Universality class in the one-dimensional localization problem

Masanori Yamanaka*
Institute for Solid State Physics, University of Tokyo, 7-22-1, Roppongi, Minato-ku, Tokyo 106, Japan

Yshai Avishai†
Institute for Solid State Physics, University of Tokyo, 7-22-1, Roppongi, Minato-ku, Tokyo 106, Japan and Department of Physics, Ben Gurion University of the Negev, Beer-Sheva 84 105, Israel

Mahito Kohmoto
Institute for Solid State Physics, University of Tokyo, 7-22-1, Roppongi, Minato-ku, Tokyo 106, Japan
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Weak disorder behavior of the Lyapunov exponent is investigated for a one-dimensional disordered system whose band structure and transfer matrix form are manifestly different from the standard ones encountered in tight-binding models. For diagonal disorder, the critical exponents governing the divergence of the localization length at zero disorder are identical with those predicted for tight-binding models. For off-diagonal disorder, a new exponent is found in one of the band edges, indicating a different universality class. The scaling functions near the different band edges are displayed, and their values for zero arguments are not identical at all edges. [S0163-1829(96)01725-0]

I. INTRODUCTION

The present work is concerned with localization of a quantum particle in an effective one-dimensional disordered system, which has two distinct properties. First, its spectrum contains a finite gap and second, its transfer matrices are generically correlated (albeit it is short ranged).

Localization of electrons within the single-particle approximation is best illustrated in one-dimensional disordered systems. In most cases the problem is reduced to the computation of the Lyapunov exponent γ resulting from a product of random transfer matrices. Despite the apparent elementary nature of the underlying physics, some subtle points emerge when the precise dependence of the localization length ξ on energy E and disorder strength W is looked for. Even for the simplest model of a tight-binding Hamiltonian with constant nearest-neighbor hopping terms and random site energies, one does not know the functional form of $\gamma(E,W)=\xi^{-1}(E,W)$. On the other hand, for weak disorder, self-consistent perturbation theory has been worked out, revealing a rich structure of $\gamma(E,W)$ in both its arguments.[1,2] It can be briefly summarized as follows: (a) If E is well inside the energy band $[-2,2]$, then $\gamma(E,W)\approx C(E)W^2$ where the function $C(E)$ is not specified but has an infinite number of singular points (Kappus-Wegner singularities.[3]) (b) When E is close to the band edge (say the point $E_b=2$), $\gamma(E,W)$ is nonanalytic in W and can be represented as $\gamma(E,W)=W^{2/3}f(x)$ where $f(x)$ with $x=(E-2)W^{-4/3}$ is a scaling function which is regular and finite at $x=0$. For $x\to\infty$ $f(x)\approx\sqrt{x}$ while for $x\to-\infty$ $\mathrm{Re}[f(x)]\approx 1/x$.

Unfortunately, this method works under the restrictive condition that the transfer matrices are uncorrelated. Furthermore, it is applicable only as long as a single channel is open. Recently, some progress has been made in circumventing the later constraint.[4–6] On the other hand, if the transfer matrices are correlated, an analytic expression for the coefficients in a weak disorder expansion is apparently out of question. Yet, it is interesting to test, at least numerically, the question of how robust the results mentioned above are, and whether different behavior of the localization length arises once the condition of uncorrelated transfer matrices is relaxed. In particular, if a new power of W will arise, it points toward a different universality class. This is the main motivation for the present work.

Our starting point is a one-dimensional tight-binding model on a decorated lattice composed of two sublattices, A and B, such that nearest neighbor hopping is allowed within the sublattice A and between sublattices, but not within sublattice B. Incidentally, this special geometry is also used in the construction of a solvable model in the study of strongly correlated systems.[7–10] As far as the single-particle approximation is concerned, this model can be mapped on a purely one-dimensional one by eliminating the sublattice B. When the system is clean (constant hopping and zero site energies) the spectrum contains two bands with a finite gap between them. Furthermore, the bands are not symmetric with respect to each other and have different extents. This means that all four band edges are different from each other as far as the behavior of the Lyapunov exponent is concerned. When the system is disordered, the 2×2 transfer matrix T_n depends (in an irreducible way) on random variables belonging to the n and the $n-1$ sites so that the condition for analytic expansion is not fulfilled and we have to resort to numerical calculations.

For diagonal (site) disorder, we find that the universality class is identical with the one-dimensional tight-binding model one, whereas for off-diagonal (bond) disorder, a different universality class prevails at one of the four band edges. Smooth scaling functions are calculated near the four band edges and their asymptotic behavior is extracted. It is also noticed that the vicinities of the band edges where scaling is valid are extremely small.

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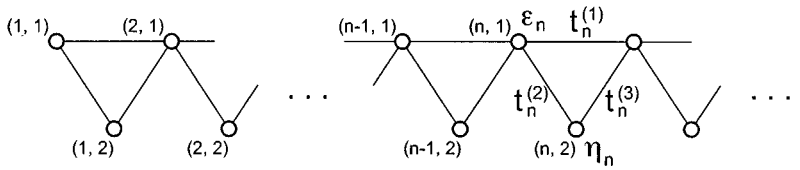

FIG. 1. Lattice structure of the model. The open circle represents a site which is labeled by (n,m). The lines represent hoppings of electrons.

In Sec. II we describe the model, and explain the method of solution. The results are displayed in Sec. III. Finally, in the Appendix we carry out weak disorder expansion of the Lyapunov exponent for a one-dimensional chain with off-diagonal disorder and zero site energies. We then confirm that the critical exponents in this case are identical with those found for the one-dimensional chain with constant hopping matrix elements and random site energies. This result corroborates our argument that the different exponent found near one of the band edges is generic for the present model.

