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Scattering of plasmons in a quasi-two-dimensional electron gas containing a fixed-point charge

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We study the scattering of two-dimensional (2D) plasmons by the carrier density nonuniformity created by a charged impurity. The plasmons are described by an integral equation obtained within the hydrodynamic model of the 2D plasma containing a fixed-point charge. We obtain the energy of the scattered plasmons, and evaluate the scattering cross section in the Born approximation. We find that at low energy it varies as the square of the frequency of the incident plasmon. This behavior is markedly different from the three-dimensional plasmon scattering where the scattering cross section goes to a nonzero value at zero momentum. In addition, we employ the random-phase approximation to treat both correlation and finite thickness effects in quantum wells. This allows us to extend these results for the 2D inhomogeneous hydrodynamic model to treat intrasubband plasmons in semiconductor quantum wells. [S0163-1829(96)02028-0]

I. INTRODUCTION

Collective plasma excitations in quasi-two-dimensional systems such as doped semiconductor quantum wells have attracted considerable interest in recent years. In particular the effects of confinement on plasmon dispersion have been well studied.[1,2] In addition, recently the interaction of quantum-well plasmons with impurities such as ionized acceptors[3] and neutral donors[4] have been of interest. An electron plasma interacting with a single ionized donor or acceptor can be represented by an electron gas containing a fixed-point charge. In a three-dimensional electron gas the interaction with a fixed charge causes scattering of the plasmons by the nonuniformity of the plasma density. Furthermore, for a negative impurity charge the plasmons in an electron gas can form a bound state localized in the vicinity of the impurity.[5]

In the case of a semiconductor quantum well with a finite electron or hole density, there are both intrasubband plasmons which are associated with a single-carrier subband and also intersubband plasmons which are associated with carrier transitions between two subbands.[2] The energy of the intersubband plasmons has a positive lower bound. We have shown in an earlier study that this feature contributes to the existence of the bound state of the intersubband plasmon.[3] On the other hand the dispersion curves of the intrasubband plasmons begin from zero energy and do not have an upper bound. These plasmons are scattered by the density inhomogeneity created by the Coulomb interactions of the plasma carriers (electrons or holes) with the charged impurity, but are not expected to form localized states. The main focus of the present work is to evaluate the scattering cross section of these plasmons. We find, in particular, that at small wave vectors the scattering cross section is linear in the wave vector, and correspondingly at low energy it varies as the square of the frequency of the incident plasmon. This behavior is very different from the three-dimensional case,[5] where the cross section goes to a nonzero constant at zero wave vector. We also study a state formed as a linear combination of plasma waves, with the frequencies below the frequencies of the two-dimensional (intrasubband) plasmon at the same wave vectors. We show that when the charge of the impurity is opposite to the charge of the carriers, such a dispersing wave-packet solution exists at densities above a certain threshold.

In Sec. II we begin by considering plasmons in a two-dimensional plasma (of electrons or holes) of zero thickness. We do it in order not to obscure the essential physics of intrasubband plasmons by finite thickness effects. We begin by using a hydrodynamic model for the electron plasma. This turns out to be an adequate approximation for the long-wavelength plasmons. In the context of this model we then correct for the fact that the speed of sound should be determined from the velocity distribution of the degenerate electron gas rather than from the compressibility of the electron fluid. Later on, in the appendixes we present a quantum-mechanical treatment using the random-phase approximation (RPA), and show that a long-wavelength approximation of the RPA produces an integral equation similar to the one obtained here in the hydrodynamic model.

The presentation is organized as follows. In Sec. II we describe the hydrodynamic two-dimensional (2D) model, and derive an integral equation that describes the interaction of plasmons with a fixed-point charge. In Sec. III we consider the appropriate scattering problem, and derive the scattering cross section for the plasmons. In Sec. IV we discuss some dispersing wave-packet solutions which are interesting because they are composed of plasmons with frequencies lower than those of free plasmons at the same values of the wave vector. In Appendix A we use the RPA to derive sum rules that allow us to obtain an integral equation for the plasmon in an inhomogeneous electron gas without appeal to the hydrodynamic model. In Appendix B we incorporate finite thickness effects to obtain the integral equation for the scattering of the intrasubband plasmons in a quantum well. In Appendix C we obtain an expression for the energy flux vector for the two-dimensional electron plasma oscillations as a zero-thickness limit of the energy flux in a thin plasma layer.

II. HYDRODYNAMIC MODEL FOR TWO-DIMENSIONAL ELECTRON GAS

We treat an electron gas as a fluid with a three-dimensional (3D) scalar density field $\rho_b(\mathbf{r},z;t)$ and a vector velocity field $\mathbf{v}_b(\mathbf{r}z;t)$, where $\mathbf{r}$ is a two-dimensional (2D) position vector, z is a coordinate on the axis perpendicular to the plasma layer, t is a time variable, and subscript b (bulk) indicate a 3D field. Define 2D and 3D gradient operators by

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}, \quad \nabla_b \equiv \nabla + \hat{z}\frac{\partial}{\partial z}. \tag{1}$$

