Quasiholes and fermionic zero modes of paired fractional quantum Hall states: The mechanism for non-Abelian statistics

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The quasihole states of several paired states, the Pfaffian, Haldane-Rezayi, and 331 states, which under certain conditions may describe electrons at filling factor $\nu=\frac{1}{2}$ or $\frac{5}{2}$, are studied analytically and numerically in the spherical geometry, for the Hamiltonians for which the ground state are known exactly. We also find all the ground states (without quasiparticles) for these systems in the toroidal geometry. In each case, a complete set of linearly independent functions that are energy eigenstates of zero energy is found explicitly. For fixed positions of the quasiholes, the number of linearly independent states is 2^{n-1} for the Pfaffian, and 2^{2n-3} for the Haldane-Rezayi state; these degeneracies are needed if these systems are to possess non-Abelian statistics, and they agree with predictions based on conformal field theory. The dimensions of the spaces of states for each number of quasiholes agree with numerical results for moderate system sizes. The effects of tunneling and of the Zeeman term are discussed for the 331 and Haldane-Rezayi states, as well as the relation to Laughlin states of electron pairs. A model introduced by Ho, which was supposed to connect the 331 and Pfaffian states, is found to have the same degeneracies of zero-energy states as the 331 state, except at its Pfaffian point where it is much more highly degenerate than either the 331 or the Pfaffian. We introduce a modification of the model which has the degeneracies of the 331 state everywhere including the Pfaffian point; at the latter point, tunneling reduces the degeneracies to those of the Pfaffian state. An experimental difference is pointed out between the Laughlin states of electron pairs and the other paired states, in the current-voltage response when electrons tunnel into the edge. An appendix contains results for the permanent state, in which the zero modes can be occupied by composite bosons, rather than by composite fermions as in the other cases; the system is found to have an incipient instability toward a spin-polarized state. [S0163-1829(96)04947-8]

I. INTRODUCTION

Over the past few years there has been renewed interest in fractional quantum Hall effect (FQHE) (Ref. 1) states involving pairing at even-denominator filling factors.[2–8] The earliest idea[2] was to generalize the Laughlin state[9] by first pairing the electrons into charge-2 bosons, then forming a Laughlin state of the bosons, for which the filling factor ν_b must be of the form $\nu_b=1/m$, $m>0$ even. Since the filling factor ν of the electrons is related to that of the bosons by $\nu=4\nu_b$,[2] one obtains ν either of the form $1/q$ or $2/q$, where in the second case q must be odd. For the cases $\nu=1/q$ with q even, this produces a fractional quantum Hall state at a filling factor not accessible in the usual hierarchy theory.[10] This idea was taken up by Haldane and Rezayi,[3] using spin-singlet pairs, to produce a candidate to explain the observed $\nu=5/2$ plateau[11] (using the usual notion that filling factors larger than 2 involve filling the lowest Landau level with electrons of both spin, and then constructing a $\nu=\frac{1}{2}$ state in the first excited Landau level). The Haldane-Rezayi (HR) wave function has a simple structure, and other paired states with analogous structures, for either spin-singlet or spin-polarized pairs, were written down in Refs. 4 and 5. In particular, the Pfaffian state of Moore and Read[5] is the simplest paired state for spinless or spin-polarized particles. The latter authors also argued that paired states exhibit pairing of composite particles, either bosons or fermions, constructed by attaching an odd or even number q, respectively, of vortices to the electrons, for filling factor $1/q$.[12,13] These objects behave like particles in zero magnetic field, and the wave functions of the paired states can be interpreted as Bardeen-Cooper-Schrieffer (BCS)-paired wave functions, in their position space form. In particular, this makes it easy to understand why the HR state is a spin singlet. It was also suggested that there should be low-energy excitations in which composite particles are unpaired (but still consist of electrons attached to vortices), as opposed to breaking the electron pairs in Halperin's picture. It was further suggested that quasiparticle excitations analogous to those of the Laughlin state,[9] which in incompressible states correspond to vortices in the order parameter,[12] would carry multiples of a half-flux quantum, and thus charges in multiples of $1/2q$, rather than $1/q$ as in the Laughlin states at filling factor $1/q$ (this also results from viewing the excitations as quasiparticles in the Laughlin states of charge-2 bosons). Finally, it was proposed[5] that these quasiparticles obey non-Abelian statistics. In brief, non-Abelian statistics requires that there be degenerate states for well-separated quasiparticles, and, when the quasiparticles are exchanged adiabatically, the effect is not merely a Berry phase representing ordinary fractional statistics, but a matrix acting within the space of degenerate quasiparticle states. In the present paper, we do not aim to exhibit this action directly, but we do aim to show that the quasihole

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states of the Pfaffian and HR states possess the necessary degeneracies, and to give a physical explanation of their origin.

In subsequent work by Greiter, Wen, and Wilczek,[6] the physics of the formation of Halperin-type paired states, that is, Laughlin states of electron pairs, was elaborated, using the Moore-Read Pfaffian state as an example, and several of the points made in Ref. 5 were repeated. Greiter, Wen, and Wilczek also introduced a three-body Hamiltonian for which the Pfaffian state at $\nu=1$ is the exact zero-energy eigenstate. As regards the statistics of the quasiparticles, however, they argued, quite reasonably, that Halperin's picture would lead to simple Abelian statistics of the quasiparticles. While we agree with much of the physical discussion by these authors (including the argument that the Halperin paired states will have $4q$-fold-degenerate ground states on the torus in the thermodynamic limit), we disagree with their use of the three-body Hamiltonian and Pfaffian-based simple wave functions to illustrate their points. Other work on this model,[7,8] and even the observation by Greiter, Wen, and Wilczek themselves[6] that there is a sixfold degeneracy of zero-energy states of the three-body interaction on the torus, are more consistent with the predictions of Ref. 5 of non-Abelian statistics and related properties that are connected with conformal field theory (CFT) in two space-time dimensions. For example, there are gapless Majorana fermion excitations at an edge of the Pfaffian state,[7,8] in addition to the usual charge-fluctuation boson excitations, while the Halperin-type state of electron pairs would be expected to possess only the latter. The results that will be obtained in the present paper lend further support to the belief that the quasiparticle states that are constructed as energy eigenstates of the three-body Hamiltonian of Ref. 6 (and its generalizations to be constructed below) do possess non-Abelian statistics.

It was also suggested[6] that non-Abelian behavior might be present only at points of special symmetry, and not be generic. Clearly, the three-body Hamiltonian might be such a point. Although it was argued in Ref. 5 that non-Abelian statistics is a topological property that cannot be altered by small perturbations because the ground states involved are assumed to have an energy gap for all excitations, this has not been tested. It is clearly an important problem, but it lies beyond the scope of the present paper.

A further development in paired FQHE states was the realization that some members (to be referred to here collectively as the 331 state) of another class of states, of which an example was introduced by Halperin,[2] also exhibit pairing.[3,14,15] These states have come under scrutiny because of their relevance for FQHE states in double-layer systems at $\nu=\frac{1}{2}$.[16,17,14,18] They can also be viewed as generalized hierarchy states,[19] and so are not expected to possess non-Abelian statistics; however, they are still distinct from the Halperin idea of a Laughlin state of charge-2 bosons. We will discuss these states, and especially recent work by Ho,[20] further in Sec. VI.

As we mentioned above, the main purpose of this paper is to check the expectation, based on the CFT ideas of Ref. 5, that the quasihole states of the Pfaffian and HR states possess degeneracies above and beyond those that would be obtained for ordinary Laughlin quasiholes, or their generalization to the Halperin-type paired states, and thus to lay the groundwork for a demonstration of non-Abelian statistics. We do this by constructing zero-energy eigenstates of those Hamiltonians for which the simple form of the ground-state wave function is correct. These systems serve as model examples, each of which we may hope is typical of a universality class (in the sense of Ref. 5), though the study of the effect of perturbations lies beyond the scope of this paper. (We should mention that quasielectrons are expected to obey non-Abelian statistics like those for the quasiholes, but it is always much more difficult to obtain energy eigenfunctions, of which the wave functions take a nice form, for quasielectrons than for quasiholes, and the energies will not be zero, nor degenerate, though presumably the degeneracies would be recovered in the thermodynamic limit for well-separated quasielectrons.) Some of the results for the Pfaffian appeared in an unpublished earlier work[21] (see also Ref. 22), but the method employed here in general is related to that used for the edge states in Ref. 8. The results are in full accord with earlier expectations. It will emerge that the degeneracies of quasihole states of the paired states can be viewed as coming from breaking pairs of composite particles and placing the unpaired (composite) particles in certain single-particle states that contribute zero to the total energy; these are ''zero modes.''

Throughout this paper we will use the terminology ''particles'' to refer to the underlying charged particles in the lowest Landau level, which could be either fermions (such as electrons) or bosons, and not to the composite particles. For a given Hamiltonian, we will also refer to energy eigenstates that have energy eigenvalue equal to zero simply as zero-energy states.

In this paper we mostly work on the sphere; we will briefly review this formalism, and results for the quasiholes of the Laughlin state.[10] One uses a uniform radial magnetic field with a total of N_ϕ flux through the surface, and in the lowest Landau level (LLL) each particle has orbital angular momentum $N_\phi/2$. The LLL wave functions on a sphere are usually written (in a certain gauge[10]) in terms of ''spinor'' (or ''homogeneous'') coordinates u_i and v_i for each particle $i=1,\ldots,N$, with $u_i=e^{i\phi_i/2}\cos\theta_i/2$, $v_i=e^{-i\phi_i/2}\sin\theta_i/2$ in terms of the spherical polar coordinates θ_i and ϕ_i, on the sphere. Since these imply that u_i and v_i are not independent complex numbers, it is often more convenient, and will simplify the writing, to use a nonredundant parametrization of the sphere by a single complex variable. This is done by stereographic projection, which gives the definition $z_i=2Rv_i/u_i$, where R is the radius of the sphere. Single-particle basis states in the LLL then take the form $z_i^m/(1+|z_i|^2/4R^2)^{1+N_\phi/2}$, where the L_z angular momentum quantum number is $L_z=N_\phi/2-m$. In this form, the rotationally invariant inner product of single-particle states on the sphere is given by multiplying one function by the conjugate of the other, and integrating over the z_i plane with no other z_i dependent factors inside the integral. Only single-particle basis states with $m\leqslant N_\phi$ correspond to LLL functions on the sphere, and can be normalized with respect to this inner product (the normalizing factors will not be needed here). [Note that when N_ϕ and $R\rightarrow\infty$ with N_ϕ/R^2 fixed, in which case the sphere becomes effectively flat, the basis functions (for m fixed) tend to $z_i^m e^{-|z_i|^2/4}$, the basis functions in the plane in the symmetric gauge[9].] Many-particle states can thus be written as

$$\Psi=\tilde{\Psi}\prod_i (1+|z_i|^2/4R^2)^{-(1+N_\phi/2)}, \tag{1.1}$$

and $\tilde{\Psi}$ must be a polynomial of degree no higher than N_ϕ in each z_i. Therefore, in the following we need specify only $\tilde{\Psi}$ in order to describe a state.

The Laughlin ground state and quasihole states are exact, zero-energy states for short-range pseudopotential Hamiltonians[10] of the general form

$$H=\sum_{i<j}\sum_{M=0}^{N_\phi} V_M P_{ij}(N_\phi - M), \tag{1.2}$$

in which $P_{ij}(L)$ is a projection operator onto the subspace in which the total orbital angular momentum of the particles i and j is L; in the summation, M can be viewed as the relative angular momentum. For the LLL states, the close approach of two or more particles occurs only when their total angular momentum is large. The parameters V_M are the pseudopotentials. Note that, for spinless particles, only even M occur for bosons, and only odd M for fermions. We will later generalize the projection operator notation to three-body operators P_{ijk}, and the subspace onto which it projects will be specified by the values of further quantum numbers of the chosen group of particles, such as total spin S, or specific values of the z component of the spin of each particle, etc. Every projection operator is always normalized in the conventional way, with $P^2=P$.

