(t)=\begin{cases}1 & if \;\; t>0\\ 0 & if \;\; t<0\end{cases}$$

Similarly, the advanced function is

$$G_A(x,t)=\frac{\theta(-t)}{2\pi}\frac{2i\alpha}{(x\mp t)^2+\alpha^2} \tag{A4}$$

The third Keldysh function in this basis for the right movers in momentum space is

$$F(\epsilon,k)=-2\pi i\;\tanh\frac{k}{2T}\delta(\epsilon-k). \tag{A5}$$

The Fourier integral is converging, and we get

$$F(x,t)=\frac{T}{\sinh\pi T(x-t)}. \tag{A6}$$

The function for the left movers has a sign opposite to that of the time variable. To be more careful with the pole one has to add $i\alpha$ to the space coordinate in Eq. (A6).

That completes the calculation of the fermions Green's functions in the rotated Keldysh basis,

$$G_{\rm rot}=\begin{pmatrix}0 & G_R\\ G_A & F\end{pmatrix}. \tag{A7}$$

To return to the original basis which is needed for bosonization, one has to perform the rotation

$$G=\begin{pmatrix}G^{++} & G^{+-}\\ G^{-+} & G^{--}\end{pmatrix}=RG_{\rm rot}R^{-1}, \tag{A8}$$

where the rotation matrix is given by

$$R=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}. \tag{A9}$$

We now calculate the boson functions. We start again with the rotated basis. The retarded Green's function in the momentum space is

$$D_R(\omega,q)=\frac{1}{\omega^2-q^2+2i\delta\omega}. \tag{A10}$$

Again we have to introduce the cutoff α. It is essential for purposes of bosonisation to do it in exactly the same way as for the case of fermions, Eq. (A2). That way we get

$$D_R(x,t)=-\frac{i}{4\pi}\theta(t)\ln\frac{(x-t+i\alpha)(x+t-i\alpha)}{(x-t-i\alpha)(x+t+i\alpha)}. \tag{A11}$$

For the advanced function we get the same logarithm,

$$D_A(x,t)=\frac{i}{4\pi}\theta(-t)\ln\frac{(x-t+i\alpha)(x+t-i\alpha)}{(x-t-i\alpha)(x+t+i\alpha)}. \tag{A12}$$

The third function in this basis, D_F, contains δ function, as does its fermion counterpart Eq. (50),

$$D_F(\omega,q)=-\frac{i\pi}{\omega}\coth\frac{\omega}{2T}[\delta(\omega-q)+\delta(\omega+q)], \tag{A13}$$

but we have to introduce the cutoff here. In real space we get

$$D_F(x,t)=\frac{i}{2\pi}\ln\frac{(\pi T\alpha)^2}{\sinh\pi T(x-t)\sinh\pi T(x+t)}, \tag{A14}$$

where the zeros of the denominator should be treated in exactly the same way as in the fermion case [see the text after Eq. (A6)].

One can clearly see that the fermion functions and the arguments of the logarithm in the boson functions are constructed from the same elements. That is why the bosonization works. To get exactly the same results in the fermion and boson perturbation series we have to turn to the original Keldysh basis Eq. (A8) and calculate physical quantities like the density-density correlation function, which is independent of Keldysh indices and therefore of the choice of the calculational technique. Then the boson and fermion versions are the same in the limit $\alpha=0$ (the physical quantities should not depend on that cutoff).

APPENDIX B

We start from the functional integral for the density-density correlation function Eq. (11), where the kinetic term now contains the nonzero imaginary part Eq. (31). We are interested in the infrared asymptotic of that correlator. Therefore we divide the boson field ϕ_μ into two part — ''fast'' and ''slow,''

$$\phi_\mu=\phi^s_\mu+\phi^f_\mu. \tag{B1}$$

The integral Eq. (11) then becomes

$$\Pi(x_1-x_2)=\frac{1}{Z}\int \mathcal{D}[\phi^s_1,\phi^s_2]\mathcal{D}[\phi^f_1,\phi^f_2]\partial_1\phi^{s*}_\alpha$$

$$\times(x_1)\tau_{\alpha\beta}\partial_1\phi^s_\beta(x_2)\exp(-A), \tag{B2}$$

where the prefactor contains only slow degrees of freedom since we are looking for the infrared asymptotic. We can separate the integral over fast variables as

$$I=\frac{1}{Z''}\int \mathcal{D}[\phi^f_1,\phi^f_2]\exp\biggl(\int\frac{d^2k'}{(2\pi)^2}[\phi^{f*}_\mu B_{\mu\nu}\phi^f_\nu -V'(\cos2\beta\phi^s_1\cos2\beta\phi^f_1-\sin2\beta\phi^s_1\sin2\beta\phi^f_1 -\cos2\beta\phi^s_2\cos2\beta\phi^f_2+\sin2\beta\phi^s_2\sin2\beta\phi^f_2)]\biggr). \tag{B3}$$