A positive background that compensates for the total charge of the electrons is assumed. For a zero-thickness plasma layer the 3D density can be written as

$$\rho_b(\mathbf{r},z)=\delta(z)\rho(\mathbf{r}). \tag{2}$$

The 3D fields ρ_b and v_b appear in the equation of continuity and the equation of motion (the Euler equation).[6,7] Projection on the $z=0$ plane gives equations for a 2D plasma:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0,$$
$$m\rho\frac{\partial \mathbf{v}}{\partial t}+m\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=-e\rho\mathbf{E}-\nabla p, \tag{3}$$

where $\mathbf{E}$ is an electric-field projection on the $z=0$ plane, $\mathbf{E}=\hat{x}E_x+\hat{\mathbf{y}}E_y=-\nabla\phi$, p is the electron fluid pressure which describes effects of the electron kinetic energy, and m is an electron effective mass. In an equilibrium $\rho(\mathbf{r})=n_0+\delta n(\mathbf{r})$, where n_0 is the 2D density of electrons in the absence of the charged impurity, and $\delta n(\mathbf{r})$ is a change in the density distribution due to the presence of the impurity. Due to the compensating positive background, $\mathbf{E}=\mathbf{0}$ in the equilibrium. Under a perturbation, we have

$$\rho(\mathbf{r},t)=n_0+\delta n(\mathbf{r})+\delta\rho(\mathbf{r},t), \tag{4}$$

where $\delta\rho(\mathbf{r},t)$ describes the time-dependent plasma oscillations, and determines the electric potential $\phi(\mathbf{r},z;t)$ through the 3D Poisson equation

$$\nabla^2\phi+\frac{\partial^2\phi}{\partial z^2}=\frac{4\pi e}{\varepsilon_s}\delta\rho(\mathbf{r})\delta(z), \tag{5}$$

where ε_s is a background dielectric constant. Define a 2D Fourier transform so that

$$\delta\rho(\mathbf{r},t)=(2\pi)^{-3}\int d^2q\int_{-\infty}^{\infty}d\omega\ e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\delta\rho(\mathbf{q},\omega). \tag{6}$$

Then, from Eq. (5) we obtain[1]

$$\phi(\mathbf{q},z)=-\frac{2\pi e}{\varepsilon_s q}\delta\rho(\mathbf{q})\exp(-q|z|). \tag{7}$$

For an adiabatic process the pressure $p(\mathbf{r},t)$ is a function of the density $\rho(\mathbf{r},t)$, and we can write[6]

$$\nabla p=\left(\frac{dp}{d\rho}\right)_0\nabla\rho=ms^2\nabla\rho, \tag{8}$$

where s is the "speed of sound." The appropriate value of s will be discussed below.

For small oscillations, $\delta\rho\ll n_0$, we linearize Eq. (3) in $\delta\rho$ and $\mathbf{v}$ to obtain

$$\frac{\partial^2\delta\rho}{\partial t^2}+(e/m)(n_0+\delta n)\nabla^2\phi+(e/m)\nabla\delta n\cdot\nabla\phi-s^2\nabla^2\delta\rho=0. \tag{9}$$

We apply the Fourier transformation defined in Eq. (6), set $z=0$ in Eq. (7), and obtain an integral equation relating the Fourier components $\delta\rho(\mathbf{q},\omega)$ and $\delta n(\mathbf{q})$:

$$\left(\omega^2-\frac{2\pi e^2 n_0}{\varepsilon_s m}q-s^2q^2\right)\delta\rho(\mathbf{q},\omega)=\frac{2\pi e^2n_0}{\varepsilon_s m}\int\frac{d^2q'}{(2\pi)^2}\frac{\mathbf{q}\cdot\mathbf{q}'}{q'}\frac{\delta n(\mathbf{q}-\mathbf{q}')}{n_0}\delta\rho(\mathbf{q}',\omega), \tag{10}$$

where $\delta n(\mathbf{q})$ is a Fourier component of the change in the electron density due to the presence of the impurity of charge Ze/ε_s, where the external charge Ze is screened by the dielectric constant ε_s. For a classical electron fluid, δn would be determined by Debye screening, whereas here we are interested in a degenerate electron gas where the Thomas-Fermi screening is an appropriate choice:[8]

$$\delta n(\mathbf{q})=\frac{Z}{\varepsilon_s}\left(1-\frac{1}{\varepsilon(\mathbf{q})}\right), \tag{11}$$

where $\varepsilon(\mathbf{q})$ is the static limit of the RPA for the dielectric function $\varepsilon(\mathbf{q},\omega)$. Therefore

$$\delta n(q)=\frac{Z}{\varepsilon_s}\frac{G(q)}{G(q)+q/q_{\mathrm{TF}}}. \tag{12}$$

