Radiation linewidth of a long Josephson junction in the flux-flow regime

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(Received 3 April 1996)

A theoretical model for the radiation linewidth in a multifluxon state of a long Josephson junction is presented. Starting from a perturbed sine-Gordon model with a temperature-dependent noise term, we develop a collective coordinate approach which allows us to calculate the finite radiation linewidth due to excitation of the internal degrees of freedom in the moving fluxon chain. At low fluxon density, the radiation linewidth is expected to be substantially larger than that of a lumped Josephson oscillator. With increasing the fluxon density, a crossover to a much smaller linewidth corresponding to the lumped oscillator limit is predicted. [S0163-1829(96)01020-X]

A long Josephson junction is an example of a distributed nonlinear oscillator, the time-dependent response of which is associated with its intrinsic spatial dynamics. An externally applied magnetic field H penetrates into the junction in the form of Josephson fluxons (solitons), each of them carrying one magnetic flux quantum Φ_0. As schematically shown in Fig. 1, fluxons move across the junction under the influence of the bias current generating an electromagnetic radiation at the junction boundary. The frequency f of the radiation is given by the Josephson relation $f=V/\Phi_0$, where V is the dc voltage induced by the fluxon motion.

For a lumped (short) Josephson junction, the linewidth δf of the emitted radiation is determined by thermal fluctuations of current passing through the junction. Assuming a Nyquist noise spectrum, for a current-biased lumped Josephson tunnel junction the full linewidth at half power is given by the expression[1,2]

$$\delta f \equiv \Delta f_T f = \frac{4\pi k_B T}{\Phi_0^2}\frac{R_D^2}{R_S}, \tag{1}$$

where k_B is Boltzmann's constant and T is the temperature. The linewidth depends on the differential resistance $R_D=dV/dI$ at the junction bias point and the static resistance $R_S=V/I$, where I is the bias current flowing through the junction.

There are two main regimes which characterize fluxon motion in long junctions. First, a shuttlelike *resonant fluxon motion* gives rise to zero-field steps (ZFS's) in the dc current-voltage (I-V) characteristics of the junction. In this regime fluxons and antifluxons undergo reflections from the junction boundaries and the radiation frequency is determined by the junction length L and the fluxon velocity v as $f_{\mathrm{ZFS}}=v/2L$. Second, in the high magnetic field, the so-called *flux-flow regime* occurs and is manifested by a flux-flow step (FFS) on the I-V curve. In this regime fluxons are created at one boundary of the junction and annihilate at the other boundary. The radiation frequency $f=v/d_{\mathrm{fl}}$ is determined by the spacing between moving fluxons d_{fl}. In general, for both ZFS and FFS regimes, the radiation linewidth δf should be related to thermal fluctuations of the fluxon velocity v. Joergensen *et al.*[3] obtained the striking general result that, in spite of the different nature of the phase slippage in small and long junctions, the linewidth of the resonant single-fluxon radiation in a long junction is given by the same Eq. (1), except for a missing factor of 4 due to the modified Josephson relation $f=V/2\Phi_0$ at the ZFS. Experiment[3] showed reasonable agreement with that theory, though some excess linewidth broadening has been seen. For the multifluxon state which forms FFS's it can be argued that internal degrees of freedom in the moving fluxon chain may yield a significant contribution in δf. Here, in contrast to the resonant single-fluxon case, local variations of the fluxon spacing d_{fl} change the radiation frequency. Recent FFS radiation linewidth measurements by Koshelets *et al.*[4] showed the scaling of δf_{FF} as predicted by Eq. (1) but with an effective temperature T_{eff} being a factor of 8 larger than the physical temperature of their experiment. Since there was no theory for the radiation linewidth δf in the flux-flow regime, the reason for the excess noise was not resolved.

In this paper we present a theoretical model for the radiation linewidth in the flux-flow regime for long Josephson junctions. We start from the perturbed sine-Gordon model with spatially and temporarily dependent noise current. Using the collective coordinate approach we calculate the finite radiation linewidth due to the internal degrees of freedom in the moving fluxon chain. The obtained analytical result is evaluated for the relevant parameter range of experimentally studied flux-flow oscillators.