II. MODEL

We consider the tight-binding Hamiltonian on a decorated lattice shown in Fig. 1 under the open boundary condition

$$H=\sum_{n=1}^{N}[(-t_n^{(1)}c_{n,1}^{\dagger}c_{n+1,1}-t_n^{(2)}c_{n,1}^{\dagger}c_{n,2}-t_n^{(3)}c_{n,2}^{\dagger}c_{n+1,1}+\mathrm{H.c.})+\epsilon_n c_{n,1}^{\dagger}c_{n,1}+\eta_n c_{n,2}^{\dagger}c_{n,2}], \tag{2.1}$$

where $c_{n,m}$ is the fermion annihilation operator at site (n,m), which satisfies the standard anticommutation relations. Here $t_n^{(j)}$ $(j=1, 2, 3)$ is the hopping matrix element and ϵ_n and η_n are the on-site potentials on the sites $(n,1)$ and $(n,2)$, respectively. They are randomly distributed according to a given probability distribution $\rho(X)$ where $X=t^{(j)}$, ϵ, or η. The ground state of this model with infinitely large on-site Coulomb repulsion is investigated by several authors.[7–10]

By using a one-particle state which is written as

$$|\Psi\rangle=\sum_{n=1}^{N}\sum_{m=1,2}\psi_{n,m}c_{n,m}^{\dagger}|0\rangle, \tag{2.2}$$

where $\psi_{n,m}$ are complex coefficients, the Schrödinger equation $H|\Psi\rangle=E|\Psi\rangle$, where E is the energy eigenvalue, is

$$\epsilon_n\psi_{n,1}-t_n^{(1)}\psi_{n+1,1}-t_{n-1}^{(1)}\psi_{n-1,1}-t_n^{(2)}\psi_{n,2}-t_{n-1}^{(3)}\psi_{n-1,2}=E\psi_{n,1},$$

$$\eta_n\psi_{n,2}-t_n^{(2)}\psi_{n,1}-t_n^{(3)}\psi_{n+1,1}=E\psi_{n,2}. \tag{2.3}$$

We first show the band structure for the pure case, where we set $t_n^{(1)}=t_n^{(2)}=t_n^{(3)}=1$ and $\epsilon_n=\eta_n=0$. We consider the system with an even number of unit cells under the periodic boundary condition $c_{N+1,1}=c_{1,1}$. From the Fourier transformation $\psi_{n,m}=(1/\sqrt{N})\Sigma_k e^{ikn}\psi_m(k)$, where

$$k=0,\pm\frac{2\pi}{N},\pm\frac{4\pi}{N},\ldots,\pm 2\pi\frac{N/2-1}{N},\pi, \tag{2.4}$$

the Schrödinger equation in the momentum space is

$$E\psi_1(k)=-(2\cos k)\psi_1(k)-(1+e^{-ik})\psi_2(k),$$

$$E\psi_2(k)=-(1+e^{ik})\psi_1(k). \tag{2.5}$$

The eigenenergies are

$$E_{\pm}(k)=-\cos k\pm\sqrt{1+(1+\cos k)^2}, \tag{2.6}$$

where $-$ and $+$ are the band indices. The dispersion relations are shown in Fig. 2. The edges of the bands are at $E=-1-\sqrt{5}$, 0, $-1+\sqrt{5}$, and 2.

Let us now move on to study the disordered system, for which we employ the transfer matrix method. The following Schrödinger equation is obtained by eliminating the variables $\psi_{n,2}$ using the second equation in (2.3):

$$\left(t_n^{(1)}+\frac{t_n^{(2)}t_n^{(3)}}{\eta_n-E}\right)\psi_{n+1,1}=\left(\epsilon_n-E-\frac{(t_n^{(2)})^2}{\eta_n-E}-\frac{(t_{n-1}^{(3)})^2}{\eta_{n-1}-E}\right)\psi_{n,1}-\left(t_{n-1}^{(1)}+\frac{t_{n-1}^{(2)}t_{n-1}^{(3)}}{\eta_{n-1}-E}\right)\psi_{n-1,1}, \tag{2.7}$$

which can be written as

$$\Psi_{n+1}=T_n\Psi_n, \tag{2.8}$$

where

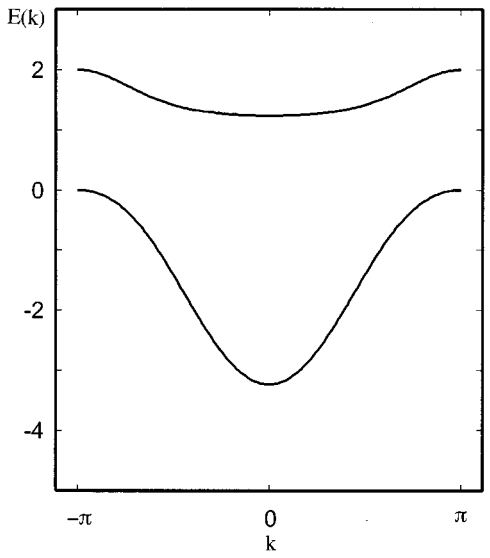

FIG. 2. The dispersion relations E_+ and E_-. The parameters are $t_n^{(1)}=t_n^{(2)}=t_n^{(3)}=1$ and $\epsilon_n=\eta_n=0$. The momentum k is expressed in units of the inverse lattice constant, which is taken to be equal to unity.

