Bulk versus edge in the quantum Hall effect

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The manifestation of the bulk quantum Hall effect on edge is the chiral anomaly. The chiral anomaly *is* the underlying principle of the ''edge approach'' of quantum Hall effect. In that approach, σ_{xy} *should not* be taken as the conductance derived from the space-local current-current correlation function of the pure one-dimensional edge problem. [S0163-1829(96)05547-6]

The question of whether the quantum Hall effect is a bulk or edge phenomena is often raised.[1] It is the purpose of this paper to address this question. Through this paper we use the unit $c=\hbar=k_B=1$.

First, we point out the connection between the Laughlin gauge argument and the edge chiral anomaly. Let us imagine sitting on a Hall plateau, where $\sigma_{xx}=0$ and σ_{xy} $=$a quantized value, and let us ask what is the response of such a system to Laughlin' s flux threading.[2] [The geometry that we consider is the ''ribbon'' used in Laughlin's original paper (see Fig. 1).] In response to the EMF (E_x) induced by the changing flux, a current $\int dy\, J_y=\sigma_{xy}\dot{\Phi}$ is induced.[3] Applying the Su-Schrieffer counting argument,[4,5] the net charge transfer is determined to be

$$\delta Q=\int dt\,dx\,J_y=\sigma_{xy}\int dt\,\dot{\Phi}=\frac{2\pi}{e}\sigma_{xy}. \tag{1}$$

Thus, the manifestation of the quantized σ_{xy}, *is a quantized charge transfer* $\delta Q=2\pi\sigma_{xy}/e$.

From the edge point of view, the world is chiral. Indeed, even in the absence of an applied electric field, there is a current flowing. Of course, from the two-dimensional (2D) point of view, this is simply due to the combined effects of (a) the slope of the spatial confining potential, and (b) the Hall effect. During Laughlin's gedanken experiment, a time-dependent electric field is observed along the edge

$$E_x=\frac{\dot{\phi}}{L}, \tag{2}$$

where L is the circumference of the ribbon. Moreover, accompanying the appearance of E_x, an influx of charge, i.e., an anomaly, occurs. The total amount of charge that flows in is given by Eq. (1). Thus a relation between ΔQ and E_x can be established:

$$\Delta Q=\sigma_{xy}\int dx\,dt\,E_x. \tag{3}$$

Equation (3) is the integral form of the ''chiral anomaly,''

$$\partial_\mu J^E_\mu=\sigma_{xy}E_x. \tag{4}$$

Here $J^E_\mu=\binom{\rho_E}{J_E}$ is the 1+1 edge current. (ρ_E and J_E are the edge charge and current density, respectively. Dimensionwise, J_E is the same as the total current I in 2D.)

Thus *the 2D quantum Hall effect is in one-to-one correspondence with the 1D chiral anomaly. Moreover, the 2D Hall conductance is identical to the coefficient in front of* E_x *in* Eq. (4). From now on we shall refer to the latter as the ''coefficient of chiral anomaly.'' The correspondence between the chiral anomaly in one dimension and the Chern-Simons effective action (i.e., quantum Hall effect) in two dimensions has already been emphasized by Callan and Harvey.[6]

The chiral anomaly is also the underlying principle of the ''edge approach'' of the quantum Hall effect.[7–9] To see that we consider the case where the current is uniform along the edge. In that case $\partial_x J_E=0$ and Eq. (4) becomes

$$\partial_t\rho_E=\sigma_{xy}E_x. \tag{5}$$

Multiplying Eq. (5) by the (constant) edge velocity $-v$, we obtain[10]

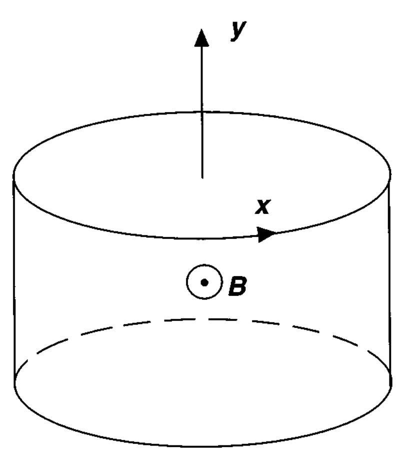

FIG. 1. The upper edge (situated at $y=0$) of the ribbon is under consideration. The edge velocity is along $-\hat{x}$, the induced EMF is along $\hat{x}$, and $\dot{\Phi}$ is along $-\hat{y}$.