The Laughlin states are zero-energy states for the pseudopotential Hamiltonian in which $V_M\neq 0$ for $M<q$, and zero otherwise (in fact, the nonzero V_M are usually taken to be positive). The Laughlin-Jastrow wave function is

$$\tilde{\Psi}_{\rm LJ}=\prod_{i<j} (z_i-z_j)^q. \tag{1.3}$$

Clearly q must be even when the particles are bosons, and odd when they are fermions. The number of flux is then $N_\phi=q(N-1)$, and the filling factor, $\nu=N/N_\phi$, tends to $1/q$ as the number of particles $N\to\infty$. We will always use the integer $q>0$ as the parameter specifying the filling factor $\nu=1/q$. In this state, any two particles have relative angular momentum $M\geqslant q$,[10] so it is annihilated by H. This property is preserved if the state is multiplied by the quasihole factors $U(w)=\Pi_i(z_i-w)$, which change the flux by one quantum. These factors can be expanded in powers of each w to obtain the elementary symmetric polynomials in the z_i's,

$$e_m=\sum_{i_1<i_2<\cdots<i_m} z_{i_1}z_{i_2}\ldots z_{i_m} \tag{1.4}$$

which are linearly independent operators, and the states obtained by multiplying in several of these factors span the full space of zero-energy states for each number of flux $N_\phi=q(N-1)+n$, where n is the number of quasiholes.[9,10] This results from the standard fact about symmetric polynomials that they can all be obtained as sums of products of the elementary symmetric polynomials. The space of states obtained in this way is equivalent to that for N bosons on the sphere in the lowest Landau level with n flux, or $n+1$ orbitals. This can be viewed as the $q=0$ case of Laughlin's states, which applies since the dimension of the desired space of states is independent of q. The dimension of the space is therefore a binomial coefficient

$$\binom{N+n}{n}. \tag{1.5}$$

Also, the expansion of

$$\prod_i\prod_k (z_i-w_k) \tag{1.6}$$

in sums of products of symmetric polynomials in the w's shows that, when the w's are viewed as the coordinates of bosons,[10] the space of available states for these bosons, which behave as if in their LLL with N as the number of flux, exactly coincides with the space of zero-energy quasihole states. The dimension of this space is then given by the formula for n bosons in $N+1$ orbitals, which is the same binomial coefficient (1.5). The equivalence of these viewpoints is the basic duality between bosonic particles and vortices within the LLL; it is analogous to the particle-hole transformation for fermions. The count can also be performed by using the $q=1$ (fermion) case instead of the $q=0$ (boson) case. It then gives the number of states for N fermions in $N_\phi+1=N+n$ orbitals, or for n holes obeying Fermi statistics in the same number of orbitals, and these are the same number (1.5). We will often refer to the dimension of the spaces of zero-energy states we find in this paper simply as the number of zero-energy states.

We now summarize the contents of the remainder of this paper. In Secs. II, III, and IV we study the quasihole states of the Pfaffian, HR, and 331 states on the sphere, for the Hamiltonians for which these ground states are exact. We find explicit wave functions for all the quasihole states, and count them to exhibit in particular the degeneracy that occurs even when the positions of the quasiholes are fixed. For the Pfaffian and HR states, this is related to non-Abelian statistics, while for the 331 states it results simply from a layer quantum number of the quasiholes. The analytical results are confirmed numerically. In Sec. V, we consider the ground states of the same Hamiltonians on the torus, that is with periodic boundary conditions, and obtain the wave functions of the zero-energy states in all cases. In Sec. VI, we make a modest attempt to discuss the effects of perturbations on the states considered, especially the Zeeman term (for systems like HR that include particles of both spin) and tunneling (for double-layer systems like 331). We make a full analysis of the zero-energy states of a model proposed by Ho,[20] which we show to be compressible and thus pathological at the point where the spin-polarized Pfaffian is among the zero-energy states. We also rectify this problem by adding further terms to the Hamiltonian. These results are again checked numerically. In addition, we mention an experimental test that can distinguish the Halperin and other paired states, by using electron tunneling into the edge, for example via a point contact. Appendix A contains definitions used in Sec. V, and Appendix B analyzes a further paired state, the permanent state, in which there are spin-singlet pairs of spin-$\frac{1}{2}$ composite bosons;[5] this state is found to be at a transition point to a ferromagnetically ordered state.

II. QUASIHOLES OF THE PFAFFIAN STATE ON THE SPHERE

In this section we will obtain all the energy eigenstates at zero energy for the three-body Hamiltonian for which the Pfaffian ground state is exact, for arbitrary numbers of added flux, that is, for any number of quasiholes. The following sections generalize the results to the Haldane-Rezayi and 331 states, and (partially) to the torus.

The Pfaffian state,[5] for even particle number N, is defined by the wave function

$$\tilde{\Psi}_{\rm Pf}(z_1,\ldots,z_N)=\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\prod_{i<j}(z_i-z_j)^q, \qquad (2.1)$$

where the Pfaffian is defined by

$$\mathrm{Pf}M_{ij}=\frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\mathrm{sgn}\sigma\prod_{k=1}^{N/2}M_{\sigma(2k-1)\sigma(2k)}$$

for an $N\times N$ antisymmetric matrix whose elements are M_{ij}; S_N is the group of permutations of N objects. The filling factor is $1/q$. The Pfaffian state is totally antisymmetric for q even, so could describe electrons, while for q odd it describes charged bosons in a high magnetic field. For $q=1$, it is the zero-energy state of the lowest flux of the Hamiltonian[6]

$$H=V\sum_{i<j<k}\delta^2(z_i-z_j)\,\delta^2(z_i-z_k), \qquad (2.2)$$

where the sum is over distinct triples of particles.

For numerical purposes on the sphere, it is more convenient to work with a projection operator form of the three-body Hamiltonian, instead of the δ functions in Eq. (2.2). The closest approach of three particles on the sphere corresponds to the state of maximum possible total angular momentum for the three. If the particles are bosons, the largest possible total angular momentum is $3N_\phi/2$ (recall that each particle has angular momentum $N_\phi/2$). Then, for the $q=1$ case, the Hamiltonian may be taken as proportional to the projection operator onto the (unique) multiplet of maximum angular momentum for each triple of bosons:

$$H=\sum_{i<j<k}VP_{ijk}(3N_\phi/2). \qquad (2.3)$$

The same trick works for the three-body interaction of fermions giving the $q=2$ case; in this case, the maximum total angular momentum of three particles is $3N_\phi/2-3$. Some numerical results for such Hamiltonians were already given in Ref. 7. For larger q, these Hamiltonians can be generalized in such a way that the zero-energy states are obtained from those for $q=1$ by multiplying by $\Pi(z_i-z_j)^{q-1}$ (it is assumed that for q odd, we are discussing bosons, and for q even, fermions). The presence of the latter factor implies that they are all zero-energy eigenstates of the projection operators for any two particles onto relative angular momentum

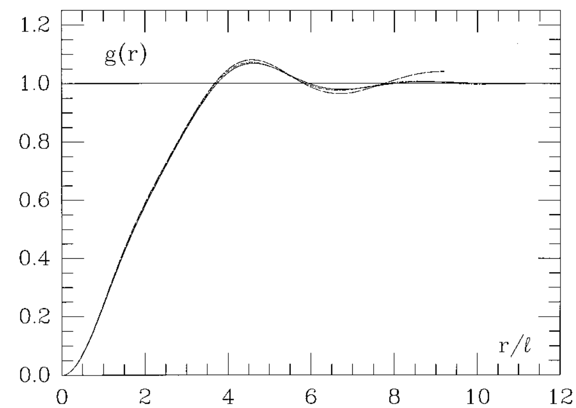

FIG. 1. Two-particle correlation function $g(r)$ for the Pfaffian state, with $N_\phi=2(N-1)-1$ (i.e., $\nu=\frac{1}{2}$), for $N=10$ (dashed line), 12 (dot-dashed line), and 14 (solid line) vs the great circle distance on the sphere. The curves for $N=12$ and 14 are almost indistinguishable.

$M=0,2,\ldots,q-3$ (q odd), or $M=1,3,\ldots,q-3$ (q even) [or the corresponding total angular momenta N_ϕ, $N_\phi-2,\ldots,N_\phi-q+3$, ($q$ odd), etc.]. The space of states annihilated by such projections is in one-to-one correspondence with the full space of states of the $q=1$ case, and the desired three-body projection operator [onto angular momentum $3N_\phi/2-3(q-1)$] is the unique one that corresponds under this mapping to that already mentioned for $q=1$. For each q, the Hamiltonian can then be taken to be the sum of the three-body and all of these two-body projection operators. A very similar approach works for the other Hamiltonians studied in this paper, so that results for higher q can be deduced easily from those for the minimal q. These Hamiltonians can also be written in terms of δ functions and their derivatives, so as to arrive at a form suitable for use in geometries other than the sphere. An attempt at a Hamiltonian appropriate for the Pfaffian at $q=2$ in the second paper in Ref. 6 is invalid as it annihilates all states.

In Fig. 1 we show the two-particle correlation function $g(r)$ for the Pfaffian state on the sphere with $q=2$ for three sizes, $N=10$, 12, and 14. We plotted the function versus the great circle separation r (in units of magnetic length) on the sphere, so that the largest possible value of r is half the circumference, and we normalized the curves in such a way that in an infinite system they would approach 1 at infinity. We see that, although for $N=10$ an exponential decay at large distances is not apparent, for $N=12$ and 14 the curve appears to be rapidly approaching 1 at large separation, and these two curves are almost indistinguishable in the region where both are defined. The correlation length in the Pfaffian, which would be defined as the length over which $g(r)-1$ decays by a factor of e, is apparently quite large.

The Pfaffian state is the only zero-energy eigenstate of H at $N_\phi=q(N-1)-1$. Zero-energy quasihole excitations can be obtained only by increasing N_ϕ, as for the quasiholes of the Laughlin state as discussed in Sec. I, but in this case the basic objects contain a half flux quantum each and must be created in pairs. A wave function for two quasiholes was proposed in Ref. 5; it is

$$\tilde{\Psi}_{\text{Pf + 2 qholes}}(z_1,\ldots,z_N;w_1,w_2)=\frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\text{sgn}\sigma\frac{\prod_{k=1}^{N/2}[(z_{\sigma(2k-1)}-w_1)(z_{\sigma(2k)}-w_2)+(w_1\leftrightarrow w_2)]}{(z_{\sigma(1)}-z_{\sigma(2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})}\prod_{i<j}(z_i-z_j)^q. \tag{2.4}$$

It is clearly the pairing structure built into the ground state which allows insertion of Laughlin-like factors

$$f(z_i,z_j;w_1,w_2)=(z_i-w_1)(z_j-w_2)+(z_i-w_2)(z_j-w_1), \tag{2.5}$$

which act only on one member of each pair, and, since the f's increase the maximum angular momentum for each z_i by 1, N_ϕ increases by 1. As the quasiholes are, at least approximately, located at w_1 and w_2, they effectively contain a half-flux quantum each, unlike the usual Laughlin quasihole that corresponds to a full flux quantum. The same structure requires that quasiholes are made in pairs, since the wave function must be totally symmetric or antisymmetric. When quasiholes coincide, that is, when $w_1=w_2$, a Laughlin quasihole is recovered.

It is clear that by inserting more factors f, with different w's, into the sum over permutations, a whole host of zero-energy eigenfunctions can be obtained. However, this involves dividing the quasihole coordinates $w_1,\ldots,w_{2n}$ into pairs in an arbitrary way; the resulting functions are invariant only under exchanges of the two quasihole coordinates in each of these pairs, or under permutatations of the pairs. One must then ask whether all these states, of which there are $(2n)!/(2^n n!)=(2n-1)(2n-3)\cdots\equiv(2n-1)!!$, are linearly independent, and also whether all zero-energy eigenfunctions can be obtained in this way.

For four quasiholes, the three distinct functions obtained from dividing the w's into pairs in three distinct ways obey just one linear relation, as we will now show. (These methods and results for four quasiholes appeared previously in Ref. 21.) For more than four quasiholes, the following method becomes increasingly impractical, and we will instead use a more direct method, inspired by the results for edge states of the Pfaffian in Ref. 8.