The parameter q_{TF} is the 2D Thomas-Fermi wave vector[9]

$$q_{\mathrm{TF}}=\frac{1}{a_2}=\frac{2}{a_0\varepsilon_s(m_0/m)}, \tag{13}$$

where a_2 is 2D Bohr radius equal to the one-half of the 3D Bohr radius a_3, $a_0=\hbar^2/m_0e^2$, m_0 is the vacuum electron mass, and m is the effective mass of the carrier in the 2D plasma. The function $G(q)$ accounts for the finite size of the Fermi surface,[10]

$$G(q)=1 \quad \text{if} \quad q<2q_F$$
$$G(q)=1-[1-(2q_F/q)^2]^{1/2} \quad \text{if} \quad q>2q_F, \tag{14}$$

where q_F is the 2D Fermi wave vector. For electrons in two dimensions with two spin degrees of freedom, q_F is given by

$$q_F=(2\pi n_0)^{1/2}. \tag{15}$$

Within the hydrodynamic model the speed of sound s in Eq. (8) is determined from the inverse compressibility of the degenerate electron gas, which gives $s^2=v_F^2/2$. The hydrodynamic model uses average quantities ρ and $\mathbf{v}$, and does not take into account correlation effects. A full derivation of s^2 involves the use of the Boltzman-Vlasov equation for the velocity distribution, and the long-wavelength expansion should involve its higher moments.[7] This expansion gives[7] $s^2=3\langle v_x^2\rangle=3/2\langle v^2\rangle$, where v_x is a one-dimensional component of the velocity. Using a normalized velocity distribution for the 2D degenerate electron gas, $f(v)=(1/\pi v_F^2)\Theta(v_F-v)$ where v_F is the Fermi velocity, we obtain

$$s^2=\tfrac{3}{4}v_F^2. \tag{16}$$

This is identical to the result obtained with the RPA in a self-consistent quantum model,[1] and is different from the value obtained from the inverse compressibility. In the following we use the form in Eq. (16) for s^2.

Let us define an auxilliary 3D density n_3 and dimensionless 2D density N as

$$n_3\equiv n_0/a_3, \quad N\equiv n_0 a_2^2. \tag{17}$$

We define an auxilliary frequency ω_p to be the plasmon frequency of the 3D gas with the density n_3, and define the dimensionless frequency Ω:

$$\omega_p^2\equiv\frac{4\pi e^2 n_3}{\varepsilon_s m}, \quad \Omega^2\equiv\frac{\omega^2}{\omega_p^2}. \tag{18}$$

We also define a dimensionless wave vector and a dimensionless Fermi wave vector:

$$\mathbf{p}\equiv\mathbf{q}/q_{\mathrm{TF}}=\mathbf{q}a_2, \quad p_F\equiv(2\pi N)^{1/2}. \tag{19}$$

Now we can rewrite the integral equation (10) in a dimensionless form,

$$(\Omega^2-p-\tfrac{3}{2}p^2)\,\delta\rho(\mathbf{p})=\int d^2p'\,K(\mathbf{p},\mathbf{p}')\,\delta\rho(\mathbf{p}'),$$
$$K(\mathbf{p},\mathbf{p}')=\frac{Z}{(2\pi)^2N\varepsilon_s}\,\frac{\mathbf{p}\cdot\mathbf{p}'}{p'}\,\frac{G(\mathbf{p}-\mathbf{p}')}{G(\mathbf{p}-\mathbf{p}')+|\mathbf{p}-\mathbf{p}'|}. \tag{20}$$

In Sec. III we will consider the scattering states of this equation, and calculate the scattering cross section in the Born approximation.

III. PLASMON SCATTERING

In order to simplify the derivation, in this section we keep only the leading term in the long-wavelength expansion of the free-plasmon dispersion relation, and so set $\Omega_0^2=p$. Then we rewrite Eq. (20) for the scattering problem in the following form:

$$(p-k)f_{\mathbf{k}}(\mathbf{p})=Q(\mathbf{k};\mathbf{p}), \tag{21}$$

where

$$Q(\mathbf{k};\mathbf{p})=-\int d^2p'\,K(\mathbf{p},\mathbf{p}')f_{\mathbf{k}}(\mathbf{p}'), \tag{22}$$

with K given in Eq. (20).

For the free plasmons with wave vector $\mathbf{k}$ and frequency Ω,

$$f_{\mathbf{k}}^0(\mathbf{p})=(2\pi)^2\delta(\mathbf{p}-\mathbf{k}), \quad k=\Omega^2. \tag{23}$$

The dependence of the free-plasmon frequency on the square root of the wave vector leads to a scattering problem with a linear spectrum.

The propagator for Eq. (21) is

$$g^{\pm}(\mathbf{p})=\frac{1}{p-k\mp i\gamma}. \tag{24}$$

In real space the propagator is obtained from Eq. (24) by a Fourier transformation. Integrating over the angles, we have

$$2\pi g^{\pm}(\mathbf{r})=\frac{1}{r}+k\int_0^{\infty}dp\,\frac{J_0(pr)}{p-k\mp i\gamma}, \tag{25}$$

where J_0 is a Bessel function of the first kind,[11,12] and $\mathbf{r}$ here is understood to be a dimensionless 2D vector $\mathbf{r}/a_2$. The integral can be further transformed by using a particular integral representation of the Bessel function,

$$J_0(x)=(2/\pi)\int_1^{\infty}dt(t^2-1)^{-1/2}\sin(xt). \tag{26}$$

The integral in Eq. (25) can be separated into a sum of two integrals, one of which has a nonsingular integrand, and another which has an integrand with poles at $k\pm i\gamma$ and can be evaluated in a complex plane. We obtain

$$2\pi g^{\pm}(\mathbf{r})=\pm i\pi H_0^{(1)}(kr)+u(kr), \tag{27}$$

where for large kr, $u(kr)=1/k^3r^3+\cdots$, and $H_0^{(1)}$ is a Hankel function of the first kind.[12] Thus for large kr, asymptotically,

$$2\pi g^{\pm}(\mathbf{r})\approx\pm i\pi\left(\frac{2}{\pi kr}\right)^{1/2}e^{-i\pi/4}e^{\pm ikr}+\cdots\;. \tag{28}$$