$$\Psi_n=\begin{pmatrix}\psi_{n,1}\\ \psi_{n-1,1}\end{pmatrix},\quad T_n=\begin{pmatrix}\dfrac{\epsilon_n E(t_n^{(2)})^2/(\eta_n-E)(t_{n-1}^{(3)})^2/(\eta_{n-1}-E)}{t_n^{(1)}+t_n^{(2)}t_n^{(3)}/(\eta_n-E)} & \dfrac{t_{n-1}^{(1)}+t_{n-1}^{(2)}t_{n-1}^{(3)}/(eta_{n-1}-E)}{t_n^{(1)}+t_n^{(2)}t_n^{(3)}/(\eta_n-E)}\\ 1 & 0\end{pmatrix}. \tag{2.9}$$

This defines the transfer matrix of the effective one-dimensional problem. It has three properties which makes it entirely different from the usual 2×2 transfer matrices encountered in one-dimensional tight-binding models. (1) Its determinant is not equal to unity. This defect can easily be circumvented though. (2) Its elements have a complicated dependence on energy, and (3) the T_n depends on random variables defined on the nth and the $(n-1)$th unit cells. In particular, the (1,1) element contains a sum of such random terms. This prevents any attempt to factorize it into a product of the form $T_n=V_nU_{n-1}$ which enables the recasting of the problem in terms of uncorrelated transfer matrices.

The quantity in which we are interested is the Lyapunov exponent γ. In one dimension, all the eigenstates are exponentially localized even in infinitesimally weak disorder,[11] and the localization length ξ is the inverse of γ. The method that we choose in order to evaluate the Lyapunov exponent has been used in the context of the Anderson localization.[11] The exponent is given by

$$\gamma=\lim_{N\to\infty}\frac{1}{N}\ln||\mathbf{T}_N\Psi_0||, \tag{2.10}$$

where

$$\mathbf{T}_N=\prod_{n=1}^{N}T_n, \tag{2.11}$$

$||\mathbf{T}_N\Psi_0||$ denotes the norm of the vector, and Ψ_0 is a initial vector. Since the determinant of the matrix T_n is not 1, we define the ''normalized'' transfer matrix by

$$\widetilde{T}_n=\sqrt{\frac{t_n^{(1)}+t_n^{(2)}t_n^{(3)}/(\eta_n-E)}{t_{n-1}^{(1)}+t_{n-1}^{(2)}t_{n-1}^{(3)}/(\eta_{n-1}-E)}}T_n, \tag{2.12}$$

whose determinant is 1, and define the product of the matrices by

$$\widetilde{\mathbf{T}}_N=\prod_{n=1}^{N}\widetilde{T}_n. \tag{2.13}$$

The relation between (2.11) and (2.13) is

$$\mathbf{T}_N=\sqrt{\frac{t_0^{(1)}+t_0^{(2)}t_0^{(3)}/(\eta_0-E)}{t_N^{(1)}+t_N^{(2)}t_N^{(3)}/(\eta_N-E)}}\widetilde{\mathbf{T}}_N. \tag{2.14}$$

Therefore, the Lyapunov exponent (2.10) is written by using the transfer matrices with the determinant 1 as

$$\gamma=\lim_{N\to\infty}\frac{1}{N}\ln\left[\sqrt{\frac{t_0^{(1)}+t_0^{(2)}t_0^{(3)}/(\eta_0-E)}{t_N^{(1)}+t_N^{(2)}t_N^{(3)}/(\eta_N-E)}}||\widetilde{\mathbf{T}}_N\Psi_0||\right]. \tag{2.15}$$

III. RESULTS

Using the numerical procedure explained above we calculate the Lyapunov exponent for two types of randomness, diagonal disorder, and off-diagonal disorder. Although the random matrices are correlated, it is anticipated that the Lyapunov exponent is a self-averaging quantity and hence, the accuracy of the numerical calculation is limited only by statistical errors which decay as $\sqrt{N}$. We perform the calculation for chain length up to $N=10^8$ by normalizing the vector Ψ_n in every step of the product. Only at $E=1$ the elements of the transfer matrix diverge and we cannot obtain solution by this method. Recall our anticipation, namely, that inside the band, the Lyapunov exponent is analytic in W, so that $\gamma\approx W^\alpha$ with an integer critical exponent. On the other hand, near a band edge E_b, the usual weak disorder perturbation expansion breaks down, since, beside W there is a second small parameter, namely, $E-E_b$. Guided by the results of Ref. 2 we seek a parametrization of the form $\gamma(E,W)=W^\alpha f(x)$ where $f(x)$ with $x=(E-E_b)W^{-2\alpha}$ is a scaling function which is regular and finite at $x=0$. For x large and E outside the band, $f(x)\approx\sqrt{x}$ while for x large and E inside the band, $\mathrm{Re}[f(x)]\approx 1/x$.

Notice however that the definition of being ''inside the band'' applies here also to points which are close to the band edge. Thus, the point $E=1.99$ is considered here as being inside the band, although the band edge is $E=2$. The true neighborhood of the band edge (judging from the power behavior of the Lyapunov exponent) is extremely small in the present model. Yet, it is unambiguously characterized by a different power and a scaling function, as we shall show below.