It is convenient to write the functions in the more general form

$$\tilde{\Psi}_p(z_1,\ldots,z_N;w_1,\ldots,w_{2n})=\text{Pf}\{\Phi_p(z_i,z_j;w_1,\ldots,w_{2n})/(z_i-z_j)\}\tilde{\Psi}_{\text{LJ}}. \tag{2.6}$$

Here Φ_p must be symmetric and of degree n in z_1 and z_2 in order to represent $2n$ quasiholes, that is so that $N_\phi=q(N-1)-1+n$. We could use products of the f's in Eq. (2.5), but it is convenient to use the following (these choices are clearly related by taking linear combinations). For $n=2$, define

$$\Phi_1(z_1,z_2;w_1,\ldots,w_4)=(z_1-w_1)(z_1-w_2)(z_2-w_3)\times(z_2-w_4)+(z_1-w_3)(z_1-w_4)\times(z_2-w_1)(z_2-w_2), \tag{2.7}$$

$$\Phi_2(z_1,z_2;w_1,\ldots,w_4)=\Phi_1(z_1,z_2;w_1,w_3,w_2,w_4), \tag{2.8}$$

$$\Phi_3(z_1,z_2;w_1,\ldots,w_4)=\Phi_1(z_1,z_2;w_1,w_4,w_2,w_3). \tag{2.9}$$

The following identity is useful: For any set of complex numbers a_i, $i=1,\ldots N$, N even, >2,

$$\text{Pf}(a_i-a_j)=0, \tag{2.10}$$

since the square of the Pfaffian is a determinant in which any three rows or columns obey a linear relation. Set $\Phi=\Phi_1+\Phi_2+\Phi_3$, then using Eq. (2.10) it can be shown that

$$\Phi_1'\equiv\Phi_1-\frac{1}{3}\Phi=\frac{1}{3}(z_1-z_2)^2[(w_1-w_4)(w_2-w_3)+(w_1-w_3)(w_2-w_4)]. \tag{2.11}$$

Hence $\Phi_2-\Phi_3=x(\Phi_1-\Phi_2)$, where

$$x=(w_1-w_2)(w_3-w_4)/(w_1-w_4)(w_2-w_3) \tag{2.12}$$

is the cross ratio. Thus as functions of z_1 and z_2, Φ_1, Φ_2, and Φ_3 are linearly related. To show there are no further relations, consider the limit $w_1\to w_2$, $w_3\to w_4$. We find

$$\Phi_1\sim(z_1-w_1)^2(z_2-w_3)^2+(z_1-w_3)^2(z_2-w_1)^2,$$
$$\Phi_2\sim\Phi_3\sim2(z_1-w_1)(z_1-w_3)(z_2-w_1)(z_2-w_3), \tag{2.13}$$

which are clearly linearly independent.

For $N=2$ particles, it now follows immediately that there are only two linearly independent states of the type shown. For an arbitrary even N, we still have to prove that $\text{Pf}\{\Phi_p(z_i,z_j;w_1,w_2,w_3,w_4)/(z_i-z_j)\}$ gives only two linearly independent states for fixed w's. For any $p=1$, 2, and 3, we use Eq. (2.11) and expand the Pfaffian in powers of Φ_p'. All terms containing more than one factor of Φ_p' cancel using identity (2.10), since they contain factors (z_i-z_j). Therefore *the N-particle wave functions satisfy the same linear relation as the* Φ_p for all N. A similar argument shows that use of linear combinations of the Φ_p inside the Pfaffian leads only to linear combinations of the same states. We note that the linearly-independent states can be taken to be the unique state where Φ_p in Eq. (2.6) is replaced by Φ, and that where only one factor of Φ_p in the expansion of the wave function is replaced by Φ_1', the other Φ_p being replaced by Φ. The effect of Φ_1' is to cancel the pairing factor $(z_{\sigma(2k-1)}-z_{\sigma(2k)})^{-1}$ for the pair on which it acts. Thus there is a "broken pair" in the wave function, as in Ref. 8. This observation provides the method to generalize these results to any number of quasiholes. As the w_i vary, these states span a space of zero-energy four-quasihole states, whose dimension we will find below when we have results for general n.

We now turn to the method for arbitrary numbers of quasiholes or added flux quanta. We will first write down the functions, then explain why they both span the full vector space of zero-energy eigenstates for each n and N, and are linearly independent. The functional form was inspired by those in Ref. 8. The functions are defined as

$$\tilde{\Psi}_{m_1,\ldots,m_F}(z_1,\ldots,z_N;w_1,\ldots,w_{2n})=\frac{1}{2^{(N-F)/2}(N-F)/2!}\sum_{\sigma\in S_N}\operatorname{sgn}\sigma\prod_{k=1}^{F} z_{\sigma(k)}^{m_k}\prod_{\ell=1}^{(N-F)/2}\times\frac{\Phi(z_{\sigma(F+2\ell-1)},z_{\sigma(F+2\ell)};w_1,\ldots,w_{2n})}{(z_{\sigma(F+2\ell-1)}-z_{\sigma(F+2\ell)})}\prod_{i<j}(z_i-z_j)^q. \tag{2.14}$$

In this equation, $\Phi(z_1,z_2;w_1,\ldots,w_{2n})$ is defined so as to be symmetric in the w's, and is a generalization of the function Φ used in the $n=2$ case:

$$\Phi(z_1,z_2;w_1,\ldots,w_{2n})=\frac{1}{(n!)^2}\sum_{\tau\in S_{2n}}\prod_{r=1}^{n}(z_1-w_{\tau(2r-1)})\times(z_2-w_{\tau(2r)}). \tag{2.15}$$

Clearly the integers m_k must obey $0\leqslant m_k\leqslant n-1$ for each $k=1,\ldots,F$, since the flux is $N_\phi=q(N-1)-1+n$, and can be taken to be ordered and distinct, $0\leqslant m_1<m_2<\cdots<m_F\leqslant n-1$, because of the antisymmetrization by the sum over permutations; thus $0\leqslant F\leqslant n$. Clearly we must also have $F\leqslant N$; for $N\geqslant n$ this restriction can be ignored, and the analytic formulas below apply in this limit. A similar caveat applies to the other paired states below, but will not be explicitly mentioned after this section. Functions (2.14) represent pairing, but with F fermions left unpaired. One could think of the unpaired fermions as resulting from breaking pairs for N even, but the states also make sense for N odd (note that N-F is always even). These functions are closely analogous to the excited quasiparticle states of a BCS-paired system, where the unpaired particles usually occupy plane waves. A whole spectrum of such excitations is expected also in the paired FQHE states[5,6] for any number of quasiholes (including zero), but these generally have nonzero energy. Here we are interested only in the subset of states that have zero energy for the three-body Hamiltonian, which occur only when quasiholes are present. These states contain fermions occupying a certain set of single-fermion wave functions $1, z, z^2, \ldots, z^{n-1}$, which can be viewed as the LLL wave functions for a flux of $n-1$, or as an angular momentum multiplet of angular momentum $(n-1)/2$. However, the actual spatial distribution of the unpaired fermions in these states is hard to calculate, since it must take into account the whole many-particle wave function. Since the effective magnetic field seen by the fermions is essentially zero except in the quasiholes where the density is lower, we expect that the orbitals have weight concentrated on the quasiholes. The occupation of these orbitals contributes nothing to the energy for our Hamiltonian, so these are ''zero modes.'' The number of zero modes is n, the number of added (real) flux quanta, and not $2n$, the number of quasiholes. Thus one cannot say that there is a zero mode locally bound to each quasihole. Instead the zero-mode wave functions are shared among the quasiholes.

We should point out that Greiter, Wen, and Wilczek[6] also stated that in the presence of quasiholes there are wave functions with broken pairs. However, the functions they published for states with one broken pair, both with and without quasiholes [see Eqs. (9) and (10) in the first paper in Ref. 6, and Eqs. (6.2), (6.8), and (6.9) in the second], vanish identically when antisymmetrized. Probably for this reason, the counting of the number of unpaired fermions that can be accomodated ''naturally,'' i.e., in zero-energy eigenstates, in the presence of quasiholes, is stated incorrectly to be n broken pairs, when the correct answer is n fermions. Note also that the states for unpaired fermions without quasiholes that they give are not zero-energy eigenstates (and thus not obviously eigenstates at all).

The linear independence of the states (2.14) is easily established for fixed w's. After removing the factor $\Pi(z_i-z_j)^q$, we arbitrarily divide the particles into pairs, and let the members of each pair approach each other, one after the other, say $z_1\to z_2$, then $z_3\to z_4$, and so on. For each limit we examine the leading behavior; clearly the leading behavior for each limit may be a single pole, in view of the paired form of the function, or it may be nonsingular. If it is a pole, we take the function that multiplies the pole (its residue), which is a function of the remaining coordinates to which the procedure has not yet been applied, as well as of the coordinates $z_1=z_2$, $z_3=z_4,\ldots$, to which it has, and we repeat the process. If the first limit is nonsingular, we call the function we started with the zeroth nonsingular residue; if the first limit is singular, but the second is not, we call the first residue the first nonsingular residue, and so on. Thus the mth residue, obtained after the mth limit, may be singular or nonsingular in the next limit; if it is nonsingular we can identify the original states as having $m=(N-F)/2$ unbroken pairs. Then the $(N-F)/2$th nonsingular residue will, by definition, be a nonsingular function of the paired coordinates $z_1=z_2$, $z_3=z_4,\ldots,$ $z_{N-F-1}=z_{N-F}$, the unpaired coordinates $z_{(N-F)+1},\ldots,z_N$, and of $w_1,\ldots,w_{2n}$. Since two functions, one of which is singular and the other nonsingular in a given limit, are linearly independent of each other, it follows that states with different numbers F of unpaired particles are linearly independent. For states with the same F, we consider the $(N-F)/2+1$th residue, as a function of the remaining F coordinates $z_{(N-F)+1},\ldots,z_N$. It is just a Slater determinant in these variables, and these determinants for distinct sets of m_k are obviously linearly independent. This concludes the proof.

From wave functions (2.14), it is straightforward to enumerate the number of states that satisfy the conditions, for given positions of the w's. First we note that for zero quasiholes, a state (the original Pfaffian ground state) exists for N even, but not for N odd. For $2n=2$ quasiholes, there is a unique possibility, both for N even (with $F=0$) and N odd (with $F=1$ fermion, in the $m_1=0$ state). For $2n=4$, there are two states both for N even and odd; for N even these are the same as the two independent states found before. In general, for given $n>0$ and F, there are clearly

$$\binom{n}{F} \tag{2.16}$$

independent states. Summing over the allowed values of F, which are those with the same parity as N, we obtain, whether N is even or odd, 2^{n-1}, by a well-known formula for binomial coefficients. This number, which is valid for $N\geqslant n$, is exactly the number of conformal blocks for $2n$ spin fields in the Majorana conformal field theory;[5] see also Ref. 22.

Just as for the quasiholes of the Laughlin state (see Sec. I), there is a (finite) positional degeneracy associated with the positions of the quasiholes. The functions for fixed w's are analogous to coherent states formed out of the linearly independent quasihole states. In the present case, this degeneracy can be calculated, for a given F and set of m_k's, by expanding all the Φ's in powers of the w's:

$$\Phi(z_1,z_2;w_1,\ldots,w_{2n})=\frac{(2n)!}{(n!)^2}\bigg[z_1^n z_2^n+\frac{1}{2}(z_1^{n-1}z_2^n+z_1^n z_2^{n-1})e_1(-w)+\cdots+e_{2n}(-w)\bigg]. \tag{2.17}$$

Here $e_m(-w)$ is shorthand for the elementary symmetric polynomials in the w's, with each w_i replaced by $-w_i$:

$$e_m(w)=\sum_{i_1<i_2<\cdots<i_m} w_{i_1}w_{i_2}\ldots w_{i_m}, \tag{2.18}$$

which arise since each w appears at most once in any term resulting from the expansion of Φ. It is known that linear combinations of products of the elementary symmetric polynomials e_m, $m=0,\ldots,2n$ yield all the symmetric polynomials in $2n$ variables. Thus when the functions in Eq. (2.14) are expanded in powers of the w's, we obtain all the symmetric polynomials in $w_1,\ldots,w_{2n}$, in which the degree in any one w is not greater than $(N-F)/2$. The total number of linearly independent states (as functions of the z's), for a fixed F and a fixed set of m_k's, cannot be greater than the number of linearly independent symmetric functions of the w's obtained in this expansion. Notice that, if the w's are regarded as coordinates of some kind of particles, the symmetric polynomials in the w's can be interpreted as the states for $2n$ bosons, each of which can occupy one of $(N-F)/2+1$ orbitals [the orbitals having the single-particle wave functions $1, w,\ldots,w^{(N-F)/2}$; they form an angular momentum multiplet of angular momentum $(N-F)/4$)]. The number of such symmetric polynomials in the w's for the Pfaffian case is thus

$$\binom{(N-F)/2+2n}{2n}. \tag{2.19}$$

It is our claim that, for each of these linearly independent symmetric polynomials, we have a linearly independent many-particle state (for each set of m_k's), and so the upper bound just obtained is in fact the answer. To establish the truth of this claim, we again make use of the residues of the successive limits $z_1\to z_2$, $z_3\to z_4,\ldots$, as defined above. The nonsingular residues have a simple form; they are proportional to

$$\prod_{i=1}^{(N-F)/2}\prod_{r=1}^{2n}(z_{2i}-w_r)\sum_{\sigma\in S_F}\operatorname{sgn}\sigma\prod_{k=1}^{F}z_{\sigma(N-F+k)}^{m_k}. \tag{2.20}$$

The last factor, involving the unpaired particles, is a Slater determinant; the first factor is simply a product of Laughlin-like quasiholes acting on the coordinate of each pair. We have thus reduced the analysis to the case of the Laughlin states, where all the quasihole states are linearly independent (see Sec. I), and this establishes our claim.