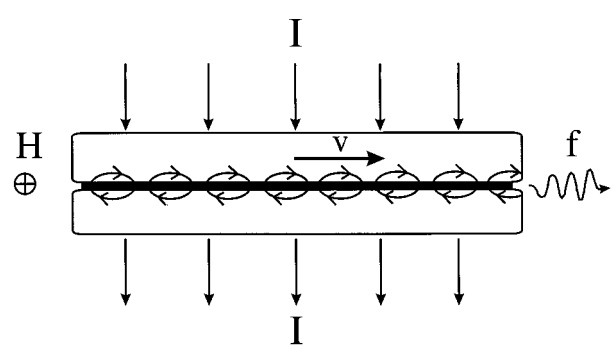

FIG. 1. Schematic cross section of a long Josephson junction in the flux-flow state. The electromagnetic radiation emitted at the junction boundary depends on the variations of the fluxon spacing due to temporaly and spatially dependent noise current through the junction.

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First, we introduce the main characteristic quantities for the system. The mean frequency of Josephson oscillations for a long junction in the flux-flow regime is

$$\langle f\rangle = \frac{v\langle H\rangle\Lambda}{\Phi_0} \equiv \beta\langle H\rangle, \tag{2}$$

where v is the velocity of the fluxon chain, $\langle H\rangle$ is the average magnetic field, and $\Lambda = 2\lambda_L + t$ is the effective magnetic thickness of the junction (λ_L is the London penetration depth, and t is the insulator thickness). The quantities essential for the linewidth problem are the mean square deviation of the frequency f,

$$\sqrt{\langle \delta f^2\rangle} = \beta\sqrt{\langle [H-\langle H\rangle]^2\rangle} = \beta\sqrt{\langle H^2\rangle - \langle H\rangle^2},$$

and reduced rms linewidth in the flux-flow regime,

$$\Delta f_{\mathrm{FF}} \equiv \frac{\sqrt{\langle \delta f^2\rangle}}{\langle f\rangle} = \sqrt{\frac{\langle H^2\rangle}{\langle H\rangle^2} - 1}. \tag{3}$$

In addition to the thermal fluctuations of the fluxon velocity,[3] there exists an additional mechanism of line broadening in the FFS regime. According to Eq. (3), this mechanism is directly related to the irregularities of the magnetic field distribution in the junction, i.e., to the fluxon density fluctuations in the moving fluxon chain. The most general physical origin for these fluctuations is the chain's deformation under the influence of thermal noise. Therefore, the thermal noise causes two different contributions to the total linewidth in the FFS regime. In the following, the dependence of the fluxon chain deformations on the velocity and magnetic field and their contribution to the linewidth are calculated.

The magnetic field distribution in the junction is proportional to the gradient of the phase difference $\phi(x,t)$ between superconducting electrodes, $H(x,t) = (\Phi_0/2\pi\Lambda\lambda_J)\phi_x(x,t)$, where λ_J is the Josephson penetration depth. In the following we denote the normalized magnetic field as $h(x,t) \equiv \phi_x(x,t)$. The phase evolution is governed by the perturbed sine-Gordon equation[5] (written in laboratory coordinates)

$$\phi_{xx} - \phi_{tt} - \sin\phi = \gamma + \alpha\phi_t + n(x,t), \tag{4}$$

where the length x and time t are normalized to λ_J and to the inverse plasma frequency ω_p^{-1}, respectively. The first term on the right hand side of Eq. (4), γ, represents the external bias current density J normalized to the critical current density J_c, α is the damping coefficient, and the term $n(x,t)$ represents the thermal fluctuations[3] with the white-noise spatio temporal correlator[6]

$$\langle n(x_1,t_1)n(x_2,t_2)\rangle = \frac{16\alpha k_B T}{E_0}\delta(x_1-x_2)\delta(t_1-t_2) \equiv \alpha\tau\delta(x_1-x_2)\delta(t_1-t_2). \tag{5}$$

Here the brackets $\langle\cdots\rangle$ stand for statistical averaging (over the ensemble), $E_0 = 8\hbar J_c W\lambda_J/2e$ is the rest energy of a fluxon, W is the junction's width, and e is the electron charge.