Notice that asymptotically the propagator for the problem with a linear spectrum turns out to be twice the propagator for the problem with a quadratic spectrum in two dimensions.[11]

Using Eq. (23) in Eq. (21), we can obtain a series expansion for the scattering states in momentum space. To identify a scattering amplitude, we need solutions in real space. From Eq. (21) we obtain

$$f_{\mathbf{k}}^{\pm}(\mathbf{r})=\frac{1}{(2\pi)^2}\int_0^{\infty}dp\,\frac{p}{p-k\mp i\gamma}\int_0^{2\pi}d\theta\,e^{ipr\cos\theta}$$
$$\times Q(\mathbf{k};p\cos\theta,p\sin\theta). \tag{29}$$

Omitting the explicit dependence on $\mathbf{k}$, let us define the even and odd functions $F_0(p)$ and $F_1(p)$:

$$2F_0(p)=Q(\mathbf{k};p,0)+Q(\mathbf{k};-p,0),$$
$$2F_1(p)=Q(\mathbf{k};p,0)-Q(\mathbf{k};-p,0). \tag{30}$$

Notice that $Q(\mathbf{k};p,0)=Q(\mathbf{k};p\hat{\mathbf{r}})$ because $\theta=\mathbf{p}\wedge\mathbf{r}$. By expansion of Q in a Taylor series, we can show that

$$2\pi f_{\mathbf{k}}^{\pm}(\mathbf{r})=\int_0^\infty dp\ p\ \frac{J_0(rp)F_0(p)+iJ_1(rp)F_1(p)}{p-k\mp i\gamma}. \tag{31}$$

We write J_1 as a derivative of J_0, use the integral representation (26), and evaluate the integral in Eq. (31) similarly to the evaluation of the propagator. Assume that $F_0(p)$ and $F_1(p)$ are analytical in the upper half-plane. Then we obtain

$$2\pi f_{\mathbf{k}}^{\pm}(\mathbf{r})=\pm i\pi[H_0^{(1)}(\pm kr)F_0(k)\pm iH_1^{(1)}(\pm kr)F_1(k)]+u(kr)\quad\text{where}\quad u(kr)=1/k^3r^3+\cdots. \tag{32}$$

From this equation we obtain the asymptotic behavior of the scattered wave for $kr\gg 1$:

$$2\pi f_{\mathbf{k}}^{+}(\mathbf{r})\approx ie^{-i\pi/4}\left(\frac{2\pi}{kr}\right)^{1/2}e^{ikr}Q(k\hat{\mathbf{k}};k\hat{\mathbf{r}}). \tag{33}$$

If F_0 and F_1 are singular in the upper half-plane but the singularities are confined to poles and branch cuts away from the real axes, then these poles and cuts will make contributions to $f_{\mathbf{k}}(\mathbf{r})$ that decay exponentially as exp(-Ar) for large r and do not contribute to the asymptotic behavior in Eq. (32).

The density variation $\delta\rho(\mathbf{r})$ is obtained as the real part of $f_{\mathbf{k}}^{+}(\mathbf{r})$. Equation (33) describes the scattering of the plasma density oscillations with $|f|^2$ per unit length being constant at large r. We will show now that Eq. (33) describes the scattering of plasmon energy by the density variation $\delta n(\mathbf{r})$ induced by the impurity charge. In order to do this we look at the real-space equation describing plasma oscillations in a 2D electron gas. Unlike the bulk case, the 2D real-space equation contains a term singular at the origin as a result of the linear p term in $\Omega^2(p)$. In order to obtain a physically sensible result for the energy in the scattered wave, we consider classical plasma oscillations in an electron layer of finite thickness d. This problem can be analyzed similarly to the surface-plasmon problem.[13] The equation is the bulk plasma equation with boundary conditions of zero normal velocity at the two surfaces $z=0$ and $z=d$. In order not to burden the main text, we analyze the plasmon problem in a layer of finite thickness in Appendix C.[14]

Taking the limit $d\to 0$ in the finite layer plasmon problem and using Eq. (C13), we can evaluate the energy flux. Prior to the evaluation of the energy flux J_H, we consider the Born approximation to the scattering problem,[11] setting $f_{\mathbf{k}}(\mathbf{p}')=(2\pi)^2\delta(\mathbf{p}'-\mathbf{k})$ and $k=\Omega^2$ in Eq. (22). We obtain

$$-Q_B(k\hat{\mathbf{k}};k\hat{\mathbf{r}})=\frac{Zk\cos\theta}{N\varepsilon_s}\,\frac{G(2k\sin\theta/2)}{G(2k\sin\theta/2)+2k\sin\theta/2}, \tag{34}$$

where $\theta=\mathbf{r}\wedge\mathbf{k}$. In the same approximation,

$$f_{\mathbf{k}}^{+}(\mathbf{r})\approx f(\theta)\,\frac{e^{ikr}}{\sqrt{r}}, \tag{35}$$

where the scattering amplitude is given by

$$f(\theta)=-\frac{Zie^{-i\pi/4}}{N\varepsilon_s\sqrt{2\pi}}\sqrt{k}\cos\theta\,\frac{G}{G+2k\sin\theta/2}. \tag{36}$$