The density integral becomes now

$$\Pi(x_1-x_2)=\frac{1}{Z'}\int \mathcal{D}[\phi^s_1,\phi^s_2]\partial_1\phi^{s*}_\alpha(x_1)\tau_{\alpha\beta}\partial_1\phi^s_\beta(x_2) \exp\biggl(\int\frac{d^2k'}{(2\pi)^2}[\phi^{s*}_\mu B_{\mu\nu}\phi^s_\nu]\biggr)\exp(\ln M), \tag{B4}$$

where $\ln M$ plays now a role of renormalisation of the imaginary part of the kinetic operator.

We now calculate the integral over the fast degrees of freedom, Eq. (62). We expand the exponent in Eq. (62) up to the first nontrivial order in perturbation (which is actually all we need, noting the results of Sec. IV) and get the Gaussian integral without any prefactor,

$$M=\frac{V'^2}{2}\int \mathcal{D}[\phi^f_1,\phi^f_2]\exp\biggl(\int\frac{d^2k'}{(2\pi)^2}[\phi^{f*}_\mu B_{\mu\nu}\phi^f_\nu]\biggr)\int d^2x_1 d^2x_2[\cos2\beta\phi^s_1(x_1)\cos2\beta\phi^f_1(x_1)-\sin2\beta\phi^s_1(x_1)\sin2\beta\phi^f_1(x_1) -\cos2\beta\phi^s_2(x_1)\cos2\beta\phi^f_2(x_1)+\sin2\beta\phi^s_2(x_1)\sin2\beta\phi^f_2(x_1)][\cos2\beta\phi^s_1(x_2)\cos2\beta\phi^f_1(x_2)-\sin2\beta\phi^s_1(x_2)\sin2\beta\phi^f_1(x_2) -\cos2\beta\phi^s_2(x_2)\cos2\beta\phi^f_2(x_2)+\sin2\beta\phi^s_2(x_2)\sin2\beta\phi^f_2(x_2)]. \tag{B5}$$

We can now calculate the functional integral. The result is

$$\begin{aligned}M=\frac{V'^2}{4}\int d^2x_1d^2x_2\biggl\{&[\cos2\beta\phi^s_1(x_1)\cos2\beta\phi^s_1(x_2)+\sin2\beta\phi^s_1(x_1)\sin\mathrm{s}in2\beta\phi^s_1(x_2)]\\&\times\exp\biggl(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}D^{--}_k(1-e^{ikx})(1-e^{-ikx})\biggr)+[\cos2\beta\phi^s_2(x_1)\cos2\beta\phi^s_2(x_2)+\sin2\beta\phi^s_2(x_1)\sin2\beta\phi^s_2(x_2)]\\&\times\exp\biggl(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}D^{++}_k(1-e^{ikx})(1-e^{-ikx})\biggr)-[\cos2\beta\phi^s_1(x_1)\cos2\beta\phi^s_2(x_2)+\sin2\beta\phi^s_1(x_1)\sin2\beta\phi^s_2(x_2)]\\&\times\exp\biggl(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}(D^{++}_k+D^{--}_k-D^{-+}_ke^{ikx}-D^{+-}_ke^{-ikx})\biggr)-[\cos2\beta\phi^s_2(x_1)\cos2\beta\phi^s_1(x_2)\\&+\sin2\beta\phi^s_2(x_1)\sin2\beta\phi^s_1(x_2)]\exp\biggl(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}(D^{++}_k+D^{--}_k-D^{-+}_ke^{-ikx}-D^{+-}_ke^{ikx})\biggr)\biggr\},\end{aligned} \tag{B6}$$

where $x=x_1-x_2$ and k_0 is the momentum cutoff, delimiting fast variables from slow.