The total number of linearly independent quasihole states, fixing only N and n, is then

$$\sum_{F,(-1)^F=(-1)^N}\binom{n}{F}\binom{(N-F)/2+2n}{2n}. \tag{2.21}$$

(Notice that this expression incorporates both restrictions $F\leqslant n$, $F\leqslant N$.) This number is clearly larger than the number that would be expected if the quasiholes behaved like the quasiholes of the Laughlin states. In the present case the expectation, based on assuming Abelian fractional statistics as for the Laughlin states, would be (in view of the half flux in each quasihole) that the number would be given by the formula for $2n$ bosons which may each occupy any of $N/2+1$ orbitals (for N even). We can compare our result with this number, which is

$$\binom{N/2+2n}{2n}. \tag{2.22}$$

For n fixed and N tending to infinity, the ratio tends to 2^{n-1}. Again, this represents the degeneracy necessary for non-Abelian statistics.

We now give arguments that the states found above are a complete set of zero-energy states. The general construction of zero-energy states for the three-body Hamiltonian for $q=1$, or its generalizations for $q>1$, was given in the first part of Appendix A of Ref. 8. It shows that without loss of generality, zero-energy states are linear combinations of the forms (from which we have removed the ubiquitous factor $\Pi(z_i-z_j)^q$)

TABLE I. Numbers of multiplets of states of total angular momentum L at zero energy for the three-body Hamiltonian for the $q=2$ Pfaffian state on the sphere, for $N_\phi=2(N-1)-1+n$, i.e., $2n$ quasiholes.

N	n	$L=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4	1	1		1																		
4	2	1		2		1																
4	3	1	2	1	2		1															
4	4	2	2	1	3	1	2		1													
4	5	1	3	1	3	2	3	1	1		1											
6	1		1		1																	
6	2	2		2	1	2		1														
6	3		3	1	4	2	3	2	2		1											
6	4	3	1	5	3	7	3	6	3	4	2	2		1								
6	5		5	3	9	6	9	7	9	5	7	4	4	2	2		1					
8	1	1		1		1																
8	2	2		3	1	3	1	2		1												
8	3	3	1	5	4	7	4	6	3	4	2	2		1								
8	4	5	2	10	7	14	10	14	9	12	7	8	4	5	2	2		1				
8	5	6	5	16	14	23	20	26	21	25	19	20	14	15	9	9	5	5	2	2		1
10	1		1		1		1															
10	2	2		4	1	4	2	3	1	2		1										
10	3		6	4	10	7	11	8	10	6	7	4	4	2	2		1					

$$\sum_{\sigma\in S_N} \operatorname{sgn}\sigma \frac{\sum_{\tau\in S_{N/2}}\prod_{k=1}^{N/2} f_k(z_{\sigma(2\tau(k)-1)},z_{\sigma(2\tau(k))})}{(z_{\sigma(1)}-z_{\sigma(2)})\cdots(z_{\sigma(N-1)}-z_{\sigma(N)})}, \quad (2.23)$$

where f_k are symmetric polynomials in two variables. For N odd we can write a similar form with $k=1,\ldots,(N-1)/2$, and include for the unpaired particle an arbitrary polynomial factor $f_0(z_{\sigma(N)})$. In order to represent states with $2n$ quasiholes, the number of flux added to the Pfaffian ground state must be n, and so the f_k (and f_0 for N odd) must be of degree at most n in each coordinate z_i; these symmetric polynomials must then be linear combinations of the linearly independent forms

$$z_1^{n_1}z_2^{n_2}+z_1^{n_2}z_2^{n_1}, \quad (2.24)$$

in which $0\leqslant n_1\leqslant n_2\leqslant n$. These clearly span a vector space of dimension $\frac{1}{2}(n+1)(n+2)$. As in Ref. 8, if any $f_k(z_1,z_2)$ vanishes at $z_1=z_2$, then it must contain a factor $(z_1-z_2)^2$, and this pair is broken and will contribute to the unpaired-fermion part of functions (2.14). The subspace of symmetric polynomials for which this occurs is spanned by $(z_1-z_2)^2$ times those in Eq. (2.24), here with $0\leqslant n_1\leqslant n_2\leqslant n-2$; this subspace has dimension $\frac{1}{2}n(n-1)$. The quotient of these two spaces, which represents the symmetric polynomials in two variables which do not vanish at $z_1=z_2$, therefore has dimension $2n+1$. But we have already found a set of such functions while expanding Φ, Eq. (2.17), and there are exactly $2n+1$ terms in this series, which form a linearly independent set for the required symmetric polynomials. Therefore we may now argue that, in the general form (2.23), we may choose the f_k to be in either of the two sets, that is, those that vanish at $z_1=z_2$ and those appearing in Eq. (2.17). The unpaired fermion part takes the given form, after use of the antisymmetrization among the F particles involved. If f_k does not vanish at $z_1=z_2$, then it is part of the expansion of Φ for that pair. Thus we obtain exactly all the terms that occur on expanding Eq. (2.14). This shows that the count of states given is correct, and that they can be viewed as arising from the states with quasiholes at fixed positions.

The numbers for the total number of zero-energy states, which are not resolved into angular momentum multiplets, are convenient for comparison with numbers of zero-energy eigenstates obtained numerically [using the same three-body Hamiltonian (2.3)], when these are summed over all L and L_z. We can also work out the decomposition of functions (2.14) into angular momentum multiplets, using wave functions (2.14) and applying the Clebsch-Gordan decomposition to the angular momenta of the bosons that represent the quasiholes in the Φ factors, and the unpaired fermions. These numbers, and the angular momentum decomposition, are in perfect agreement for moderate sizes. The cases that have been checked numerically are shown in Table I. In the

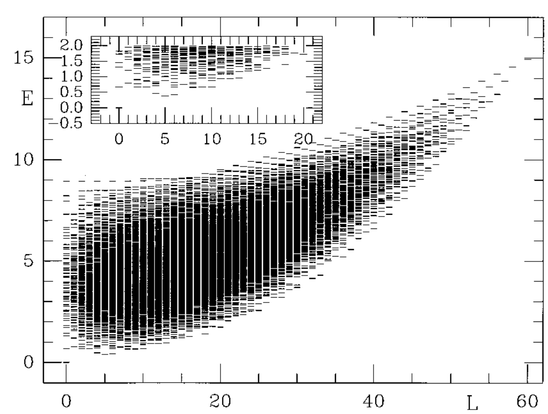

FIG. 2. Spectrum of the three-body Hamiltonian for the Pfaffian state of fermions with $\nu=\frac{1}{2}$, for $N=12$ and $N_\phi=21$; that is, no quasiholes. The inset enlarges the low-lying levels.

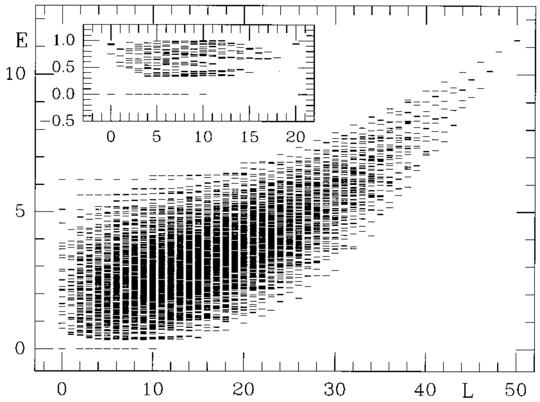

FIG. 3. As in Fig. 2, but with $N=10$ and $N_\phi=19$; that is, four quasiholes.

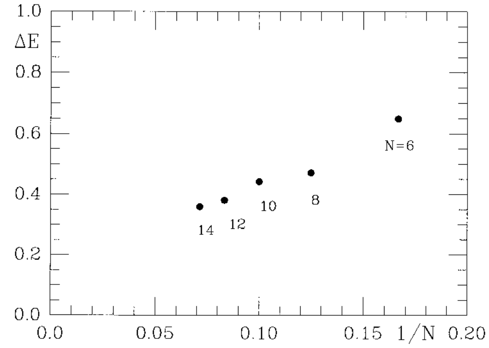

FIG. 5. As in Fig. 4, but for the energy ΔE of the lowest-energy excited state vs $1/N$.

table, we show only the results for N even; similar results, in complete agreement with the analytic formulas, are also found for N odd.

As an aside, it is interesting that in this case we established the linear independence of the functions in general, whereas in Ref. 8 we were forced to resort to a case-by-case analysis. In fact, the results given here now suffice to complete the proof of linear independence of the edge states of the Pfaffian state on a disk. We first note that if we place many (or all) quasiholes at the same place by setting all the w's equal, then there are no particles in that region of the sphere, and the fluid has an edge there. If we take the limit of these states as $N_\phi \to \infty$ with N fixed, then the sphere becomes an infinite plane, and the particles are concentrated in a disk at the origin if the quasiholes are at the position on the sphere that is mapped to the point at infinity by the stereographic projection. Thus the problems of finding the zero-energy quasihole and edge states are essentially the same. The general edge states, that include charge-fluctuation excitations at the edge, are obtained by letting some of the w's deviate from infinity, and expanding in $1/w$'s gives states that contain symmetric polynomials in the paired-particle coordinates, rather than in all the coordinates as in Ref. 8. This is merely another basis for the edge states; the number of states at each angular momentum level is easily seen to be the same, for sufficiently large N. In the limit, the complete and linearly independent zero-energy bulk quasihole states, for the case where all w's are nearly equal, yield all the edge states of the disk, and this gives proof that the latter are linearly independent. To obtain results for the cylinder, with two edges, we place half the quasiholes at each pole of the sphere and take a similar limit. In the limit, the particles occupy a narrow band around the equator of the sphere, and the infinite flux through the system makes it equivalent to the cylinder, if we consider states where the particles are close to the equator, which again means the w's must not deviate far from the poles. Note that the fact that the particles are spread very thinly along this band in the limit we have taken does not affect the construction or counting of edge states, which is independent of the aspect ratio of the occupied region of the cylinder. Also, the operation that shifts charge from one edge to the other[8] is obtained by removing a quasihole from one pole and placing it at the other.

In Fig. 2 we show numerical spectra for the $q=2$ three-body Hamiltonian, described after Eq. (2.3) (in which the projection is onto $L=3N_\phi/2-3$), with $V=\frac{1}{6}$, for $N=12$ electrons and $N_\phi=2N-3$ flux, that is, no quasiholes, and $\nu=\frac{1}{2}$. In Fig. 3 we show the same but with $N=10$ and 2 flux added, so there are $2n=4$ quasiholes. The zero-energy states, of which the degeneracies were given in Table I, can be seen at $E=0$, as can the set of angular momentum values obtained in this case. The figure also shows that all higher-energy states are separated by a significant gap that we expect will survive in the thermodynamic limit, as needed for the arguments for non-Abelian statistics. To investigate further the claim that this system is incompressible, we include

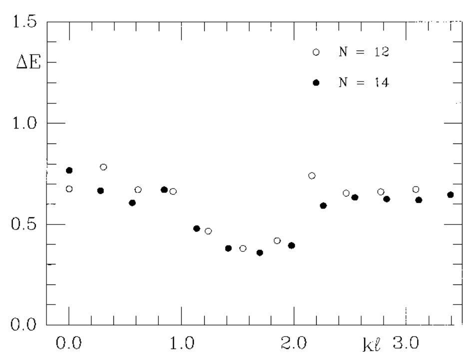

FIG. 4. Low-lying excited states of the three-body Hamiltonian for the Pfaffian ground state at $\nu=\frac{1}{2}$ $[N_\phi=2(N-1)-1]$ for $N=12$ and 14, plotted against $k=L/R$.

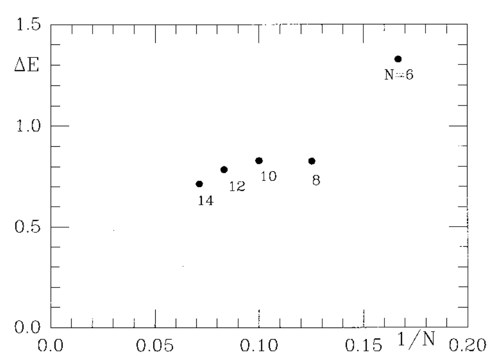

FIG. 6. Ground-state energy of the three-body Hamiltonian for the Pfaffian state at $N_\phi=2(N-1)-2$ (that is, $\nu=\frac{1}{2}$ with two quasielectrons added), plotted against $1/N$.

some finite-size scaling results. In Fig. 4, we show the low-lying excited energy levels above the ground state for $N=12$ and 14, versus $k=L/R$. It can be seen that the spectra lie almost on top of one another. In Fig. 5, we show the size dependence of the lowest excited state, versus $1/N$, for several values of N, including those in Fig. 4. Although it is not quite clear by inspection that this energy gap is converging to a constant as $N\rightarrow\infty$, because of the behavior of the points for the sizes $N=10$, 12, and 14, we note that in Fig. 4 the lowest-energy points for $N=12$ seem to lie on two sides of a minimum, while for $N=14$ one lies near a minimum. Hence we expect that, if results for still larger sizes were available, convergence to a finite gap would be apparent. In Fig. 6, we show the ground-state energy for systems with $N_\phi=2(N-1)-2$, that is, two *quasielectrons*, where there are no zero-energy states for the three-body Hamiltonian. If these energies converge to a finite gap as $N\rightarrow\infty$, then this energy determines the slope in the total-energy density versus density in the thermodynamic limit, on the higher-density side of $\nu=\frac{1}{2}$. Since the energy density is zero on the lower-density side, this would represent a discontinuity in the chemical potential at $\nu=\frac{1}{2}$ for this Hamiltonian, showing that the system is incompressible. From the results shown, we cannot be sure that the points approach a nonzero limiting value as $N\rightarrow\infty$, but they are nonetheless consistent with this hypothesis.