$$\partial_t J_E = -\sigma_{xy} v E_x, \tag{6}$$

which implies

$$J_E(t) - J_E(0) = -\sigma_{xy} \int_0^t dt' v E_x(t'). \tag{7}$$

Let us now consider the case where E_x is produced by Laughlin's flux threading in time interval $(0,t)$. Moreover, let us assume that initially $J_E(0)=0$. At the end of the flux threading, a current

$$|J_E| = \sigma_{xy} V \tag{8}$$

is established, where $V=\int_0^t dt' v E_x$ is the amount of work the electric field does to every unit of charge during $(0,t)$. Another way of stating Eq. (8) is that if we raise the edge electrochemical potential by V, a current given by Eq. (8) will flow in the new ground state. The last statement is the building block of the edge approach used in Refs. 7–9. A formula similar to Eq. (8), $I=(e^2/2\pi)V$, also appears in the one dimensional conduction of *free electrons* with no impurity scattering. In that case chiral anomaly also provides a natural intepretation of the often-confused quantized conductance.

Next, we demonstrate that the chiral anomaly is a *constraint* on the edge dynamics of a quantum Hall droplet. First, we look at the primary quantum Hall liquid (QHL) $(\sigma_{xy}=e^2/2\pi m)$, so that there is only one edge. We recall that the bulk effective gauge action of a QHL is

$$S_{\mathrm{eff}} = \int dt\, d^2r \left[\frac{\sigma_{xy}}{2}\epsilon_{abc} A_a \partial_b A_c + J_a^0 A_a\right]. \tag{9}$$

Throughout this paper Roman letters, e.g., a,b,c, are used to label the $2+1$ space time, while Greek indices are reserved for the $1+1$ space time. In Eq. (9) $J_a^0 = (\overline{\rho}, -v\overline{\rho}, 0)$ is the ground state $2+1$ current,[10] and $\epsilon_{abc}\partial_b A_c$ is the *perturbing* part of the external EM field. When the Hall liquid is spatially finite, the above becomes

$$S_{\mathrm{eff}} = \int dt\, d^2r \mathcal{M}(t,\vec{r}) \left[\frac{\sigma_{xy}}{2}\epsilon_{abc} A_a \partial_b A_c + J_a^0 A_a\right]. \tag{10}$$

In the above $\mathcal{M}$ describes the dynamic shape of the Hall droplet, and $\mathcal{M}(t,\vec{r})=1$ or 0 depending on whether at time t the spatial point $\vec{r}$ is inside or outside the droplet. The dynamics of $\mathcal{M}(t,\vec{r})$ is *determined* by the requirement of gauge invariance of the S_{eff} in Eq. (10).[11,12] Trivial manipulation gives

$$J_a^0 \partial_a \mathcal{M} + \frac{\sigma_{xy}}{2}\epsilon_{abc}\partial_a \mathcal{M} \partial_b A_c = 0. \tag{11}$$

We emphasize that Eq. (11) is a *constraint* on the edge dynamics.

Now consider the simple case where $\epsilon_{0ab}\partial_a A_b = \epsilon_{1ab}\partial_a A_b = 0$, and a striplike Hall droplet (Fig. 1). Let $u(x,t)$ be the normal displacement of the upper liquid boundary from the straight line, Eq. (11) implies

$$J_\mu^0 \partial_\mu u = \frac{\sigma_{xy}}{2} E_x, \tag{12}$$

where $J_\mu^0 \equiv (\overline{\rho}, -\overline{\rho v})$. By identifying the chiral current (not the total edge current) as