A. Critical exponents

Starting with diagonal disorder, we sample the site energies from the uniform rectangular distribution

$$\rho(\epsilon)=1,\quad \text{for } -\frac{W}{2}\leqslant\epsilon_n\leqslant\frac{W}{2}, \tag{3.1}$$

$$\rho(\eta)=1,\quad \text{for } -\frac{W}{2}\leqslant\eta_n\leqslant\frac{W}{2}, \tag{3.2}$$

and set the hopping matrix element

$$t_n^{(j)}=1,\quad \text{for } j=1,2,3. \tag{3.3}$$

In Fig. 3 we show the global behavior of the Lyapunov exponent as a function of E with various values of the width (disorder strength) W. This figure clearly indicates that the power of W in the weak disorder expansion of γ depends on whether one is inside the band or close to the band edge. To investigate it more quantitatively, we plot the Lyapunov exponents inside the band and near the band edge as a function

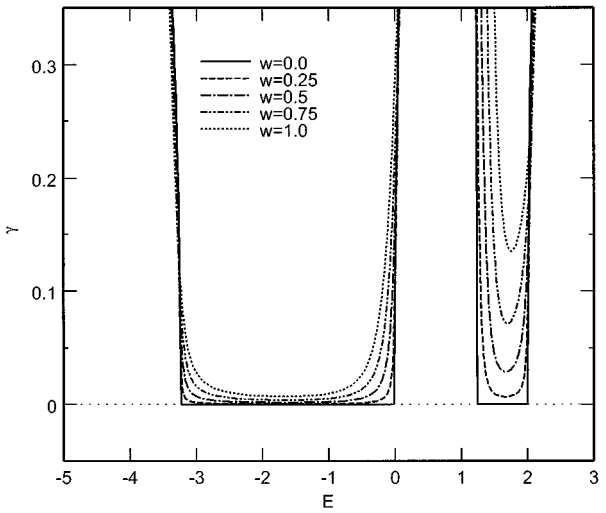

FIG. 3. Estimates of the Lyapunov exponents γ as a function of energy E, for several values of the width W. The error is smaller than the width of the line. The inverse localization length γ is expressed in units of the inverse lattice constant.

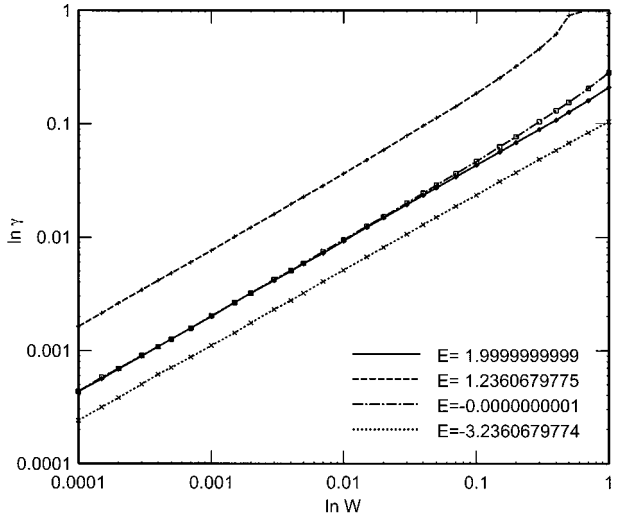

FIG. 5. Estimates of the Lyapunov exponents γ at the band edges on a logarithmic scale vs the width W, for several values of energy E. The error is smaller than the size of the plotted point. The lines are guides to eyes.

of W on a log-log plot in Figs. 4 and 5, respectively. As long as W is small enough, the behavior is definitely a power W^{α}. We estimate the critical exponent α by a least squares fit. The smallest 10 points are used for fitting. The estimates and the χ^2 are shown in Table I. The results suggest that for diagonal disorder, the exponents are 2 inside the band and 2/3 at the band edge.

This behavior is identical in the two bands so we can conclude that as far as diagonal disorder is concerned, our result for the critical exponents is the same as that of Ref. 2. This is valid despite the distinct properties of the transfer matrices discussed above.

Now let us investigate the case of off-diagonal disorder. For the random variables t_n we choose the rectangular distribution

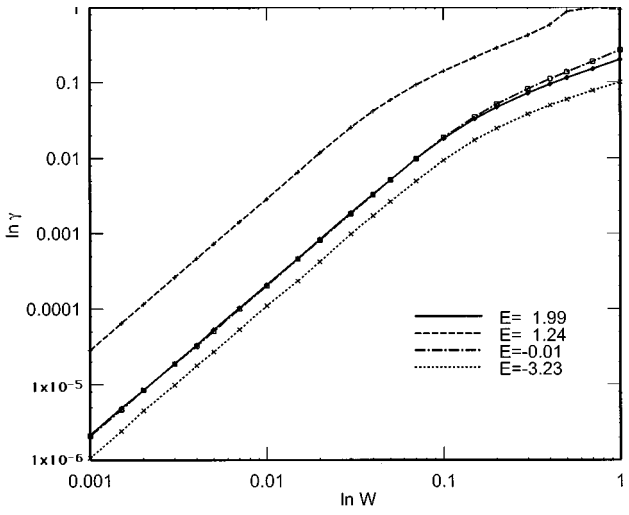

FIG. 4. Estimates of the Lyapunov exponents γ inside the band on a logarithmic scale vs the width W, for several values of energy E. The error is smaller than the size of the plotted point. The lines are guides to eyes.

$$\rho(t^{(j)})=1, \quad \text{for } 1-\frac{W}{2}\leqslant t_n^{(j)}\leqslant 1+\frac{W}{2}, \quad (j=1,2,3), \tag{3.4}$$

and set the on-site potentials

$$\epsilon_n=\eta_n=0. \tag{3.5}$$

In Fig. 6 we show the global behavior of the Lyapunov exponent as a function of E with various values of the width W. This shows a behavior similar to the one encountered in Fig. 3 for diagonal disorder. To elucidate the pertinent powers, we display the Lyapunov exponents inside the band as a function of W in Fig. 7. An estimate of the exponent in terms of the slope yields again $\alpha\approx 2$, in agreement with Ref. 2. We are then tempted to conjecture that the behavior $\gamma(E,W)\approx W^2$ inside the band is robust for any tight-binding model with disorder. Surprisingly, this is not true in general when the Lyapunov exponent in the neighborhood of the band edge is analyzed. In Fig. 8 the critical exponent is calculated near the first, the third, and the fourth band edges (counting from below) and our χ^2 indeed suggests that the exponent α is equal to 2/3 in all three band edges. In Fig. 9 we elucidate the critical exponents α at the vicinity of the second band edge $E=0$, and find that here $\alpha\approx 4/3$. Hence,

TABLE I. The estimates of the exponent α for diagonal (site) disorder case.