III. QUASIHOLES OF THE HALDANE-REZAYI STATE ON THE SPHERE

The Haldane-Rezayi (HR) state[3] can be written in terms of the coordinates of $N/2$ up-spin particles at $z_1^\uparrow,\ldots,$ and $N/2$ down-spin particles at $z_1^\downarrow,\ldots,$ as

$$\tilde{\Psi}_{\rm HR}(z_1^\uparrow,\ldots,z_{N/2}^\uparrow,z_1^\downarrow,\ldots,z_{N/2}^\downarrow)=\sum_{\sigma\in S_{N/2}}{\rm sgn}\sigma\frac{1}{(z_1^\uparrow-z_{\sigma(1)}^\downarrow)^2\cdots(z_{N/2}^\uparrow-z_{\sigma(N/2)}^\downarrow)^2}\prod_{i<j}(z_i-z_j)^q. \tag{3.1}$$

Here $q\geqslant 2$ is even for electrons, and odd for bosons, and the filling factor is $1/q$. The first factor is of course just a determinant. The product over z_i's with no spin labels attached is over all particles. The fact that this describes a singlet is discussed carefully in Ref. 3. In Ref. 5 it was pointed out that this state can be regarded as a BCS-type condensate of spin-singlet pairs of spin-$\frac{1}{2}$ composite fermions that consist of a particle and q vortices, from which the spin-singlet property can be more easily understood. Some further discussion related to the edge states is contained in Ref. 8. The HR state is the unique zero-energy state at $N_\phi=q(N-1)-2$ flux of a ''hollow-core'' pseudopotential Hamiltonian that gives any two particles a nonzero energy when their relative angular momentum is either $q-1$ or $\leqslant q-3$,[3] again with the assumption that the particles are bosons for q odd, fermions for q even, and that $q\geqslant 2$.

In exact analogy with the Pfaffian state, the wave function for two quasiholes is

$$\tilde{\Psi}_{\rm HR}(z_1^\uparrow,\ldots,z_{N/2}^\downarrow;w_1,w_2)=\sum_{\sigma\in S_{N/2}}{\rm sgn}\sigma\frac{\Pi_{k=1}^{N/2}[(z_k^\uparrow-w_1)(z_{\sigma(k)}^\downarrow-w_2)+(w_1\leftrightarrow w_2)]}{(z_1^\uparrow-z_{\sigma(1)}^\downarrow)^2\cdots(z_{N/2}^\uparrow-z_{\sigma(N/2)}^\downarrow)^2}\prod_{i<j}(z_i-z_j)^q. \tag{3.2}$$

Due to the spin-independence of the newly inserted factors acting on each pair inside the sum over permutations, the state is still a spin singlet, and this suggests that the quasiholes carry no spin. The two-quasihole state is again a zero-energy eigenstate of the hollow-core Hamiltonian. To see this fact, expand the inserted factors for each pair in terms of powers of $z_k^\uparrow\pm z_{\sigma(k)}^\downarrow$. Due to the symmetry between $z_k^\uparrow$ and $z_{\sigma(k)}^\downarrow$ in each factor, it is easy to see that $z_k^\uparrow-z_{\sigma(k)}^\downarrow$ must occur to an even power. Thus in the complete wave function, the absence of $(z_k^\uparrow-z_{\sigma(l)}^\downarrow)^{q-1}$ for any k and l, and hence the zero-energy property of the ground state, is preserved in the quasihole states.

It is possible to write down directly the forms of all the zero-energy states of the hollow-core Hamiltonian, in analogy with those for the Pfaffian and those in Ref. 8. In terms of the coordinates of $N_\uparrow$ up particles and $N_\downarrow$ down particles, the wave functions are linear combinations of

$$\frac{1}{(N_\uparrow-F_\uparrow)!}\sum_{\substack{\sigma\in S_{N\uparrow}\\ \rho\in S_{N\downarrow}}}{\rm sgn}\sigma{\rm sgn}\rho\prod_{k=1}^{F_\uparrow}(z_{\sigma(k)}^\uparrow)^{n_k}\prod_{l=1}^{F_\downarrow}(z_{\rho(l)}^\downarrow)^{m_l}\prod_{r=1}^{N_\uparrow-F_\uparrow}\frac{\Phi(z_{\sigma(F_\uparrow+r)}^\uparrow,z_{\rho(F_\downarrow+r)}^\downarrow;w_1,\ldots,w_{2n})}{(z_{\sigma(F_\uparrow+r)}^\uparrow-z_{\rho(F_\downarrow+r)}^\downarrow)^2}\prod_{i<j}(z_i-z_j)^q. \tag{3.3}$$

Here $N_\uparrow-F_\uparrow=N_\downarrow-F_\downarrow$ is the number of unbroken pairs, and we may assume the n_k's and m_k's are strictly increasing, as for those in the Pfaffian quasihole states, with $0\leqslant n_1<n_2<\cdots<n_{F_\uparrow}\leqslant n-2$, $0\leqslant m_1<m_2<\cdots<m_{F_\downarrow}\leqslant n-2$; consequently, $0\leqslant F_\sigma\leqslant n-1$ for each $\sigma=\uparrow$ or $\downarrow$. As written, these states do not have definite spin, but eigenstates of $\mathbf{S}^2$ and S_z can be constructed. Since the paired particles form singlets, the spin is determined by the spin-$\frac{1}{2}$ unpaired fermions in the sums over σ and ρ, which behave identically to ordinary spin-$\frac{1}{2}$ fermions. Hence the possible spin states are determined by adding the spins of particles in different orbitals (labeled by n_k or m_k), with the only constraint that an orbital occupied with both an up and a down fermion must form a singlet. The total spin S of the zero-energy states therefore obeys $S\leqslant(n-1)/2$. Notice that the number of flux in these states is $N_\phi=q(N-1)-2+n$. Arguments

TABLE II. Numbers of multiplets of states of total angular momentum L and total spin S at zero energy for the hollow-core Hamiltonian for the $q=2$ HR state on the sphere, for $N_\phi=2(N-1)-2+n$, i.e., $2n$ quasiholes.

N	n	S	$L=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	1	0	1		1													
4	2	0	1		2		1											
4	3	0	2		2	1	2		1									
4	3	1				1												
4	4	0	2		3	1	4	1	2		1							
4	4	1		1		1	1	1										
4	5	0	2		4	1	4	2	3	1	2		1					
4	5	1		1	2	2	1	2	1	1								
4	5	2	1															
6	1	0		1		1												
6	2	0	2		2	1	2		1									
6	3	0		3	1	5	2	3	2	2		1						
6	3	1	1		1		1		1									
6	4	0	4	1	6	4	8	4	7	3	4	2	2		1			
6	4	1		2	1	3	2	3	1	2	1	1						
6	5	0		7	4	12	8	12	9	11	6	8	10	4	2	2		1
6	5	1	3	1	6	4	7	5	7	5	2	3	1	1				
6	5	2						1										

that the states given are both complete and linearly independent, so that our count of states is correct, can be constructed straightforwardly as a combination of those in Appendix A of Ref. 8 and in Sec. II above; we omit the details.

We may now count the number of linearly independent states as for the Pfaffian. We see that for $n=0$ and 1, we must have $F_\uparrow=F_\downarrow=0$, and such states exist only for $N=N_\uparrow+N_\downarrow$ even; they are the ground and two-quasihole states written down above. For $n=2$, we find two possibilities for both N even or odd, like the Pfaffian case. For the general case, we can write the number of states for fixed n, $F_\uparrow$, $F_\downarrow$, and w's in two ways. One is

$$\binom{n-1}{F_\uparrow}\binom{n-1}{F_\downarrow}. \tag{3.4}$$

If we sum over $F_\uparrow$ with $F=F_\uparrow+F_\downarrow$ fixed, we obtain a second form,

$$\binom{2(n-1)}{F}, \tag{3.5}$$

which is the number of states for F fermions in $2(n-1)$ orbitals. The sum over F satisfying $(-1)^F=(-1)^N$ then yields a total of $2^{2(n-1)-1}=2^{2n-3}$ zero-energy quasihole states of all spins for fixed N (either even or odd, except for small n as already shown) and w's.

We now find the dimension of the space of quasihole states at all spins for fixed N, including the positional degeneracy due to the w's as for the Pfaffian. Then the total number of linearly independent quasihole states, fixing only N and n, is

$$\sum_{F,(-1)^F=(-1)^N}\binom{2(n-1)}{F}\binom{(N-F)/2+2n}{2n}. \tag{3.6}$$

Again the ratio of this number to that for positional degeneracy only is 2^{2n-3} as $N\to\infty$. In this case, the factor 2^{2n} might give the impression that there is a factor 2 for each quasihole, perhaps because each carries spin $\frac{1}{2}$. But the result is in fact 2^{2n-3}, which indicates the connection with non-Abelian statistics. There are only $n-1$ zero modes available, similarly to the Pfaffian, which can be occupied with either spin, with a final condition on the parity of the number of unpaired fermions.

The two-particle correlation functions for the HR ground state with $q=2$ have been published previously,[3] for six particles; they suggest that the correlation length is quite large in this system also. In Table II we show results obtained numerically for zero-energy quasihole states of the hollow-core Hamiltonian for $q=2$, which agree exactly with the general formula, as do the orbital and spin angular momenta. In Fig. 7, we show the spectrum of the HR state at $\nu=\frac{1}{2}$,

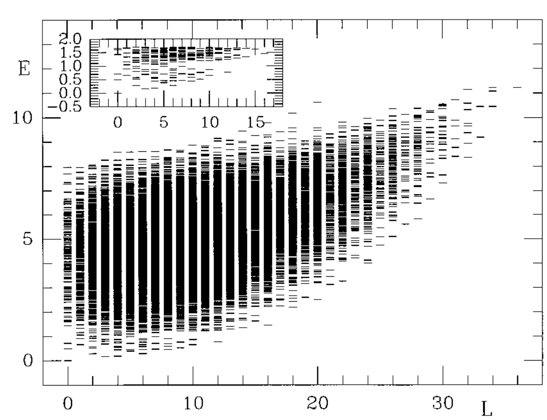

FIG. 7. Spectrum of the hollow-core model for the HR state of fermions with $\nu=\frac{1}{2}$, for $N=8$ and $N_\phi=12$; that is, no quasiholes. The inset enlarges the low-lying levels.

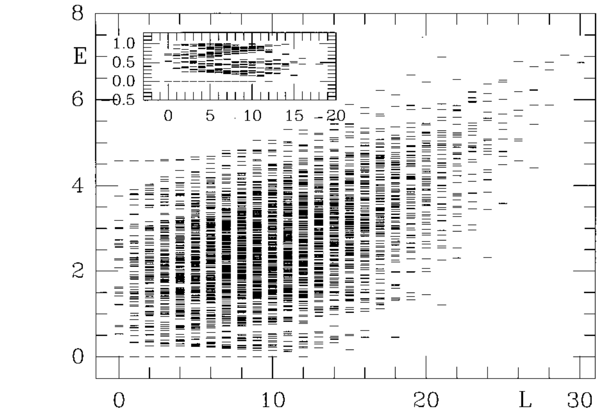

FIG. 8. As in Fig. 7, but with $N=6$ and $N_\phi=12$; that is, eight quasiholes.

when no quasiholes are present, for $N=8$ particles. In the spectra, we have chosen $V_1=1$. There does appear to be a gap above the ground state, and the form of the low-lying excitation spectrum is similar to that of the Pfaffian. In Fig. 8, we show the spectrum when 8 quasiholes are present, for $N=6$ and $N_\phi=12$. However, we note that because the particles carry spin in this system, it is harder to reach sufficiently large sizes to ensure that the results are converging to the thermodynamic limit, and larger sizes may be needed in order to prove that the energy gaps are approaching constants in this limit.

IV. QUASIHOLES OF THE 331 STATE ON THE SPHERE

It is of interest to perform a similar calculation for the quasiholes of the 331 state, even though the excitations of this state are known to have Abelian statistics. The relation to paired states has been discussed in Refs. 14,15,20 and 8.