The phase distribution in the flux-flow regime is given by the cnoidal-wave solution written in laboratory coordinates as

$$\phi = \phi_{\mathrm{cn}}(x - vt - \xi) \quad \text{where } \phi_{\mathrm{cn}}(z) = \pi - 2\,\mathrm{am}(z/k\sqrt{1-v^2}). \tag{6}$$

Here $\mathrm{am}(z)$ is the elliptic amplitude, v is the velocity of the fluxon chain, and the slowly varying shift ξ describes the chain's deformation. In the coordinate frame moving at the velocity v, $x' = (x-vt)/\sqrt{1-v^2}$, and $t' = (t-vx)/\sqrt{1-v^2}$, the deformation is governed by the linear equation[7]

$$\xi_{t't'} - \xi_{x'x'} + \alpha\xi_{t'} = \rho^{-1}n(x',t'), \tag{7}$$

where $\rho = 4E(k)/k^2K(k)$ is the density of the fluxon chain, and the elliptic modulus k is related to the chain's period $l' = 2kK(k)$ in the moving frame, $K(k)$ and $E(k)$ being the complete elliptic integrals of the first and second kinds, respectively. The average velocity of the fluxon chain v under the external bias current γ is given by the Marcus-Imry perturbation approach[8] as

$$v = \left[1 + \left(\frac{4\alpha}{\pi\gamma}\frac{E(k)}{k}\right)^2\right]^{-1/2}. \tag{8}$$

This relation determines the form of the current-voltage ($I-V$) characteristics in the flux-flow regime. In the laboratory coordinate frame k in Eq. (8) is determined by given fluxon spacing l as a root of the transcendental equation $l/\sqrt{1-v^2} = 2kK(k)$.[8]

The local magnetic field in the junction $h(x,t)$ is given by the expression

$$h(x,t) = \phi'_{\mathrm{cn}}(x-vt-\xi)(1-\xi_x) \simeq [\phi'_{\mathrm{cn}}(x-vt) - \phi''_{\mathrm{cn}}(x-vt)\xi](1-\xi_x), \tag{9}$$

where, using the smallness of ξ, we have Taylor expanded the first multiple in Eq. (9), keeping the first two terms. Next we define the mean and mean squared magnetic fields as

$$\langle h\rangle = \phi'_{\mathrm{cn}}(x-vt), \quad \langle h^2\rangle = \langle(\phi'_{\mathrm{cn}} - \phi''_{\mathrm{cn}}\xi - \phi'_{\mathrm{cn}}\xi_x)^2\rangle. \tag{10}$$

Substituting (10) into Eq. (3), we obtain the following relation between the reduced rms linewidth and the deformation of the fluxon chain ξ:

$$\Delta f_{\mathrm{FF}} = \sqrt{\langle\xi_x^2\rangle + \left\langle\left[\xi\frac{\phi''_{\mathrm{cn}}}{\phi'_{\mathrm{cn}}}\right]^2\right\rangle}. \tag{11}$$

Thus, in order to calculate Δf_{FF} we need to find the solution of Eq. (7) which can be done by means of the Fourier transform. First, we write Eq. (7) in the laboratory reference frame:

$$\xi_{tt} - \xi_{xx} + \frac{\alpha v}{\sqrt{1-v^2}}\xi_x + \frac{\alpha}{\sqrt{1-v^2}}\xi_t = \rho^{-1}n(x,t). \tag{12}$$

Using the Fourier transform $\xi(x,t) = \int_{-\infty}^{+\infty}\xi(q,\omega)\exp(iqx - i\omega t)\,dq\,d\omega$ it is straightforward to find the solution to Eq. (12):

$$\xi(q,\omega) = \frac{\rho^{-1}n(q,\omega)}{q^2 - \omega^2 - i\alpha(\omega - vq)(1-v^2)^{-1/2}}, \tag{13}$$

where $n(q,\omega)$ is the Fourier transform of the thermal noise $n(x,t)=\int_{-\infty}^{+\infty} n(q,\omega)\exp(iqx-i\omega t)\,dq\,d\omega$ which is subject to the correlations

$$\langle n(q,\omega)n(q',\omega')\rangle=\frac{\alpha\tau}{(2\pi)^2}\delta(q+q')\delta(\omega+\omega'). \quad (14)$$