The remaining integrals are similar to those calculated in the regular perturbation series. We have the Fourier transform of the boson Green's functions to the real space in the exponent; then we take the exponential and Fourier transform back to the momentum space. The difference is that we now have another set of Green's functions — with a nonzero imaginary part — and the momentum integration in limited by the cutoff k_0. The result of this integration will now depend on the cutoff. For the retarded and advanced functions we get

$$D_R(x,t;k_0)=\frac{i}{4\pi}\theta(t)e^{-\eta' t}2i\ \mathrm{Im}(E_1\{k_0[\eta t-i(x+t)]\} -E_1\{k_0[\eta t+i(x-t)]\}), \tag{B7}$$

$$D_A(x,t;k_0)=-\frac{i}{4\pi}\theta(-t)e^{-\eta'|t|}2i\ \mathrm{Im}(E_1\{k_0[\eta t -i(x+t)]\}-E_1\{k_0[\eta t+i(x-t)]\}), \tag{B8}$$

where E_1 is the exponential integral. The result for the third Keldysh function D_F depends on whether the cutoff is larger or smaller than the temperature. For $k_0 > T$ we get

$$D_F(x,t;k_0>T) = -\frac{i}{2\pi}\frac{1}{2}\mathrm{Re}(E_1\{k_0[\alpha+i(x+t)]\} + E_1\{k_0[\alpha+i(x-t)]\}), \tag{B9}$$

so that at the origin

$$D_F(0;k_0>T) = \frac{i}{2\pi}\ln(k_0\alpha) \tag{B10}$$

which is a small number. For the other case, $k_0 < T$, in the limit of large x and t, and assuming that the original η is much less than the temperature, we have

$$D_F(x,t;k_0<T) = -\frac{iT}{\pi k_0}\left(\frac{\sin k_0(x+t)}{k_0(x+t)} + \frac{\sin k_0(x-t)}{k_0(x-t)}\right), \tag{B11}$$

and at the origin is

$$D_F(0;k_0<T) = -\frac{i}{\pi}\frac{T}{k_0}, \tag{B12}$$

which is extremely large.

We can now proceed with the renormalization of the imaginary part coefficient η. To do that we expand Eq. (B6) in boson fields, and get their bilinear combination

$$M = \frac{(V'\beta)^2}{4}\int d^2x_1 d^2x_2 \Big\{ \phi_1^s(x_1)\phi_1^s(x_2)\exp\left(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}D_k^{--}(1-e^{ikx})(1-e^{-ikx})\right) + \phi_2^s(x_1)\phi_2^s(x_2)\exp\left(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}D_k^{++}(1-e^{ikx})(1-e^{-ikx})\right) - \phi_1^s(x_1)\phi_2^s(x_2)\exp\left(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}(D_k^{++}+D_k^{--} - D_k^{-+}e^{ikx} - D_k^{+-}e^{-ikx})\right) - \phi_2^s(x_1)\phi_1^s(x_2)\exp\left(i\beta^2\int_{k'>k_0}\frac{d^2k'}{(2\pi)^2}(D_k^{++}+D_k^{--} - D_k^{-+}e^{-ikx} - D_k^{+-}e^{ikx})\right)\Big\}, \tag{B13}$$

which can be abbreviated as

$$M = \frac{(V'\beta)^2}{4}\int d^2x_1 d^2x_2[\phi_\alpha^s(x_1)\mathcal{R}_{\alpha\beta}(x)\phi_\beta^s(x_2)], \tag{B14}$$

where as usual $x = x_1 - x_2$.

The imaginary part coefficient η is most transparent in the rotated Keldysh basis Eq. (A7). The element of the matrix $\mathcal{R}_{\alpha\beta}(x)$ in the rotated basis which corresponds to the renormalization of η in the retarded function in terms of original elements is

$$\mathcal{R}_R = \tfrac{1}{2}[\mathcal{R}_{11}(x) - \mathcal{R}_{22}(x) - \mathcal{R}_{12}(x) + \mathcal{R}_{21}(x)]. \tag{B15}$$

For the case of the large cutoff this yields, after the Fourier transform, $i\omega\eta(k_0/k_0^4)\exp[-(\beta^2/2\pi)\ln k_0\alpha]$, so that the change in η is

$$\Delta\eta = \eta\frac{V^2}{4(k_0\alpha)^6}, \tag{B16}$$

since $\beta = \sqrt{4\pi}$. Here, as always, α is the lattice spacing which cuts off the large momentum integration. So here we have a power-law rise in η. When the cutoff is less that the temperature we gain a different exponential, so that now

$$\Delta\eta = \eta\exp\left(4\frac{T}{k_0}\right)\frac{V^2}{4(k_0\alpha)^4}. \tag{B17}$$

This means that, on scales of momenta less than that of the temperature, the imaginary part coefficient η experiences a tremendous growth, and we immediately get the diffusive dynamics.

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