	E	α	χ^2
	1.99	1.990	0.0002144
Inside the band	1.24	2.009	0.00003568
	−0.01	1.988	0.0001957
	−3.23	1.990	0.001002
	1.9999999999	0.6698	0.00003796
Band edge	1.2360679775	0.6698	0.000008140
	−0.0000000001	0.6641	0.00003694
	−3.2360679774	0.6626	0.00008829

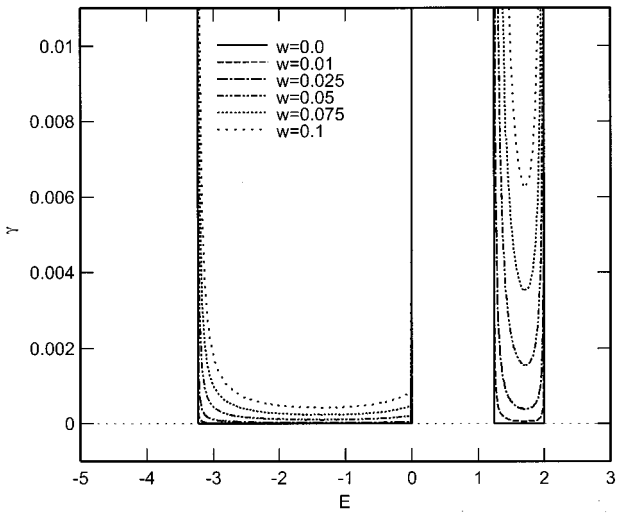

FIG. 6. Estimates of the Lyapunov exponents γ as a function of energy E, for several values of the width W. The error is smaller than the width of the line.

the present model with diagonal disorder belongs to a different universality class. Our precise numerical results for all the four critical exponents near the band edges for off-diagonal disorder are summarized in Table II.

B. Scaling functions

Near a band edge E_b with critical exponent α the scaling argument is defined as $x=(E-E_b)/W^{2\alpha}$ and the corresponding scaling function is defined as $f(x)=\gamma/W^{\alpha}$. Far outside the band, it is anticipated that the Lyapunov exponent does not depend on disorder at all, so that $f(x)\approx x^{1/2}$. Deep inside the band it is expected that the transition to the region where the exponent is $\alpha=2$ will be smooth, which requires that $\mathrm{Re}[f(x)]\approx x^{-1}$. This is not a general rule, and in some cases[5] the decay of $f(x)$ inside the band is exponential and the power behavior inside the band masks its true falloff.

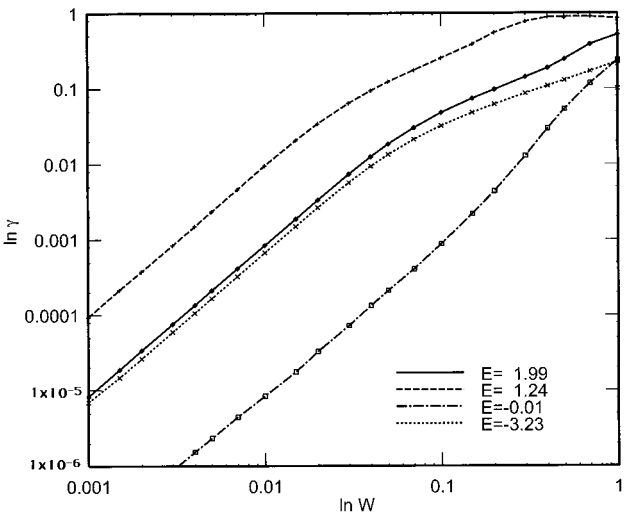

FIG. 7. Estimates of the Lyapunov exponents γ inside the band on a logarithmic scale vs the width W, for several values of energy E. The error is smaller than the size of the plotted point. The lines are guides to eyes.

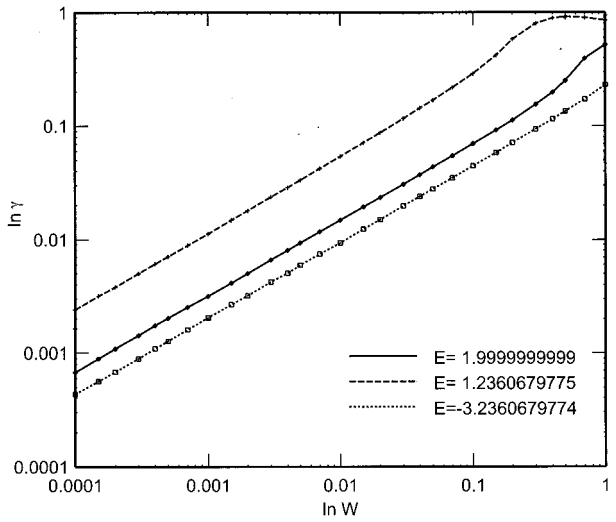

FIG. 8. Estimates of the Lyapunov exponents γ at the band edges on a logarithmic scale vs the width W at $E=1.999\,999\,999\,9$, $1.236\,067\,977\,5$, and $-3.236\,067\,977\,4$. The error is smaller than the size of the plotted point. The lines are guides to eyes.