The density variations $\delta\rho(\mathbf{r},t)$ are obtained as $\mathrm{Re}\{f_k^{+}(\mathbf{r})\exp(i\omega_k t)\}$. Equation (C13) can be evaluated in polar coordinates to give the radial component of the energy flux at large r,

$$\mathbf{j}_r=B^2\hat{\mathbf{r}}k\omega_k\,\frac{|f(\theta)|^2}{r}\sin^2(kr-\omega_k t), \tag{37}$$

where B is defined in Eq. (C14). The energy flux in a scattered wave through the circle of radius r is given by

$$\mathbf{J}_r=\int \mathbf{j}\cdot\hat{\mathbf{r}}\,dl=\int \mathbf{j}\cdot\hat{\mathbf{r}}\ r\ d\theta. \tag{38}$$

Taking an average over many time periods, we obtain

$$\langle\mathbf{J}_r\rangle=\lim(T\to\infty)\ \frac{1}{T}\int_0^T dt\ \mathbf{J}_r=\int d\theta\ \tfrac{1}{2}B^2\,\frac{\hat{\mathbf{r}}}{r}\,k\omega_k|f(\theta)|^2. \tag{39}$$

The 2D scattering cross section is, by definition, a ratio of scattering flux to the incident flux per unit length. We have been using dimensionless units above, and we now restore the original units by taking $r\to ra_2$. Therefore

$$d\sigma/a_2=\frac{\langle rj_r\rangle}{\langle j_{\mathrm{inc}}\rangle}\,d\theta=|f(\theta)|^2d\theta, \tag{40}$$

where $f(\theta)$ is given in Eq. (36). The total cross section is obtained by integrating over the angle.

In the limit of small k, for the low-energy scattering cross section we obtain

$$\sigma/a_2\approx\frac{kZ^2}{2N^2\varepsilon_s^2}=\frac{\omega^2mZ^2}{4\pi e^2\varepsilon_s n_0^3a_2^3}. \tag{41}$$

This is markedly different both from bulk-plasmon scattering[5] and intersubband plasmon scattering[3] because in those cases σ goes to a nonzero value at zero wave vector. At large wave vectors the scattering cros section obtained from Eqs. (40) and (36) decreases as $1/k$ for $q_{\mathrm{TF}}\ll k<2q_F$, and as $1/k^5$ for $k\gg 2q_F$.

IV. WAVE-PACKET SOLUTIONS OF THE INTEGRAL EQUATION

Let us consider Eq. (6) as a wave packet of plasma oscillations with a dispersion $\omega(q)$,

$$\delta\rho(\mathbf{q},\omega)=(2\pi)^3\delta\rho(\mathbf{q})\delta[\omega-\omega(q)],$$
$$\delta\rho(\mathbf{r},t)=\int d^2q\ \exp[i(\mathbf{q}\cdot\mathbf{r}-t\omega(q))]\delta\rho(\mathbf{q}). \tag{42}$$

For the 3D plasmons[5] and also for intersubband plasmons in quantum wells,[3] if Z has the same sign as the plasma carriers, one finds localized solutions (''bound states'') with $\omega(q)=\omega_b$, which is a constant frequency below the minimum frequency of the free plasmons. There are no such solutions for the 2D case or for intrasubband plasmons in a quantum well. However, we can look for the solutions of the form (42) with $\omega(q)<\omega_0(q)$, where $\omega_0(q)$ is the dispersion law of the free 2D plasmons. In particular, we take

$$\Omega^2(p)=\alpha^2(p+\tfrac{3}{2}p^2) \tag{43}$$

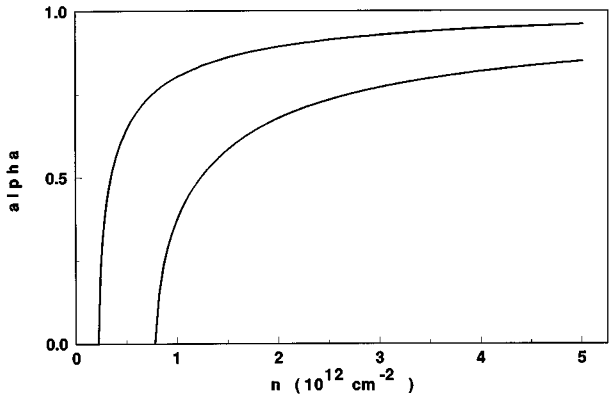

FIG. 1. The parameter α, defined in Eq. (43), is shown as a function of the 2D density n. It relates the frequencies of the oscillations contributing to the wave-packet solution of the integral equation (45) to the dispersion law of the free plasmons $\omega_0(q)$ through $\omega = \alpha^2 \omega_0(q)$. At each value of the density the two lowest allowed values of α are shown for $m = 0.0665 m_0$, $\epsilon_s = 12.35$, and $Z = 40$.

in Eq. (20), with $\alpha^2 < 1$. The integral equation will have solutions for $Z < 0$, i.e., for a negative fixed charge in the electron gas or for a positive fixed charge in a hole gas.