Here, we consider only the case of a sufficiently dense fluxon chain in the form

$$\phi_{\rm cn}\simeq\langle h\rangle(x-vt)-\langle h\rangle^{-2}(1-v^2)^{-1}\sin[\langle H\rangle(x-vt)], \quad (15)$$

which assumes

$$\langle h\rangle^2(1-v^2)\gg 1. \quad (16)$$

In this case, the coefficient which enters Eq. (11) is $(\phi''_{\rm cn}/\phi'_{\rm cn})^2\simeq\langle h\rangle^{-2}(1-v^2)^{-2}$. Note that according to Eq. (12) we have $\xi_x^2/\xi^2\sim\alpha^2v^2(1-v^2)^{-1}$. It is easy to see that, if we add to the chain stiffness condition (16) the additional condition

$$\alpha^2\langle h\rangle^2v^2(1-v^2)\ll 1, \quad (17)$$

the first term under the square root in Eq. (11) may be neglected as compared to the second term. The condition (17) holds in the most practically important case of the underdamped junction. Since $v^2(1-v^2)$ is always below its maximum value of 1/4, conditions (16) and (17) can be combined as

$$\frac{1}{1-v^2}\ll\langle h\rangle^2\ll\frac{4}{\alpha^2}. \quad (18)$$

For the region (18) we thus obtain

$$\Delta f_{\rm FF}=\langle h\rangle^{-1}(1-v^2)^{-1}\sqrt{\langle\xi^2(x,t)\rangle}, \quad (19)$$

where we have inserted the above approximation for $(\phi''_{\rm cn}/\phi'_{\rm cn})^2$. For the stiff fluxon chain its elliptic modulus k and mass density ρ are given by particularly simple expressions $k=2/\langle h\rangle\sqrt{1-v^2}$ and $\rho=\langle h\rangle^2(1-v^2)$.

Now, what remains is to calculate the rms value $\sqrt{\langle\xi^2\rangle}$. Using Eqs. (13) and (14) this quantity can be calculated explicitly. However, we encounter a divergence when performing integration over small wave numbers q. This divergence is regularized by the fact that, in a Josephson junction of a finite length L, the smallest wave number is π/L. Taking this into account we finally obtain

$$\Delta f_{\rm FF}=\frac{4}{\pi(1-v^2)^{3/2}\langle h\rangle^3}\sqrt{\frac{L}{\lambda_J}\frac{1}{\alpha}\frac{k_BT}{E_0}}. \quad (20)$$

This expression provides the linewidth of a Josephson flux-flow oscillator. It is related to the intrinsic mechanism of the fluxon chain deformation under the influence of thermal noise.

The calculation of another noise contribution Δf_T due to the thermal motion of the fluxon chain as a whole (similar to that of Ref. 3) gives a result identical to that of Eq. (1). Thus, the total reduced linewidth of the Josephson flux-flow oscillator is given by the expression

$$\Delta f_\Sigma=\Delta f_T+\Delta f_{\rm FF}. \quad (21)$$

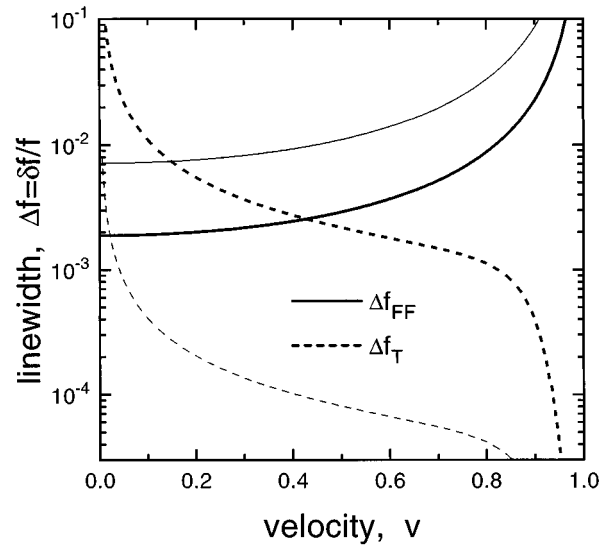

FIG. 2. Reduced flux-flow oscillations linewidth $\delta f\equiv\Delta f/f$ given by Eq. (20) (solid lines) and Eq. (1) (dashed lines) as a function of the fluxon chain velocity v. Thick and thin lines correspond to two typical parameter sets accounting for Refs. 9 and 4, respectively.