Two typical examples of scaling functions, one for diagonal and one for off-diagonal disorder, are displayed in Figs. 10 and 11. While we cannot predict the constant multipliers, the corresponding log plots indicate that the pertinent powers are indeed 1/2 outside the band and -1 inside the bands, as anticipated. Another interesting question is whether band edges with identical critical exponents are completely equivalent in the sense that they have the same value of $f(0)$. The eight values of the scaling functions at $x=0$ corresponding to four band edges and two types of disorder are summarized in Table III. From this table we conclude that some pairs of bands are equivalent while some are not. For diagonal disorder, no value of $\gamma/\langle W^2\rangle^{\alpha/2}$ coincided with the corresponding figure $0.289\ldots$ found in Ref. 2 for the one-chain tight-binding model with diagonal disorder.

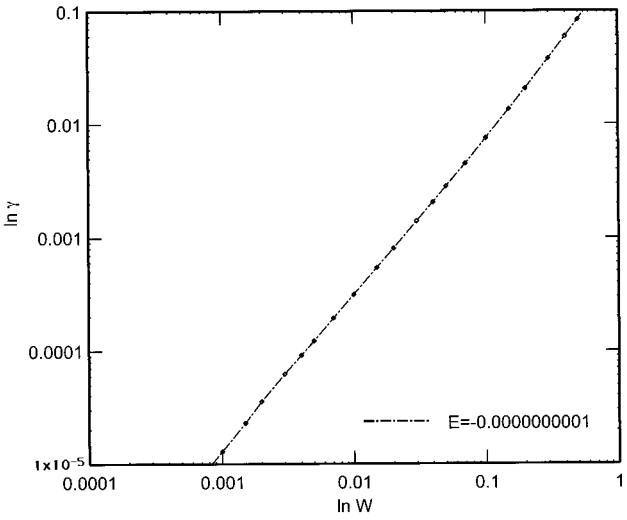

FIG. 9. Estimates of the Lyapunov exponents γ at the band edges on a logarithmic scale vs the width W at $E=-0.000\,000\,000\,1$. The error is smaller than the size of the plotted point. The lines are guides to eyes.

TABLE II. The estimates of the exponent α for off-diagonal (bond) disorder case.

	E	α	χ^2
	1.99	2.002	0.00008475
Inside the band	1.24	1.980	0.0009219
	−0.01	1.846	0.002842
	−3.23	1.996	0.0004974
	1.9999999999	0.6714	0.0001213
Band edge	1.2360679775	0.6708	0.000008547
	−0.0000000001	1.369	0.0008678
	−3.2360679774	0.6750	0.0001417

In conclusion we have investigated numerically the weak disorder behavior of the Lyapunov exponent for a special one-dimensional model whose single-particle spectrum contains a gap and whose transfer matrix is a complicated function of energy. The critical exponents are calculated in all the relevant energy domains and the scaling functions at the four band edges are displayed. For off-diagonal disorder, a new critical exponent is found, whose value differs from the ones predicted in Ref. 2.

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APPENDIX: ONE-DIMENSIONAL CHAIN WITH OFF-DIAGONAL DISORDER

In order to show that the occurence of an exponent 4/3 in our decorated lattice is not trivial, we briefly derive the weak disorder expansion of the Lyapunov exponent for a one-dimensional simple chain with off-diagonal disorder and zero site energies. The procedure follows closely the one of Ref. 2 with a few changes due to the different kind of disor-