We can expand the density oscillations in 2D harmonics,

$$\delta\rho(\mathbf{q}) = \sum_{n=0}^{\infty} e^{in\theta} \psi_n(q), \tag{44}$$

and consider the $n=0$ solution, the s wave. It is obtained as a solution of the following integral equation:

$$\psi_0(p) = \frac{|Z|}{1-\alpha^2} \frac{1}{N\varepsilon_s} \int_0^{\infty} \frac{dp'}{(2\pi)^2} \frac{p'}{1+\frac{3}{2}p} \int_0^{2\pi} d\theta \times \frac{G(\Delta p')}{G(\Delta p') + \Delta p'} \psi_0(p'),$$
$$\Delta p' \equiv (p^2 + p'^2 - 2pp' \cos\theta)^{1/2}. \tag{45}$$

This equation can be symmetrized and cast in the form of a one-dimensional integral equation with a symmetric real kernel. The resulting equation can be solved numerically by the method described in our study of the intersubband plasmons.[3] We find that there is a discrete set of values of α for which a wave packet in Eq. (42) is a solution of the integral equation.

As an example we consider a 2D electron gas with material parameters appropriate for GaAs: $m/m_0 = 0.0665$ and $\varepsilon_s = 12.35$. The resulting values of α start at zero at some value of the density, and then approach 1. In order to obtain a value of α substantially different from 1 at reasonably high densities, one needs a high value of Z. Physically this may be realized by a small cluster of charged acceptors. As an illustration we show the two lowest s-wave solutions for $Z = 40$ in Fig. 1. This shows the existence of the wave-packet solutions whose components have lower phase velocities than free plasmons.

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APPENDIX A: RANDOM-PHASE APPROXIMATION FOR THE INHOMOGENEOUS 2D ELECTRON GAS

The Hamiltonian for a system of independent electrons (or holes) interacting with a charged impurity is

$$H_0 = K + H_{c\text{-}i}, \tag{A1}$$

where K is the kinetic-energy operator, and $H_{c\text{-}i}$ describes the interaction of the carriers with the fixed charge through the Coulomb interaction $V(\mathbf{r}\text{-}\mathbf{r}')$, which has a Fourier transform

$$V_q(z) = \frac{2\pi e^2}{\varepsilon_s q} \exp(-q|z|). \tag{A2}$$

For a strictly 2D gas, $z \to 0$ in Eq. (A2). The total Hamiltonian is

$$H = H_0 + H_{\text{int}}, \tag{A3}$$

where H_{int} describes carrier-carrier interactions. In our previous study of intersubband plasmons in a quantum well, explicit expressions in the second quantization were given, and the RPA and its long-wavelength approximation for an inhomogeneous gas of carriers were derived. Here we adapt that study for the cases of intrasubband plasmons and 2D plasmons.

Let $\delta\rho(\mathbf{k},\omega)$ describe a Fourier component of the variation of plasma density in response to a weak external perturbation $\delta\rho^{\text{ext}}$. In linear-response theory[9] the change of the total carrier density in the presence of an external perturbation, $\delta\rho^T = \delta\rho + \delta\rho^{\text{ext}}$ is related to $\delta\rho^{\text{ext}}$ through the inverse dielectric operator:

$$\delta\rho^T(\mathbf{k},\omega) = \sum_{\mathbf{k}'} \varepsilon^{-1}(\mathbf{k},\mathbf{k}';\omega)\, \delta\rho^{\text{ext}}(\mathbf{k}',\omega). \tag{A4}$$

Inverting this equation and defining collective excitations by the condition $\delta\rho^T \neq 0$, while $\delta\rho^{\text{ext}} = 0$, we obtain the following equation for the collective excitations of the system H:

$$\int d^2k'\, \varepsilon(\mathbf{k},\mathbf{k}';\omega)\, \delta\rho^T(\mathbf{k}',\omega) = 0. \tag{A5}$$

The Dyson equation in the RPA can be derived in close analogy to that for a homogeneous 2D system.[2] The carrier density propagator is defined as a time-ordered density-density correlation function. The lowest-order term in the perturbation expansion of its irreducible part is given by

$$\Pi^0(\mathbf{k},\mathbf{k}';\omega)=(2\pi)^{-4}\int d^2q\; d^2q'\int_{-\infty}^{\infty}d\omega' \times G(\mathbf{q},\mathbf{q}';\omega+\omega')G(\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q}';\omega'), \tag{A6}$$

where $G(\mathbf{k},\mathbf{k}';\omega)$ is a one-particle propagator for the system of independent electrons interacting with the impurity, i.e., for the Hamiltonian H_0 in Eq. (A1).