Let us discuss the physical meaning of the result (20). First, the term k_BT/E_0 is typically small, of the order of magnitude 10^{-4}–10^{-5}. Therefore, a comparison of Eq. (1) with Eq. (20) shows that, due to the square-root dependence of $\Delta f_{\rm FF}$ on k_BT/E_0, the linewidth in the flux-flow regime can be essentially broader as compared to the resonant flux motion regime. Moreover, the flux-flow linewidth scales with the junction length as $\sqrt{L}$, which means additional broadening for longer junctions. Physically, this dependence is related to a formal divergence of the chain's deformation in the infinitely long junction.

According to Eq. (20), the linewidth is rapidly reduced with an increase of the magnetic field h which makes the fluxon chain more stiff. In sufficiently high magnetic fields $h>h^*$ the contribution determined by Eq. (20) may become smaller than that due to thermal fluctuations of the chain velocity given by Eq. (1). In this case the crossover to the standard mechanism discussed in Ref. 3 takes place and the total linewidth is determined by Eq. (1).

The crossover magnetic field h^* depends essentially on the dimensionless ratios $k_BT/\alpha E_0$, L/λ_J, and the fluxon chain velocity v. We note that Eqs. (1) and (20) predict quite different dependences on v. Namely, Δf_T decreases with v and becomes very small in the relativistic regime $v\to 1$. In contrast to that, $\Delta f_{\rm FF}$ increases with v.

In order to compare Δf_T and $\Delta f_{\rm FF}$ it is useful to estimate them for typical parameters of practical Josephson tunnel junctions. Figure 2 shows the calculated reduced linewidths given by Eqs. (1) and (20) as functions of the fluxon velocity for two typical sets of experimental parameters.[9,4] In order to determine R_D and R_S in Eq. (1), the form of the current-voltage (I-V) characteristics for the fluxon spacing $l=2\pi/h$ was calculated using the Marcus-Imry perturbation formula (8) which with given parameters was independently checked by numerical simulations for periodic boundary conditions. In both cases we took the dissipation coefficient

$\alpha = 0.02$ and the normalized magnetic field $h = 4$ (which is approximately twice the first critical field of the fluxon penetration into the junction) at $T = 4.2$ K. In the case of low-J_C junctions[9] ($J_C = 200$ A/cm^2) we used $L = 200$ μm, $W = 10$ μm, and $\lambda_J = 35$ μm, and in the other case of high-J_C junctions[4] ($J_C = 8000$ A/cm^2) the values $L = 450$ μm, $W = 3$ μm, and $\lambda_J = 4$ μm were used. In Fig. 2 one can see that in both cases the effect of the spatially dependent fluxon chain noise contribution Δf_{FF} given by Eq. (20) is dominating, in particular at high fluxon velocities. At low velocities the reduced linewidth Δf_T (1) is formally diverging due to finite value of δf_T at $f = 0$, while the intrinsic flux-flow linewidth Δf_{FF} saturates at the finite level according to Eq. (20). In both cases the rest fluxon energy $k_B T/E_0$ is of the order of 10^{-5} J. For the used magnetic field $h = 4$ the stiff fluxon chain assumption (16) strictly holds only for $v < v_s = 0.6$. Since at $v > v_s$ the fluxon chain can only become more soft, Eq. (20) gives the lowest limit for the expected Δf_{FF}.

We note, however, that in experiments the flux-flow oscillators are often operated close to a resonant regime. In such cases the junction frequency spectrum is modulated by characteristic eigenfrequencies which determine the oscillation amplitudes (13). In the resonant states the linewidth can be expected to be narrowed by cavity resonances (known as Fiske steps) in the junction. Thus, the nonresonant flux-flow frequency linewidth given by Eq. (20) can be taken as the *upper* limit for the real flux-flow oscillator of the finite length.

Finally, we would like to mention that an additional mechanism of the flux-flow oscillator linewidth broadening may arise from technological inhomogeneities in the junction. Such a contribution is nonuniversal and depends strongly on the particular type of disorder (local imperfections in the tunnel barrier, the precision of the photolithography defined junction width, etc.), and therefore this mechanism was not discussed in the present paper. This issue will be the subject of further investigation.

We would like to thank M. Cirillo and V. P. Koshelets for useful discussions. One of the authors (B.A.M.) appreciates the hospitality of KFA Jülich. This work was partially supported by INTAS Grant No. 94-1783.

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