$$J_\mu^C \equiv J_\mu^0 u, \tag{13}$$

Eq. (12) becomes[12]

$$\partial_\mu J_\mu^C = \frac{\sigma_{xy}}{2} E_x. \tag{14}$$

The fact that the *chiral* current anomaly is only half of that of the *total* edge current is well understood.[14–17] The reason is that the total edge current is the sum of the chiral current and an additional piece. Indeed, if we solve $\mathcal{M}$ in terms of $\epsilon_{abc}\partial_b A_c$ via Eq. (11) and substitute the answer back into Eq. (10), we obtain a gauge-invariant effective action $S_{\mathrm{eff}}(A_a)$. The total current $J_a^{\mathrm{tot}} = \partial S_{\mathrm{eff}}/\partial A_a$ contains a bulk term and an edge one, i.e., $J_a^{\mathrm{tot}} = J_a^{\mathrm{bulk}} + J_a^{\mathrm{edge}}$. The $1+1$ dimensional edge current $J_\mu^E(t,x)$ is obtained from the $2+1$ dimensional $J_a^{\mathrm{edge}}(t,x,y)$ via $J_\mu^E(t,x) = \int dy\, J_{a=\mu}^{\mathrm{edge}}(t,x,y)$. It can easily be shown that

$$\partial_\mu J_\mu^E = \partial_\mu J_\mu^C + \frac{\sigma_{xy}}{2}\epsilon_{\mu\nu}\partial_\mu A_\nu. \tag{15}$$

Equations (14) and (15) are of course equivalent to Eq. (4). [In the literature $\partial_\mu J_\mu^E = \sigma_{xy} E_x$ is called the ''covariant anomaly'' while $\partial_\mu J_\mu^C = (\sigma_{xy}/2) E_x$ is called the ''consistent anomaly''] *In the following we shall concentrate on the consistent anomaly* [Eq. (14)]. To obtain the covariant anomaly (i.e., the total edge current anomaly) we simply multiply the anomaly coefficient by 2.

Following the approach used by Wen,[11] we now construct an edge action, so that the exact equation of motion reproduces Eq. (12). In order to get a local action, it is convenient to introduce the so-called ''chiral boson'' field ϕ so that

$$\overline{\rho u} = \frac{1}{2\pi}\partial_x \phi. \tag{16}$$

In terms of ϕ the answer is (remember that $\sigma_{xy}=e^2/2\pi m$)

$$S = \int dt\, dx \left[\frac{m}{4\pi}\partial_x \phi(\partial_t - v\partial_x)\phi + \frac{e}{4\pi}\phi(\partial_x A_t - \partial_t A_x)\right]. \tag{17}$$

Since Eq. (17) is quadratic in ϕ, the saddle-point equation given by

$$\partial_x(\partial_t - v\partial_x)\phi = \frac{e}{2m}(\partial_x A_t - \partial_t A_x) = -\frac{e}{2m}E_x, \tag{18}$$

is exact. Due to Eqs. (13) and (16), the above is identical to Eq. (14). Although Eq. (17) is derived in the spirit followed by Wen,[11] its gauge coupling differs from that used by Wen in important ways. The gauge coupling that we use is dictated by the chiral anomaly [Eq. (14)]. We emphasize that the gauge action resulting from integrating out ϕ in Eq. (17) is *not* the edge effective action. Instead, the latter is obtained by solving $\mathcal{M}$ in terms of $\epsilon_{abc}\partial_b A_c$ via Eq. (11), substituting the answer back into Eq. (10) and extracting the terms that localize on the edge.[12]

The above result can be easily generalized to hierarchical QHL's. The effective edge action is

$$S=\frac{1}{4\pi}\int dt\,dx\sum_{ij}\,(K_{ij}\partial_t\phi_i\partial_x\phi_j-V_{ij}\partial_x\phi_i\partial_x\phi_j)+\frac{e}{4\pi}\int dt\,dx\sum_i\,t_i\phi_i(\partial_xA_t-\partial_tA_x). \tag{19}$$