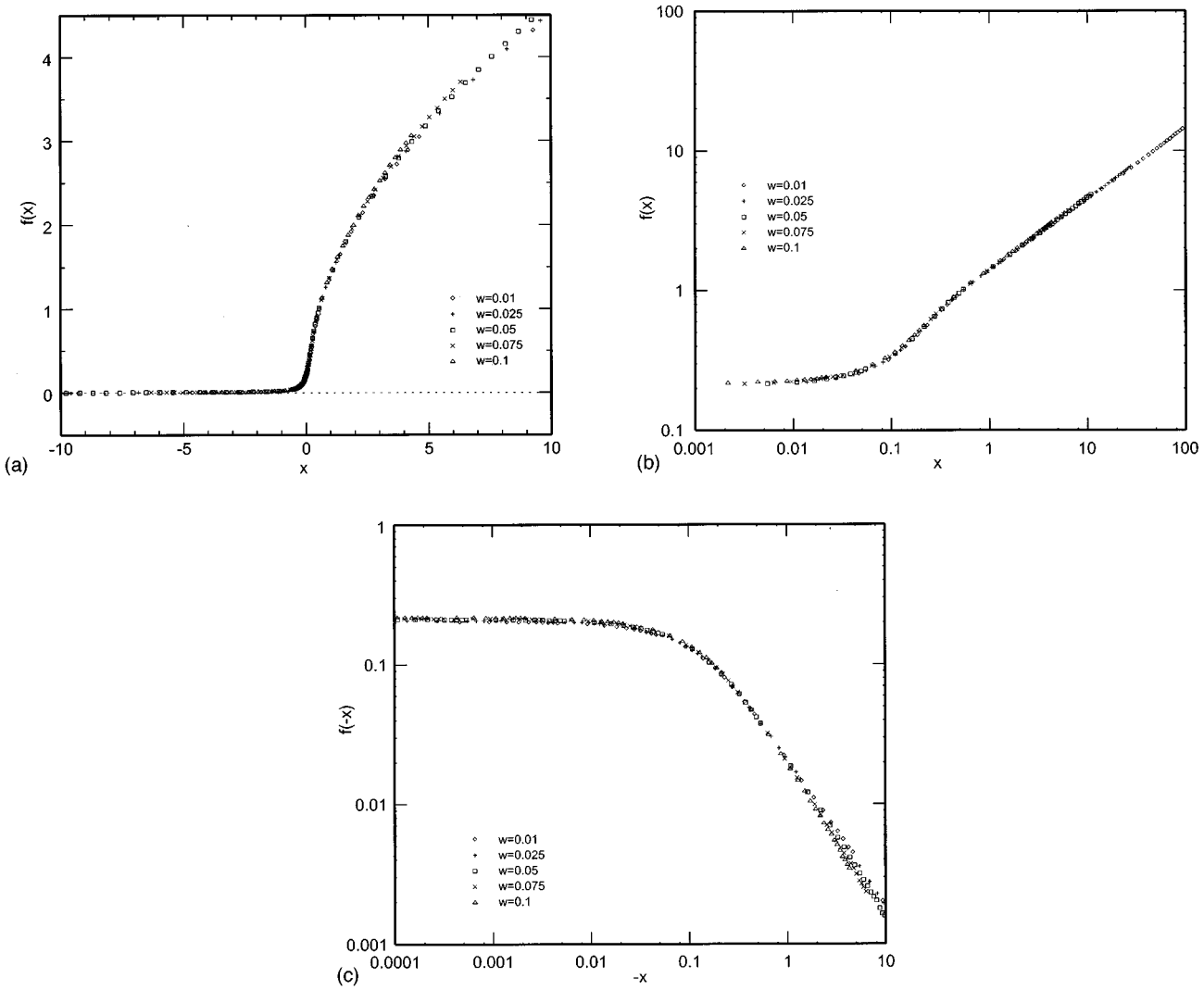

FIG. 10. (a) Plot of the scaling function of the Lyapunov exponent near the second band edge (counting from below) for diagonal disorder. That on a log-log plot in the regions $x>0$ (b) and $x<0$ (c). [We show the mirror image of $f(x)$ in (c) to take logarithm.]

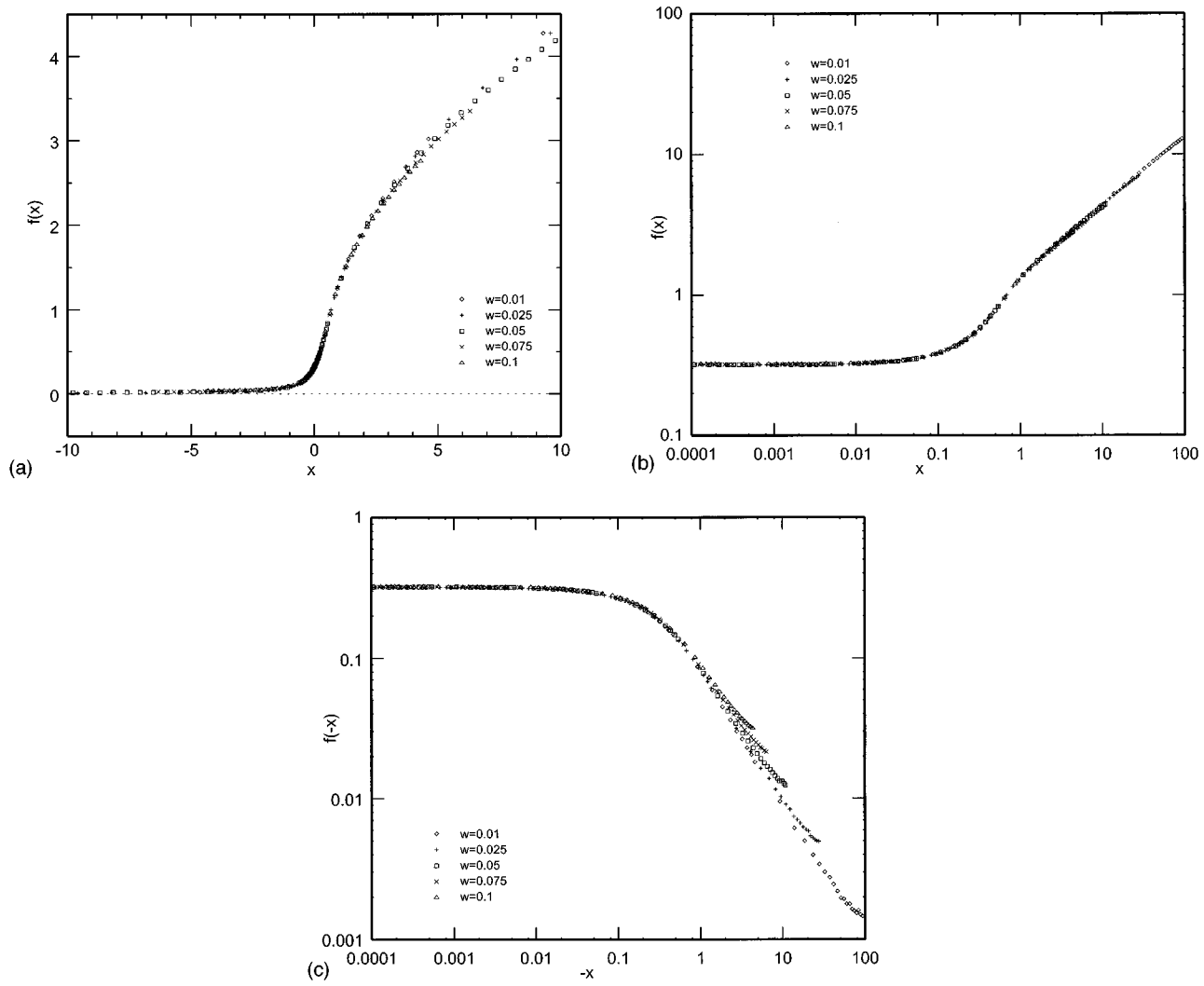

FIG. 11. (a) Plot of the scaling function of the Lyapunov exponent near the fourth band edge (counting from below) for off-diagonal disorder. That on a log-log plot in the regions $x>0$ (b) and $x<0$ (c). [We show the mirror image of $f(x)$ in (c) to take logarithm.]

der. On the basis of our analysis below it is then argued that near the band edge, the relevant exponent is indeed $\alpha=2/3$.