The 331 state is just one of a family of two-component states, the so-called $mm'n$ states, first introduced by Halperin,[2] and studied further in Refs. 10,16–18 and 14. Using notation $\uparrow$, $\downarrow$ for the two components, these states can be written:

$$\tilde{\Psi}_{mm'n}(z_1^\uparrow,\ldots,z_{N_\uparrow}^\uparrow,z_1^\downarrow,\ldots,z_{N_\downarrow}) =\prod_{i<j}(z_i^\uparrow-z_j^\uparrow)^m\prod_{k<l}(z_k^\downarrow-z_l^\downarrow)^{m'}\prod_{rs}(z_r^\uparrow-z_s^\downarrow)^n. \quad (4.1)$$

The general $mm'n$ state (for some values of $N_\uparrow$ and $N_\downarrow$) is the unique lowest total-angular-momentum ground state of a spin-dependent pseudopotential Hamiltonian, that generalizes Eq. (1.2) to the two-component case,[10]

$$H=\sum_{i<j}\left[\sum_{M=0}^{m-1}V_M P_{ij}(N_\phi-M,\uparrow\uparrow)+\sum_{M=0}^{m'-1}V'_M P_{ij}(N_\phi-M,\downarrow\downarrow)\right]+\sum_{ij}\sum_{M=0}^{n-1}V''_M P_{ij}(N_\phi-M,\uparrow\downarrow) \quad (4.2)$$

in which the projection operators $P_{ij}(L,\sigma\sigma')$ project onto the spin states σ and σ' for particles i and j, respectively, as well as onto total orbital angular momentum L. Thus this Hamiltonian gives positive energy to any state in which two $\uparrow$ or $\downarrow$ particles have relative angular momentum less than m or m', respectively, or in which an $\uparrow$ and a $\downarrow$ particle have relative angular momentum less than n.

For the case when the exponents in these states are of the form $m=m'=q+1$, $n=q-1$, and $q\geqslant 1$ (which require $N_\uparrow=N_\downarrow=N/2$, giving a filling factor $\nu=1/q$, and the partial filling factors for $\uparrow$ and $\downarrow$ are both $1/2q$; for brevity, we will continue to refer to this class of states with general q as the 331 state), then use of the Cauchy determinant identity

$$\prod_{i<j}(z_i^\uparrow-z_j^\uparrow)\prod_{k<l}(z_k^\downarrow-z_l^\downarrow)\prod_{rs}(z_r^\uparrow-z_s^\downarrow)^{-1}=\det\left(\frac{1}{z_i^\uparrow-z_j^\downarrow}\right) \quad (4.3)$$

allows the ground states to be written in a paired form (i.e., as a spin-independent Laughlin-Jastrow factor times a pairing function), similar to the Pfaffian and HR states.[3,14] In terms of BCS-type pairing, this function describes p-type spin-triplet pairing, with each pair in the $S_z=0$ state of a spin triplet.[15,20]

We will write the quasihole states immediately in terms of broken pairs (although a simple form using Laughlin quasihole operators acting on either $\uparrow$ or $\downarrow$ spins will be described later),

$$\frac{1}{(N_\uparrow-F_\uparrow)!}\sum_{\substack{\sigma\in S_{N_\uparrow}\\ \rho\in S_{N_\downarrow}}}\operatorname{sgn}\sigma\operatorname{sgn}\rho\prod_{k=1}^{F_\uparrow}(z_{\sigma(k)}^\uparrow)^{n_k}\prod_{l=1}^{F_\downarrow}(z_{\rho(l)}^\downarrow)^{m_l}\prod_{r=1}^{N_\uparrow-F_\uparrow}\frac{\Phi(z_{\sigma(F_\uparrow+r)}^\uparrow,z_{\rho(F_\downarrow+r)}^\downarrow;w_1,\ldots,w_{2n})}{z_{\sigma(F_\uparrow+r)}^\uparrow-z_{\rho(F_\downarrow+r)}^\downarrow}\prod_{i<j}(z_i-z_j)^q, \quad (4.4)$$

which is particularly similar to the HR case, except that here n_k and $m_l\leqslant n-1$, and so $0\leqslant F_\sigma\leqslant n$. The flux in these states is $N_\phi=q(N-1)-1+n$, as for the Pfaffian.

For the count of states at fixed w's we obtain, again summing over $F_\uparrow$ with $F=F_\uparrow+F_\downarrow$ fixed,

$$\binom{2n}{F}, \quad (4.5)$$

which is the number of states for F fermions in $2n$ orbitals. The sum over F satisfying $(-1)^F=(-1)^N$ then yields a total of 2^{2n-1} zero-energy quasihole states of all spins for fixed N and w's. This is valid for all n except $n=0$, in which case there is no zero-energy state for N odd.

The result 2^{2n-1} may also be understood by viewing it as coming from the choice of layer index on the Laughlin quasihole operator. Thus quasihole states of zero energy can clearly be obtained by multiplying factors $\Pi(z_i^\sigma - w^\sigma)$, that act on either particles of spin $\sigma=\uparrow$ only, or on spin-$\downarrow$ only, into the 331 ground state. Further, the numbers $N_\uparrow$ and $N_\downarrow$ need not be equal. However, the flux seen by both $\uparrow$ and $\downarrow$ particles must be equal. This leads to the condition $2(N_\uparrow - N_\downarrow) = n_\downarrow - n_\uparrow$, where n_σ are the numbers of quasiholes of the two types, in which the factors act on the particles of spin σ. For fixed $N=N_\uparrow+N_\downarrow$, there are 2^{2n-1} ways to choose the spins of the quasiholes consistent with these conditions, for $n\geqslant 1$. We will refer to this construction of the zero-energy states as bosonic, because, as explained in Ref. 8, the relation of the two approaches is related to bosonization of Fermi systems.

Returning to the paired, or fermionic, description, the total number of states resulting from the positional degeneracy is, in this case,

$$\sum_{F,(-1)^F=(-1)^N} \binom{2n}{F}\binom{(N-F)/2+2n}{2n}. \tag{4.6}$$

This is larger than for the Pfaffian or HR states. It should also be possible to obtain this formula from the bosonic description, although we have not done so. The positional degeneracy for the quasiholes of each spin is, in that approach,

$$\binom{N_\sigma + n_\sigma}{n_\sigma}, \tag{4.7}$$

and the summations are constrained by the fact that for the zero-energy states N_σ depends on n_σ.

For the 331 state, it is again not too difficult to extend the arguments for the Pfaffian or HR cases, to show the completeness and linear independence of the zero-energy states found, in the pairing or fermionic form. This is also quite clear in the bosonic form of the states.

V. GROUND STATES ON THE TORUS

In this section we consider the zero-energy eigenstates of the special Hamiltonians discussed above on a system with periodic boundary conditions (a torus), without quasiholes. Although we believe that the states we will give span the complete spaces of such states, we will not prove this, but will refer to numerical results for confirmation.

First we briefly review known results for this geometry,[23] to fix notation. In the Landau gauge $\mathbf{A}=-By\hat{\mathbf{x}}$, we will take the magnetic length to be 1, and the system to be a parallelogram with sides L_1 and L_2 and periods L_1 and $L_2e^{i\alpha}=L_1\tau$ in the complex plane; α is the angle between the sides, and τ (with $\mathrm{Im}\tau>0$) parametrizes the aspect ratio. As usual, there are N_ϕ flux through the surface. Many-particle wave functions in the LLL can be written

$$\Psi(z_1,\ldots,z_N)=f(z_1,\ldots,z_N)e^{-\Sigma_i y_i^2/2}, \tag{5.1}$$

where f is a holomorphic function; as a consequence of the boundary conditions on Ψ it is required to satisfy, for all i,

$$\frac{f(z_1,z_2,\ldots,z_i+L_1,\ldots)}{f(z_1,z_2,\ldots,z_i,\ldots)}=e^{i\phi_1},$$

$$\frac{f(z_1,z_2,\ldots,z_i+L_1\tau,\ldots)}{f(z_1,z_2,\ldots,z_i,\ldots)}=e^{i\phi_2}e^{-i\pi N_\phi(2z_i/L_1+\tau)}. \tag{5.2}$$

Here ϕ_1 and ϕ_2 represent general twisted boundary conditions, the same for all particles, and are the same for all states in the Hilbert space. They can be set to zero without any real loss of generality.

From a general symmetry analysis,[24] any state in the system can be decomposed into center of mass and relative motion, as

$$f(z_1,\ldots,z_N)=F_{\mathrm{cm}}(Z)f_{\mathrm{rel}}(z_1,\ldots,z_N), \tag{5.3}$$

where f_{rel} is invariant under shifts of all z_i by the same amount, and F_{cm} is a function of $Z=\Sigma_i z_i$ only. Given the boundary conditions on f, specified by ϕ_1 and ϕ_2, there is still some freedom of choice in the corresponding phases in the conditions on F_{cm} and f_{rel}, which is related to Haldane's **k**-vector quantum number,[24] and which will be useful in the following. In any case, there are always q solutions for F_{cm}, for filling factor $\nu=p/q$ (p, q coprime), which are mapped into each other by magnetic translations of the center of mass, and consequently are degenerate in energy for any translationally invariant Hamiltonian.

For the Laughlin state on the torus, the property of vanishing as the qth power as any $z_i\to z_j$ fixes the relative wave function to be (definitions and results for the theta and elliptic functions used in this section are given in Appendix A)

$$f_{\mathrm{rel}}=\prod_{i<j}\vartheta_1((z_i-z_j)/L_1|\tau)^q\equiv f_{\mathrm{LJ}}. \tag{5.4}$$

The basis states for the center-of-mass wave functions found in Ref. 23 can be rewritten (see Appendix A), apart from some constant factors, as[25]

$$F_{\mathrm{cm}}(Z)=\vartheta\begin{bmatrix}\ell/q+(N_\phi-q)/2q+\phi_1/2\pi q\\ -(N_\phi-q)/2-\phi_2/2\pi\end{bmatrix}(qZ/L_1|q\tau). \tag{5.5}$$

Here $\ell=0,1,\ldots,q-1$ labels the center-of-mass degeneracy, and ϕ_1 and ϕ_2 have been retained for generality. Using the properties given in Appendix A, this can be verified to obey the conditions resulting from Eq. (5.2) that are given in Ref. 23. Alternatively one can verify directly that Eq. (5.3) satisfies Eq. (5.2). Note that F_{cm} has q zeros in the unit cell for Z of sides L_1 and $L_1\tau$, and linear combinations of these functions span the Hilbert space of a charged particle on a torus in a magnetic field with q flux quanta through the torus, in the LLL. The flux seen by the particles in these states is $N_\phi=qN$.

For the paired states on the torus, the relative motion part f_{rel} must be modified from f_{LJ}, in particular to change the symmetry under permutations, which can be done at the same flux $N_\phi=qN$, by using

$$f_{\rm rel}=f_{\rm elliptic}f_{\rm LJ}\,. \tag{5.6}$$

Here $f_{\rm elliptic}(z_1,\ldots,z_N)$ is a meromorphic function, and, since $f_{\rm rel}$ must be holomorphic, any poles in $f_{\rm elliptic}$ must be of sufficiently low order that they are rendered nonsingular by the zeros of $f_{\rm LJ}$, which are located on the hyperplanes $z_i=z_j$. Further, $f_{\rm elliptic}$ must obey z_i-independent boundary conditions

$$f_{\rm elliptic}(z_1,\ldots,z_i+L_1,\ldots,z_N)\propto f_{\rm elliptic}(z_1,\ldots,z_i,\ldots,z_N), \tag{5.7}$$

and similarly for $z_i\rightarrow z_i+L_1\tau$. Thus $f_{\rm elliptic}$ is analogous to an elliptic function, but in N complex variables. In general, the phases that are the proportionality factors in Eq. (5.7) need not be equal to 1, because any phase left by translating $f_{\rm rel}$ can be absorbed (canceled) by modifying the behavior of $F_{\rm cm}$, by a shift in ϕ_1 or ϕ_2 (by π in the present case) in Eq. (5.5) before setting ϕ_1 and ϕ_2 to zero. This is the freedom of choice that is related to the **k** vector, mentioned above. For the paired states, $f_{\rm elliptic}$ is expected to be a periodic generalization of the pairing functions, such as the Pfaffian, discussed earlier on the sphere.