Here ϕ_i is the chiral boson field associated with the edge of the ith level QHL, K_{ij} is an integer-valued symmetric matrix, V_{ij} is a positive definite matrix, and t_i is the "charge vector."[11] The equation of motion implied by Eq. (19) is

$$\sum_j\,(K_{ij}\partial_t\partial_x\phi_j-V_{ij}\partial_x\partial_x\phi_j)=-\frac{1}{2}et_iE_x. \tag{20}$$

The chiral charge and current density associated with ϕ_i is

$$\rho_{c,i}=-et_i\frac{1}{2\pi}\partial_x\phi_i,$$

$$J_{c,i}=et_i\frac{1}{2\pi}\sum_{jk}\,K_{ij}^{-1}V_{jk}\partial_x\phi_k. \tag{21}$$

Substituting Eq. (21) into Eq. (20), we obtain

$$\left(\frac{1}{t_i}\right)\partial_\mu J_{c,i\mu}=\frac{e^2}{4\pi}(K^{-1}t)_iE_x. \tag{22}$$

Thus the total chiral current anomaly is

$$\partial_\mu J^C_\mu=\sum_i\,\partial_\mu J_{c,i\mu}=\frac{e^2}{4\pi}(t^TK^{-1}t)E_x. \tag{23}$$

Since

$$\sigma_{xy}=\frac{e^2}{2\pi}(t^TK^{-1}t), \tag{24}$$

Eq. (14) holds. The fact that we obtain Eq. (23) is not at all surprising, since the chiral anomaly is built in as a *constraint* on the edge dynamics.

In a recent paper,[13] Kane, Fisher, and Polchinski defined a "two terminal conductance," from the local edge current-current correlation function (following that reference we shall change to the Euclidean metric below)

$$G=\left(\frac{e}{2\pi}\right)^2|\omega|\sum_{ij}\,t_it_j\langle\phi_i(-w,x=0)\phi_j(\omega,x=0)\rangle. \tag{25}$$

In the above the average on the right hand side is performed in the absence of external electric field. In Ref. 13 it is claimed that on a Hall plateau

$$G=\sigma_{xy}/2. \tag{26}$$

Now we first show that if the QHL under consideration is primary, Eq. (26) is indeed correct. However, for general hierarchical QHL's Eq. (26) is only correct if all edge eigenmodes propagate in the same direction.

By using Eqs. (17) and (25) it is simple to show that

$$G=\frac{e^2}{2\pi m}|\omega|\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{q(-i\omega+vq)}=\frac{e^2}{4\pi m}. \tag{27}$$

Thus for primary QHL's the two terminal conductance defined in Eq. (25) agrees with $\sigma_{xy}/2$. Is this a coincidence? To shed light on that question, we consider a hierarchical QHL. Using Eqs. (19) and (25) it is simple to show that

$$G=\frac{e^2}{2\pi}|\omega|\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{q}\sum_{ij}\,t_i(-i\omega K+qV)_{ij}^{-1}t_j=\frac{e^2}{2\pi}\,\frac{|\omega|}{i\omega}(t^TK^{-1}Mt). \tag{28}$$

In the above the matrix M is given by

$$M\equiv\int_{-\infty}^{\infty}\frac{dq}{2\pi}(qI-i\omega KV^{-1})^{-1}. \tag{29}$$

Let S be the linear transformation that diagonalizes KV^{-1}. Thus

$$M=S\left[\int_{-\infty}^{\infty}\frac{dq}{2\pi}D\right]S^{-1},$$

where

$$D_{ij}=\delta_{ij}\frac{1}{q-i\omega\lambda_i}. \tag{30}$$

Here λ_i is the ith eigenvalue of KV^{-1}. Now the integral can be carried out for each individual diagonal element of D to yield

$$\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{q-i\omega\lambda_i}.=\pm\frac{i}{2}. \tag{31}$$