The Schrödinger equation is

$$-t_{n+1}\psi_{n+1}-t_{n-1}\psi_{n-1}=E\psi_n. \tag{A1}$$

We assume that t_n fluctuate slightly near 1 so we have

$$t_n=1+\lambda u_n, \tag{A2}$$

where the odd moments of u_n vanish, while the even moments are given. The small parameter λ is the pertinent expansion parameter. Setting $t_n\psi_n=\phi_n$ leads to a modified equation, which can be cast in a transfer matrix form with

$$T_n=\begin{pmatrix} -\dfrac{E}{t_n} & -1 \\ 1 & 0 \end{pmatrix}. \tag{A3}$$

Now we define the Ricatti variable

$$R_n=\frac{\phi_n}{\phi_{n-1}}, \tag{A4}$$

which satisfies the equation

$$R_{n+1}=-\frac{E}{t_n}-\frac{1}{R_n}. \tag{A5}$$

Let us write R_n as

$$R_n=Ae^{\lambda B_n+\lambda^2 C_n+\lambda^3 D_n+\lambda^4 E_n+\cdots}. \tag{A6}$$

Substitute the expansion (A.6) in (A.5) and compare equal powers of λ to get the following set of equations:

TABLE III. Estimates of $f(0)$ at the band edges for $W=0.01$. For off-diagonal case we used α shown in Table II for the estimation.

E	2	$-1+\sqrt{5}$	0	$-1-\sqrt{5}$
Diagonal disorder	0.199	0.78	0.20	0.11
Off-diagonal disorder	0.322	1.1	0.16	0.21

$$A=-E-A^{-1}, \tag{A7}$$

$$AB_{n+1}=Eu_n+\frac{B_n}{A}, \tag{A8}$$

$$A\left(\frac{B_{n+1}^2}{2}+C_{n+1}\right)=-Eu_n^2+A^{-1}\left(C_n-\frac{B_n^2}{2}\right), \tag{A9}$$

$$A\left(\frac{B_{n+1}^3}{6}+B_{n+1}C_{n+1}+D_{n+1}\right)=Eu_n^3+\frac{1}{A}\left(\frac{B_n^3}{6}-B_nC_n+D_n\right), \tag{A10}$$

$$A\left(\frac{B_{n+1}^4}{24}+\frac{B_{n+1}^2C_{n+1}}{2}+\frac{C_{n+1}^2}{2}+B_{n+1}D_{n+1}+E_{n+1}\right)=-Eu_n^4+\frac{1}{A}\left(-\frac{B_n^4}{24}+\frac{B_n^2C_n}{2}-\frac{C_n^2}{2}-B_{n+1}D_n+E_{n+1}\right), \tag{A11}$$

and so on. Using the relation between the Ricatti variable and the Lyapunov exponent γ, and the assumption that the odd moments of u_n vanish, we have

$$\gamma=\ln A+\lambda^2\langle C\rangle+\lambda^4\langle E\rangle+\cdots, \tag{A12}$$

where $\langle C\rangle=\langle C_n\rangle$, etc, (independent of n). Calculations up to fourth order then yield

$$\gamma=\ln A+\lambda^2\left(\frac{EA\langle V^2\rangle}{(A^2-1)}-\frac{E^2A^2\langle V^2\rangle}{2(A^2-1)^2}\right)+\lambda^4\left(\frac{A^2E^2(2-2A^2-2A^4+2A^6-4AE+4A^5E+3A^2E^2+2A^4E^2)\langle V^2\rangle^2}{2(1-A^2)^5(1+A^2)}\right.$$
$$\left.+\frac{AE(2-2A^2-AE)(2-4A^2+4A^4-2AE+2A^3E-A^2E^2)\langle V^4\rangle}{4(1-A^2)^4}\right). \tag{A13}$$

This equation indicates that, at least up to fourth order, the various terms in the expansion of the Lyapunov exponent for the one-chain model with off-diagonal disorder are distinct from those encountered in the one-chain model with diagonal disorder. Evidently, this is the situation to every order in the perturbation expansion. The validity of (A.13) is for energy E outside the band (the energy E may possibly be complex). To reach the physical region one needs to carry out an analytic continuation. This procedure is possible as long as the energy E is not close to the band edge and $A^{2n}\neq 1$, which indicates that inside the band $\gamma\approx\lambda^2$. Near a band edge E_b, $A\rightarrow 1$. The second term (proportional to $\langle V^2\rangle$) and the third term (proportional to $\langle V^2\rangle^2$) become comparable to the first. The important point to notice, however, is that the singularity near the band edge is dominated by the respective denominators $(A^2-1)^2$ and $(A^2-1)^5$ and not by the denominators (A^2-1) and $(A^2-1)^4$. These singular terms have the same singularity structure as in the diagonal disorder model. The second and third terms then become comparable with $\ln A$ when $(E-E_b)/\lambda^{4/3}$ is finite. This leads to the $\lambda^{2/3}$ behavior of the Lyapunov exponent near the band edges. Hence, the occurence of an exponent 4/3 in the decorated model with offdiagonal disorder is not trivial.

Notice the singularity of the Lyapunov exponent at the band center, where $E=0$ and $A=-1$. The behavior of the localization length at this special energy was studied earlier.[12]

*Electronic address: yamanaka@kodama.issp.u-tokyo.ac.jp
†Electronic address: yshai@bgumail.bgu.ac.il

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