We now review the ground states of the three-body Hamiltonian Eq. (2.2) and its generalizations on the torus. These ground states, without quasiholes, were found by Greiter, Wen and Wilczek.[6] For the relative motion part we take

$$f_{{\rm elliptic},a}={\rm Pf}\left[\frac{\vartheta_a((z_i-z_j)/L_1|\tau)}{\vartheta_1((z_i-z_j)/L_1|\tau)}\right] \tag{5.8}$$

where $a=2$, 3, or 4. [The three ratios $\vartheta_a/\vartheta_1(z|\tau)$ are essentially the three Jacobi elliptic functions ${\rm sn}(z|\tau)$, ${\rm cn}(z|\tau)$, and ${\rm dn}(z|\tau)$, up to translations of z and some factors.] These then reverse sign under exactly two of the three transformations $z_i\rightarrow z_i+L_1$, $z_i\rightarrow z_i+L_1\tau$, and $z_i\rightarrow z_i+L_1(1+\tau)$, and are invariant under the third, for any i. The change in sign can be absorbed by the effect of a modification of $F_{\rm cm}$ as explained above, such that the full wave function always obeys the same boundary conditions. This structure has the consequence that Haldane's **k**-vector quantum number,[24] which lies in a Brillouin zone, and which is zero (modulo reciprocal-lattice vectors) in all q of the periodic Laughlin states, is nonzero in these periodic Pfaffian states. The distinct nonzero values for $a=2$, 3, or 4, are determined by the behavior under the three translations already mentioned. For the most symmetrical choice of τ, $\tau=e^{i\pi/3}$, which gives the system sixfold rotational symmetry in real space and in the hexagonal Brillouin zone of **k**'s, these nonzero vectors lie halfway along the shortest nonzero reciprocal-lattice vectors; only three of these vectors are distinct modulo addition of reciprocal-lattice vectors. Thus they lie at the midpoints of the edges of the zone, for this choice of zone. Thus, all $3q$ zero-energy states found for $\nu=1/q$ have distinct quantum numbers, and so are linearly independent (indeed, orthogonal) states. Numerical calculations are in agreement with these quantum numbers.[26]

Next we turn to the HR state on the torus. For the HR state at $\nu=\frac{1}{2}$ ($q=2$), numerical calculations have revealed that there are $10=5q$ ground states of the hollow-core model,[27,28] and so we expect to find $5q$ for general q. The full set of wave functions has, to our knowledge, not been obtained previously.

Following the reasoning for the Pfaffian, we might expect $f_{\rm elliptic}$ to be a determinant of $\vartheta_a\vartheta_b/\vartheta_1^2$'s in $z_i^\uparrow-z_j^\downarrow$ with a, $b=2$, 3, and 4, or possibly an antisymmetrized combination of other functions that each obey the same boundary conditions. (This assumption may be too restrictive, since the boundary conditions in fact need only apply to the complete function. We will see that it works for HR, but not for 331.) These products of Jacobi elliptic functions are not, however, linearly independent. In general, elliptic functions are completely determined by the singular part of their behavior near the poles, and by the periodicities (see, e.g., Ref. 29). In the present case, we require that under $z\rightarrow z+1$, and $z\rightarrow z+\tau$ the elliptic function [with arguments $(z|\tau)$] be either periodic or antiperiodic, giving four possibilities which we will label $++$, $+-$, $-+$, and $--$, in an obvious notation, and we also require that there be a double pole at $z=0$, with residue zero.

For the $++$ case, there is a classic solution to these requirements: the Weierstrass function $\wp(z|\tau)$. The required functions with the other boundary conditions, which we will denote $\wp_2$, $\wp_3$, and $\wp_4$ (in the same sequence as before), can be obtained straightforwardly (see Appendix A). Defining $\wp_1=\wp$, four candidates for the relative part of the HR state on the torus are obtained:

$$f_{{\rm elliptic},a}=\det\wp_a((z_i^\uparrow-z_j^\downarrow)/L_1|\tau), \tag{5.9}$$

and so, on including $f_{\rm LJ}$ and $F_{\rm cm}$, we obtain $4q$ states. It is easy to see that they are zero-energy states of the hollow-core Hamiltonian. The **k** values for the cases $a=2$, 3, and 4 are as for the Pfaffian, while $a=1$ gives states at $\mathbf{k}=\mathbf{0}$. Since the $4q$ states have distinct quantum numbers, they are linearly independent.

To obtain the fifth set of q states, we note that, for the $++$ case only, the Weierstrass function is not the unique solution to the problem posed: we obtain another solution by adding a constant. (For the other cases, this would violate the boundary conditions.) Indeed, we could have used a constant in place of $\wp_1$, but the determinant would then vanish except in the case of $N=2$ particles. If we insert $\wp_1+c$ in place of $\wp_1$ in the determinant, and then expand in powers of c, we find that terms of order higher than one in c vanish identically because there are rows or columns in the determinant that are equal. The term of first order, however, is linearly independent of the zeroth-order term (which is $f_{\rm elliptic,1}$) and is nonzero. Further, it is a zero-energy eigenstate of the hollow-core model; it clearly has $\mathbf{k}=\mathbf{0}$. Explicitly, this function is

$$f_{\rm elliptic,5}=\frac{1}{(N/2-1)!}\sum_{\substack{\sigma\in S_{N/2}\\ \rho\in S_{N/2}}}{\rm sgn}\sigma\,{\rm sgn}\rho\prod_{r=1}^{N/2-1}\times\wp_1((z^\uparrow_{\sigma(1+r)}-z^\downarrow_{\rho(1+r)})/L_1|\tau). \tag{5.10}$$

We point out that this has an interpretation in terms of unpaired fermions. Unpaired fermions must occupy single-particle states that are holomorphic, and obey the boundary conditions on $f_{\rm elliptic}$. For any except $++$, there are no such states, and for $++$ the only state is, again, the constant function. By breaking one pair in $f_{\text{elliptic},1}$ and putting the two fermions (with opposite spin) in this constant state, we obtain $f_{\text{elliptic},5}$. We observe that all $5q$ states found are spin singlets. The first and fifth sets of q states are linearly independent because they have different numbers of poles. The **k** values found agree with the numerical results.

The existence of five sets of q states for the HR state on the torus is surprising, especially as the analysis in Ref. 8 found just $4q$ sectors of edge states, and one expected,[5] on general CFT grounds, that the number of sectors (or of primary fields of the chiral algebra of the CFT, which comes to the same thing) would be the same as the number of conformal blocks in the CFT on the torus. All we can say about this here is that the CFT described in Ref. 8 for the HR state does lead to $4q$ vacuum sectors on the torus, yet what is actually required for constructing a QHE state is a correlator containing many *insertions* of fields in the chiral algebra, corresponding to the particles in the ground state. For the HR case, these correlators are the ground states found above, and one set of q sectors has turned out to contain two conformal blocks. However one chooses to interpret this in CFT terms (it is probably related to other oddities of the CFT for the HR state[8]), the existence of $4q$ sectors in the underlying theory is not in doubt.

Finally, we turn to the 331 state. Since the pole in f_{elliptic} is expected to be first order, we try

$$f_{\text{elliptic},a}=\det\left[\frac{\vartheta_a((z_i^\uparrow-z_j^\downarrow)/L_1|\tau)}{\vartheta_1((z_i^\uparrow-z_j^\downarrow)/L_1|\tau)}\right]. \tag{5.11}$$

For $a=2$, 3, and 4, these lead to nonvanishing, zero-energy states for the pseudopotential Hamiltonian. We expect, however, that there are $4q$ states altogether, based on the general structure of this Abelian quantum Hall state (see, e.g., Ref. 8), and we expect the remaining f_{elliptic} to be $++$. Clearly the natural choice $\det\vartheta_1/\vartheta_1$ is nonvanishing only for $N=2$. From the theory of elliptic functions in one variable, the constant is the only elliptic function with at most one simple pole and these boundary conditions, so we have exhausted the possibilities of this structure. However, as pointed out above, f_{elliptic} need not be an antisymmetrized product of elliptic functions in $(z_i-z_j)/L_1$ that each satisfy the boundary condition; only f_{elliptic} itself must have this property, so we should broaden our search.

The correct solution can be obtained, no doubt, in various ways. One way is to use the expectation that f_{elliptic} is a conformal block for a Dirac Fermi field on the torus, with $++$ boundary condition in the case of interest (the other functions $f_{\text{elliptic},a}$, $a=2$, 3, and 4, can be viewed in exactly this way). This conformal block is known to exist, and could also be obtained by bosonization. We are indebted to Greg Moore, who obtained the following formula (for $N=4$) for this function, at our request, by a limiting procedure of first considering the conformal block for the fermi field with the general boundary condition, that is twisted by $e^{i\psi_1}$ and $e^{i\psi_2}$, and taking the limits ψ_1 and $\psi_2\to 0$. The result is (up to factors independent of the z_i's)

$$f_{\text{elliptic},1}=\frac{1}{(N/2-1)!}\sum_{\substack{\sigma\in S_{N/2}\\ \rho\in S_{N/2}}}\text{sgn}\sigma\,\text{sgn}\rho\prod_{r=1}^{N/2-1}\times\frac{\vartheta_1'((z_{\sigma(1+r)}^\uparrow-z_{\rho(1+r)}^\downarrow)/L_1|\tau)}{\vartheta_1((z_{\sigma(1+r)}^\uparrow-z_{\rho(1+r)}^\downarrow)/L_1|\tau)}, \tag{5.12}$$

where $\vartheta_1'(z|\tau)=d\vartheta_1(z|\tau)/dz$. Notice that, like the fifth function for the HR state, there are two unpaired fermions of opposite spins, occupying the constant single-particle state that is allowed by the boundary conditions. For $N=2$, this is all that remains, and this function was already noted above. For $N>2$ this function contains ϑ_1'/ϑ_1, which is not strictly periodic (since no such functions exist for $++$ boundary conditions), but obeys

$$\frac{\vartheta_1'(z+\tau|\tau)}{\vartheta_1(z+\tau|\tau)}=\frac{\vartheta_1'(z|\tau)}{\vartheta_1(z|\tau)}-2\pi i, \tag{5.13}$$

and is invariant under $z\to z+1$. When any $z_i^\uparrow$ ($z_j^\downarrow$) in $f_{\text{elliptic},1}$ is translated by $L_1\tau$ ($-L_1\tau$), the result is $f_{\text{elliptic},1}(z_1^\uparrow,\ldots,z_{N/2}^\downarrow)$ plus a term that vanishes because the constant $-2\pi i$ must be antisymmetrized with the constant 1 that represents the missing row and column in the determinant. Thus $f_{\text{elliptic},1}$ is invariant, and we found the fourth set of q zero-energy states. We note that $\mathbf{k}=\mathbf{0}$ for these states, and the quantum numbers of all $4q$ states are distinct, so these states are linearly independent.

In this section we implicitly assumed that the number of particles N is even. One may ask if there are also ground states on the torus for N odd. Such states will have an odd number of unpaired fermions. As we have seen, this is possible only in the $++$ (or $\mathbf{k}=\mathbf{0}$) sector. For the Pfaffian, there is no such sector, so there are no ground states for N odd, except for $N=1$. For 331 the $\mathbf{k}=\mathbf{0}$ states already include two unpaired fermions for N even. For N odd, there must be just one unpaired fermion, of either up- or down-spin, otherwise the state vanishes. But, on generalizing Eq. (5.12), one finds that it no longer satisfies the boundary condition, except, again, for $N=1$. Finally, for HR, there is no problem constructing a zero-energy state with one unpaired fermion in the $\mathbf{k}=\mathbf{0}$ sector. This gives a spin-$\frac{1}{2}$ doublet of ground-states for all odd N, and we have verified numerically that states with these quantum numbers are the only zero-energy states for N odd. If these ground state wave functions are again interpreted as CFT correlators, then they imply that in this $++$ sector, there are nonzero correlators containing an odd number of insertions of the Fermi field.

Greiter, Wen, and Wilczek[6] also found formulas for two quasiholes of the Pfaffian state on the torus. There should be no great difficulty in extending the results of the present paper to include any number of quasiholes on the torus for any of the paired states we have considered.

VI. EFFECT OF ZEEMAN TERM, TUNNELING, AND OTHER PERTURBATIONS

In this section we address the question of what happens to the degeneracies of the quasihole states and of the ground states on the torus when the Hamiltonian is varied from the exactly soluble (for the zero-energy states) forms we have considered up to now. First we consider the effect of the Zeeman term $-h\Sigma_i\sigma_{z,i}$ on the HR state; here $h>0$ is the magnetic field $g\mu_B B$, and $\sigma_{z,i}$ is the z component of the spin operator for the ith particle, which can be represented by the usual Pauli matrices. The hollow-core Hamiltonian is spin rotation invariant, and its eigenstates are also eigenstates for the total spin. Hence the Zeeman term simply has the effect of splitting the multiplets of spin states. For the quasihole states, this means that the degeneracy is partially resolved, independent of the locations of the quasiholes. The lowest-energy states are then those with the largest number of $\uparrow$ unpaired fermions, and the lowest number of $\downarrow$. Since $F_\sigma \leqslant n-1$, and F and N have the same parity, these states contain $F_\uparrow = n-1$ and $F_\downarrow = 0$ when N and $n-1$ are of the same parity, and either $F_\uparrow = n-1$, $F_\downarrow = 1$ or $F_\uparrow = n-2$, $F_\downarrow = 0$ when N and $n-1$ are of opposite parity. For fixed w's these states clearly have a residual degeneracy 1 in the first case, $2(n-1)$ in the second. [The total spin of these lowest-energy states, which is alternately $S_z=(n-1)/2$ or $S_z=(n-2)/2$, agrees with the results in Ref. 3.] In the first case in particular, the lifting of the degeneracy implies that adiabatic exchange of the quasiholes can produce only a phase factor, and so the statistics is Abelian.