In Eq. (31) the sign is plus if $i\omega\lambda_i$ lies in the upper half of the complex plane; otherwise it is minus. To understand the physical meaning of λ_i we look back at Eq. (19). In the absence of the external EM field the dispersion relation is

$$\omega K=qV, \tag{32}$$

or

$$\omega KV^{-1}=qI,\quad I=\text{identity matrix}. \tag{33}$$

If K and V are $N\times N$ matrices, there are N solutions

$$\omega=\lambda_i^{-1}q\,i=1,\dots,N. \tag{34}$$

Thus λ_i is the inverse velocity of the ith eigenmode, consequently it should be real. Therefore

$$\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{q-i\omega\lambda_i}.=\frac{i}{2}\,\frac{\omega}{|\omega|}\,\frac{\lambda_i}{|\lambda_i|}, \tag{35}$$

and

$$\left[\int_{-\infty}^{\infty}\frac{dq}{2\pi}D\right]_{ij}=\delta_{ij}\frac{i}{2}\,\frac{\omega}{|\omega|}\,\frac{\lambda_i}{|\lambda_i|}. \tag{36}$$

Substituting the above result into Eqs. (28)–(30), we obtain

$$G=\frac{e^2}{4\pi}(t^T K^{-1} S\Lambda S^{-1} t),$$

where

$$\Lambda_{ij}=\delta_{ij}\frac{\lambda_i}{|\lambda_i|}. \tag{37}$$

A great simplification occurs if all λ_i are positive. In that case $\Lambda=I$, and

$$G=\frac{e^2}{4\pi}(t^T K^{-1} t)=\frac{1}{2}\sigma_{xy}. \tag{38}$$

However, in general, when λ_i of both sign exists, $G\neq\frac{1}{2}\sigma_{xy}$. For example, as shown In Ref. 13, for the $\nu=2/3$ QHL,

$$K=\begin{pmatrix}1 & 0\\ 0 & -3\end{pmatrix},\quad V=\begin{pmatrix}v_1 & v_{12}\\ v_{12} & v_2\end{pmatrix},\quad t=\begin{pmatrix}1\\ 1\end{pmatrix}, \tag{39}$$

one can show that $G=\Delta/3\pi$, where $\Delta=(2-\sqrt{3}c)/(\sqrt{1-c^2})$ with $c=2v_{12}/[\sqrt{3}(v_1+v_2)]$. However, in Ref. 13 this result was taken as the indication that another mechanism (edge impurity scattering) has to be invoked to yield a quantized σ_{xy}. Our message is that it is the coefficient of chiral current anomaly [Eq. (14)] instead of G [Eq. (25)] that should be identified with $1/2\sigma_{xy}$. This point has already been emphasized by Haldane[18], and by Nagaosa and Kohmoto[19] and others.[20]

Thus we find that *the bulk and edge pictures of quantum Hall effect are totally consistent*. The bulk quantum Hall effect corresponds to the edge chiral anomaly. The quantization of the bulk Hall conductance is manifested as the quantization of the chiral anomaly coefficient. Finally we ask ‘‘under what condition is the edge theory used above the correct low energy description?’’ Since the edge theory is a direct consequence of the bulk quantum Hall effect [Eq. (9)], the question reduces to ‘‘to what extent is Eq. (9) the correct bulk effective action?’’ One way to view the stability of the bulk quantum Hall effect is through the boson Chern-Simons theory.[21–23] In that theory, the quantum Hall effect is explained in terms of the superconductivity of composite bosons. For example, the composite boson for the $\nu=1$ plateau is made up of an electron bound to a fictitious magnetic flux quantum. When the composite boson condenses, the $\nu=1$ quantum Hall effect is exhibited. However, when the vortices of the composite boson condense, the system becomes insulating.[24] Wen’s bulk effective gauge theory is the dual form of the boson Chern-Simons theory upon abandoning the vortices in infrared limit.[23] Thus, Wen’s action will continue be the low energy effective theory, as long as the vortices of the Chern-Simons boson do not condense. Under that condition, the effective edge theory discussed above remains valid, and the chiral anomaly coefficient remains unchanged.[25] Of course, the real tough question is whether a particular condition will cause the vortices of composite boson to condense. This is a localization issue which is beyond the scope of this paper.

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