At zero temperature in the same approximation, the irreducible retarded propagator[9] is given by

$$\mathrm{Re}\Pi^R(\omega)=\mathrm{Re}\Pi^0(\omega),\qquad \mathrm{Im}\Pi^R(\omega)=(\mathrm{sgn}\omega)\mathrm{Im}\Pi^0(\omega). \tag{A7}$$

From the Dyson equation for the density propagator and the definition of the dielectric operator, we obtain the RPA for ε, in the operator form $\hat{\varepsilon}=1\text{-}\Pi^R V$. With this we obtain the RPA for Eq. (A5). Omitting the explicit ω dependence,

$$\int d^2k'[\delta(\mathbf{k}-\mathbf{k}')-\Pi^R(\mathbf{k},\mathbf{k}')V(\mathbf{k}')]\delta\rho^T(\mathbf{k}')=0. \tag{A8}$$

Following the treatment of the bulk case,[8,3] we obtain the long-wavelength expansion for Π^R using the spectral representation

$$\hbar\Pi^R(\mathbf{k},\mathbf{k}';\omega)=\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\frac{\sigma(\mathbf{k},\mathbf{k}';\nu)}{\omega-\nu+i\gamma}, \tag{A9}$$

where $\gamma\rightarrow+0$. The spectral density is given by

$$\sigma(\mathbf{k},\mathbf{k}';\omega)=\int_{-\infty}^{\infty}dt\; e^{i\omega t}A^{-1}\langle[\hat{\rho}(\mathbf{k},t),\hat{\rho}^+(\mathbf{k}',0)]\rangle, \tag{A10}$$

where $\hat{\rho}$ is a Heisenberg operator defined with the Hamiltonian H, A is the normalization area, and at zero temperature the averaging of the commutator is performed over the ground state of H. The real part of Π will be obtained taking the principal value in Eq. (A7).

We expand $1/(\omega\text{-}\nu)$ in powers of ν, and use the following identity:

$$(2\pi)^{-1}\int_{-\infty}^{\infty}d\nu\;\nu^n\sigma(\nu)=(-i)^n\frac{\partial^n\sigma(t)}{\partial t^n}\bigg|_{t=+0}. \tag{A11}$$

The right-hand side can be evaluated from the equations of motion for the Heisenberg density operators defined with the Hamiltonian H_0 in Eq. (A1), $i\hbar(\partial\rho/\partial t)=[\rho,H_0]$. In this way an infinite sequence of the sum rules is obtained. The terms with even powers of ν vanish. The first nonvanishing term is given by the following sum rule:

$$(2\pi)^{-1}\int_{-\infty}^{\infty}d\nu\;\nu\sigma(\mathbf{q},\mathbf{q}';\nu)=\frac{\hbar^2}{m}\mathbf{q}\cdot\mathbf{q}'n(\mathbf{q}-\mathbf{q}'). \tag{A12}$$

Here $n(\mathbf{q})$ is a Fourier transform of $n_0+\delta n(\mathbf{r})$, where n_0 is a density of the uniform gas and $\delta n(\mathbf{r})$ is a change of the density due to the presence of the fixed charge Ze/ε_s.

The validity of the expansion is restricted to $(qv_F)/\omega<1$. For plasmons with a square-root dispersion law, $\omega(q)\propto\sqrt{q}$, this condition will be satisfied for sufficiently small q. From the static limit of the RPA we obtain $\delta n(q)$ in the form of the Thomas-Fermi screening as in Eq. (12). The sum rules are used in Eq. (A9). To the same order in the long-wavelength approximation as in Sec. II, and using the dimensionless variables defined in Eq. (17)–(19), from Eqs. (A8) and (A9) we obtain the following integral equation:

$$\left(\Omega^2-p-\frac{3}{2}\frac{p^3}{\Omega^2}\right)\delta\rho(\mathbf{p}) =\frac{Z}{N\varepsilon_s}\int\frac{d^2p'}{(2\pi)^2}\frac{\mathbf{p}\cdot\mathbf{p}'}{p'}\frac{G(\mathbf{p}-\mathbf{p}')}{G(\mathbf{p}-\mathbf{p}')+|\mathbf{p}-\mathbf{p}'|}, \tag{A13}$$

where function $G(p)$ is defined in Eq. (14).

For $Z=0$ the dispersion relation for free plasmons is obtained from the left-hand side of this equation as expansion in powers of p,

$$\Omega_0^2=p+\tfrac{3}{2}p^2+\cdots. \tag{A14}$$

To leading order in the wave vector, the integral equation (A13), which is derived here as the long-wavelength approximation of the RPA, reproduces Eq. (20) derived in the hydrodynamic model of Sec. II. The differences appear in higher orders of p^2/Ω^2, and are due to the neglect of correlations in the hydrodynamic approach.

APPENDIX B: SCATTERING OF INTRASUBBAND PLASMONS IN QUANTUM WELL

The RPA method discussed in Appendix A for the 2D plasmons can be extended to the case of intrasubband plasmons in the quantum well, and thereby finite thickness effects can be incorporated in the present work. In our previous study[3] we presented a detailed theory for the intersubband plasmons in the quantum-well plasma interacting with the charged impurity. In a similar fashion we now obtain an integral equation that describes the scattering of intrasubband plasmons by the density inhomogeneity due to the impurity charge. In the dimensionless form,

$$\left(\Omega^2-p-\frac{3}{2}\frac{p^3}{\Omega^2}+c_0p^2\right)\delta\rho(\mathbf{p})=\int d^2p'K(\mathbf{p},\mathbf{p}')\delta\rho(\mathbf{p}'), \tag{B1}$$

where K is given by the following form:

$$K(\mathbf{p},\mathbf{p}')=\frac{Z}{(2\pi)^2N\varepsilon_s}\frac{\mathbf{p}\cdot\mathbf{p}'}{p'}\frac{G(\Delta p')S(\Delta p')}{\Delta p'+G(\Delta p')F(\Delta p')},\qquad \Delta p'\equiv|\mathbf{p}-\mathbf{p}'|. \tag{B2}$$

$G(p)$ is given in Eq. (14), and the quantities c_0, S, and F are given by

$$c_0=\int dz\; dz'\xi_1^2(z)\xi_1^2(z')|z-z'|, \tag{B3}$$

$$S(p)=\int dz\; e^{-p|z|}\xi_1^2(z), \tag{B4}$$

$$F(p)=\int dz\, dz'\, \xi_1^2(z)\xi_1^2(z')e^{-p|z-z'|}, \tag{B5}$$

where $\xi_1(z)$ is a normalized subband wave function of the one-dimensional confining potential of the quantum well. From the free-plasmon dispersion relation (A14), it is clear that the p^3/Ω^2 term is $O(p^2)$. The derivation in Sec. III applies here, and the low-energy scattering cross section ($k\rightarrow 0$ limit) is given by Eq. (41).

APPENDIX C: HYDRODYNAMIC MODEL FOR A LAYER OF PLASMA

The equation for free plasma oscillations in a layer of thickness d can be obtained from the bulk equation

$$\delta\ddot{\rho}+\omega_p^2\delta\rho-s^2\nabla_b^2\delta\rho=0, \tag{C1}$$

with the boundary condition of zero normal velocity at the surfaces. The subscript b indicates a 3D vector. We define a dimensionless variable $\xi=zq$, and perform a Fourier transformation in time and coordinates x and y. We then have the following equation for the layer plasmons:

$$\left(\Omega^2-1-\frac{s^2q^2}{\omega_p^2}\right)\delta\rho(\mathbf{q},\xi)+\frac{s^2q^2}{\omega_p^2}\frac{\partial^2}{\partial\xi^2}\delta\rho(\mathbf{q},\xi)=0 \tag{C2}$$

for $0<\xi<qd$, with boundary conditions

$$v_z(z=0)=v_z(z=d)=0, \tag{C3}$$

where ω_p is a bulk plasma frequency and $\Omega=\omega/\omega_p$. The velocity field is determined by the Euler equation. In the proper treatment of the problem we should keep the pressure variation term, i.e., $s^2\neq 0$. If we want to keep only the leading term in the resulting dispersion law, the limit $s\rightarrow 0$ can be taken, but only in the end.[13] The solution of Eqs. (C2) and (C3) is given by

$$\delta\rho(\mathbf{q},\xi)=2A\cosh b\left(\xi-\frac{dq}{2}\right), \tag{C4}$$

$$b^2=\frac{1+s^2q^2/\omega_p^2-\Omega^2}{s^2q^2/\omega_p^2}. \tag{C5}$$

In the limit $sq\rightarrow 0$ and $dq\ll 1$, A is related to the normal component of the electric field at the $z=0$ surface:

$$A=E_z(0)\frac{\varepsilon_s\omega_p}{4\pi e^2 s}. \tag{C6}$$

The boundary conditions result in the following equation for b, and implicitly for ω:

$$\coth(bdq/2)[1-\exp(-dq)]-b[1+\exp(-dq)]$$
$$=2b(1-b^2)s^2q^2/\omega_p^2. \tag{C7}$$

For a thin layer we expand the exponents to $O(d^2q^2)$ and take into account that $b=\omega_p/sq+O(dq)$. We define a 2D density as $n_0=dn_3$, and from Eq. (C7) we obtain the dispersion relation

$$\omega^2=(2\pi e^2n_0/\varepsilon_s m)q+s^2q^2+\cdots . \tag{C8}$$

Equation (C1) can be written as a Euler equation for a scalar field $f=\delta\rho(\mathbf{r}_b,t)$ for the Lagrangian density

$$\mathcal{L}=\frac{1}{2}\dot{f}^2-\tfrac{1}{2}s^2|\nabla_b f|^2-\tfrac{1}{2}\omega_p^2 f^2. \tag{C9}$$

The corresponding canonical momentum and Hamiltonian density are given by

$$\pi=\frac{\partial\mathcal{L}}{\partial\dot{\rho}}=\dot{f},\quad H=\pi\dot{f}-\mathcal{L}. \tag{C10}$$

From Eqs. (C1) and (C10), it is simple to show that

$$\frac{\partial H}{\partial t}+\nabla_b\cdot(s^2\dot{f}\nabla_{\mathbf{b}}f)=0, \tag{C11}$$

and therefore we can define the Hamiltonian flux vector[14] as

$$\mathbf{j}_H=s^2\dot{f}\nabla f. \tag{C12}$$

In the finite layer problem we use Eqs. (C4) and (C6) with Eq. (C12) to obtain a two-dimensional vector of energy flux expressed through the boundary value of the electric field,

$$\mathbf{j}=B^2\delta\dot{\rho}(\mathbf{r},t)\nabla\delta\rho(\mathbf{r},t), \tag{C13}$$

where

$$B=\frac{E_z(0)\varepsilon_s\omega_p}{4\pi e^2}. \tag{C14}$$

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