We also note here that for the edge states,[8] the gapless spectrum of spin-carrying fermion excitations at the edge implies that the spin susceptibility of the edge is nonzero, and for finite h there will be some $\uparrow$ fermions present in the ground state, so there is a magnetization at the edge, or one could say the edge is reconstructed in this way; a confining potential is necessary to stabilize this effect. If one thinks of the $\uparrow$ fermions and their antiparticles, the $\downarrow$ fermions, as particles and holes of a chiral Fermi-Dirac sea, then the reconstructed state just corresponds to shifting the Fermi energy of the sea. There is thus still a gapless branch of fermion excitations for both spin-$\uparrow$ and -$\downarrow$ at the edge. Further, the degeneracy of $5q$ on the torus is not split by the Zeeman term, because all the states are singlets.

It is clear that similar effects may be expected for any Abelian or non-Abelian statistics state, when there is a symmetry present, particularly a continuous symmetry: A symmetry-breaking perturbation may break the degeneracies and leave some kind of Abelian statistics behavior. This does not, however, necessarily mean that non-Abelian statistics is unstable against *any* perturbation, nor does it mean that the Abelian statistics obtained is that of some simpler Abelian state, such as a Laughlin state of charge-2 bosonic pairs. We also note that, for either the Pfaffian or HR state, when there are only two quasiholes, there are no degeneracies for fixed quasihole positions on the sphere (so there is nothing to split), yet the expectation is that the Berry phase obtained on adiabatically exchanging the two is not that in the Laughlin state of charge-2 bosonic pairs,[5] at least if the quasiholes are given by the wave functions studied in this paper. In fact, while the breaking of degeneracies does strictly mean that non-Abelian statistics does not occur in adiabatic transport of quasiparticles, other associated properties, such as the spectrum of edge states, are, as we have seen, not affected (in general, a splitting of the velocities might occur due to a symmetry-breaking perturbation.) Quite similar phenomena are known in the hierarchy states, which have Abelian statistics, where, for example, SU(N) symmetry appears in certain states,[19] but the symmetry (or related degeneracies) is presumably broken by the Hamiltonian, both for the bulk quasiparticle states and for the edge spectrum. For this reason, we propose that such effects do not really represent a change of universality class in the non-Abelian systems either.

We now discuss the effect of tunneling between the layers on the 331 states; mathematically this is the same as a Zeeman-like term $-t\Sigma_i\sigma_{x,i}$ (see earlier discussions in Refs. 15 and 20). This term can be diagonalized by using a basis of σ_x eigenstates for each particle, which we will label e and o (for ''even,'' ''odd''), given by $e=(\uparrow+\downarrow)/\sqrt{2}$, $o=(\uparrow-\downarrow)/\sqrt{2}$, which have eigenvalues $+1$ and -1, respectively, under $\sigma_{x,i}$. In the literature, these states have often been denoted ''symmetric'' and ''antisymmetric,'' respectively. Unlike the HR case, the pseudopotential Hamiltonian for which 331 is exact is not spin rotation invariant, and the energy eigenstates are not eigenstates of $S_x=\frac{1}{2}\Sigma_i\sigma_{x,i}$. Thus the tunneling is a symmetry-breaking perturbation, which breaks the conservation of S_z. The effect of the tunneling term is to modify the states, not merely to split their energies. Nonetheless, when $t>0$ is small, we may try to use degenerate perturbation theory to understand its effect, which means diagonalizing the tunneling term in the subspace of the states that have zero energy when $t=0$; this would give the exact results at first order in t. We are not able to carry this out analytically in general, because we do not have the matrix elements of the perturbation among these states. We may expect, however, that the degeneracies would be at least partially lifted, in a similar way as for the HR state with Zeeman, by the following argument. In spin space, the pairs in the 331 state take the form $\uparrow_i\downarrow_j+\downarrow_i\uparrow_j=e_ie_j-o_io_j$, using an obvious notation for the spinors for the ith and jth particles in a term in which these form a pair. Halperin[15] has argued that the effect of positive tunneling t is to cause a change in the 331 ground state, which within a trial-wave-function description causes the amount of oo in the pairs to decrease. Ho[20] proposed that this occurs at first order in t. If we also write the unpaired fermions in our zero-energy states in the e-o basis, and neglect the effect of t on the paired part of the state, its effect would be to split the energies of the e and o unpaired fermions, exactly as for the $\uparrow$ and $\downarrow$ fermions of the HR state, and again with the effect of removing most of the degeneracy. For the ground states on the torus, we first note that states with $\mathbf{k}\neq\mathbf{0}$ are even under layer exchange, which has the effect of multiplying by $\Pi_i\sigma_{x,i}$, while the states at $\mathbf{k}=\mathbf{0}$ are odd (incidentally, this agrees with numerical findings[26]). The broken pair in those states contains one e and one o fermion. Application of the same naive argument as for the quasihole states then suggests that the $4q$ degeneracy is split to $3q$ by $t>0$. However, an accurate calculation should include the modification of the rest of the state, and the splitting might disappear in the thermodynamic limit.

If we consider arbitrary small changes in the Hamiltonian from the special forms considered in previous sections, then physical arguments like those for the Zeeman and tunneling terms suggest that, as the degeneracy arises from breaking pairs and putting the fermions into the zero-mode functions, there is no obvious reason why it should not be broken in general. For the bound states in the gap in the vortex cores in conventional BCS superfluids, such excited many-particle states have positive energies. However, such arguments may just be too naive, because of the modification of the ground-state wave function.

In an interesting paper, Ho[20] gave a more general interpolation between the 331 and Pfaffian ground states by varying the spin states of the pairs. Within a paired trial wave-function description, the effect of tunneling is presumably to reduce the amount of oo in the spin part of the wave function of a pair.[15] If this is done without changing the spatial factor $(z_i - z_j)^{-1}$, then when the pairs are purely ee, and the ground state is precisely the Pfaffian state. Ho claimed that this somehow contradicts the ''topological'' arguments that assert that one cannot go continuously between these distinct ground states. However, this is really a misstatement; one can *always* interpolate between any two state vectors in the same Hilbert space. The real question is whether the *properties* of the states, like those considered in this paper, can be continuously connected. If one wishes to exhibit a breakdown of the ''topological'' arguments, then it is necessary to show that the interpolation occurs without any phase transition, that is without any energy gap for local excitations going to zero (which would be a second order transition) or any level crossing of ground states (which would be a first order transition). Ho proposed a family of Hamiltonians for each of which his corresponding wave function is a zero-energy ground state, but did not show that the energy gap is maintained throughout the interpolation. We will now examine this. We will show that at the point where Ho's ground state is the Pfaffian, there is an enormous degeneracy of other zero-energy states for his Hamiltonian, implying that the energy gap collapses to zero at this point, and the system is not incompressible. We will show that, up to that point, the degeneracies of the zero-energy eigenstates of his model are the same as those of the pseudopotential Hamiltonian for the 331 state, considered earlier. We will then consider modifications of his model that remove the pathology.

Following Ho, we now fix $q=2$, so the particles are fermions, and the filling factor is $\nu=\frac{1}{2}$ (as usual, other cases are similar). To understand Ho's description, first recall that for spin-$\frac{1}{2}$ fermions in the LLL, Fermi statistics implies that if the particles have even relative angular momentum, $m=0, 2, \ldots$, then they must be in an antisymmetric spin state, which can only be a singlet, while if they are in an odd relative angular momentum state, $m=1,3,\ldots$, then they can only be in a symmetric spin state, which must be a triplet. We emphasize that these statements remain true in the presence of any number of other particles. If the two particles have total spin 1, then there is a three-dimensional complex vector space of spin states, and the spin state can be described exactly by a nonzero complex three-component vector $\mathbf{d}$, of which the magnitude and phase are irrelevant to the state. We will not need the detailed definition of $\mathbf{d}$, which can be found in Ho[20] or in references on superfluid ^{3}He (see, e.g., Ref. 30). We note only that, for the spin state $\uparrow_i\downarrow_j + \downarrow_i\uparrow_j = e_i e_j - o_i o_j$ (as in the pairs in the 331 state), $\mathbf{d} \propto \hat{\mathbf{x}}_z$, while for $e_i e_j$ (as in the pairs in the Pfaffian), $\mathbf{d} \propto (\hat{\mathbf{x}}_z - i\hat{\mathbf{x}}_y)/\sqrt{2}$ (here $\hat{\mathbf{x}}_z$ denotes a unit vector in spin space in the z direction, etc.). Notice that the transformation from one state to the other is not simply a rotation. Ho's Hamiltonian is a pseudopotential Hamiltonian

$$H_{\rm Ho}(\mathbf{d}) = \sum_{i<j} [V_0 P_{ij}(N_\phi, S=0) + V_1 P_{ij}(N_\phi - 1, \perp \mathbf{d})], \tag{6.1}$$

which gives positive energy to any two particles with (i) relative angular momentum zero and spin zero, or (ii) relative angular momentum one and triplet spin state orthogonal to a chosen state specified by $\mathbf{d}$. For a choice of $\mathbf{d}$ that corresponds to 331, this reduces to the pseudopotential Hamiltonian (4.2) used above; pairs may have zero energy and relative angular momentum 1 only if they are in the $\uparrow_i\downarrow_j + \downarrow_i\uparrow_j$ spin state. For a $\mathbf{d}$ vector that corresponds to the ee pairs in the Pfaffian, it allows two particles to have relative angular momentum 1 at zero energy if they both have spin e, but not if they are $eo+oe$ or oo. However, at this $\mathbf{d}$ vector, which we call the Pfaffian point, *all states of the electrons in which all spins are e are zero-energy eigenstates of this Hamiltonian*. This follows because there are clearly no singlets in such states, and so no relative angular momentum zero pairs either. This should be no surprise, since no convenient two-body Hamiltonian (like Ho's) giving the Pfaffian as ground-state and a sensible spectrum is known. Since there are very many spin-aligned states at $N_\phi = 2N-3$, Ho's Hamiltonian has a very large ground-state degeneracy. There might, of course, also be degenerate zero-energy states in which the spins are not all e. We will fully analyze this degeneracy below.

For $\mathbf{d} \not\propto (\hat{\mathbf{x}}_z - i\hat{\mathbf{x}}_y)/\sqrt{2}$, the degeneracies of Ho's ground state, and of quasihole states on the sphere and ground states on the torus, coincide with what was found earlier for the case of the 331 state. To construct these zero-energy states for general $\mathbf{d}$, we use a more precise notation for the wave functions that includes the spin states. We label the particles $i=1,\ldots,N$, and use $\uparrow_i$ and $\downarrow_i$ for spinors which are eigenstates of $\sigma_{z,i}$; the wave function is now in a tensor-product space of spatial wave functions and spinors, and must be antisymmetric under simultaneous exchange of coordinates and spinors of two particles. For example, the ground state on the sphere is

$$\tilde{\Psi}_{\mathbf{d}} = \sum_{\sigma \in S_N} \text{sgn}\sigma \prod_{k=1}^{N/2} \frac{\chi_{\sigma(2k-1),\sigma(2k)}(\mathbf{d})}{z_{\sigma(2k-1)} - z_{\sigma(2k)}} \prod_{i<j} (z_i - z_j)^q \tag{6.2}$$

(which is actually a Pfaffian), where $\chi_{i,j}(\mathbf{d})$ is the spin state of two particles i and j that corresponds to $\mathbf{d}$, and the product of these factors is actually a tensor product, so that $\tilde{\Psi}_{\mathbf{d}}$ is a multispinor function of the coordinates. This clearly has zero energy. We will call it the Ho state, as it appears in Ref. 20 (in a different notation). In the presence of quasihole factors Φ in the pairing factors, one may construct zero-energy states with unpaired fermions, as for the 331 state, but with the pairs taking the same form as in Eq. (6.2). The unpaired fermions can be in either spin state. Even though the ground state (6.2) does not have a definite total S_z, except in the 331 special case, the counting of zero-energy quasihole states proceeds just as for the 331 state, and the results are identical to those in Sec. IV. Similar results are found for the edge states, which are in one-to-one correspondence with those in Ref. 8, and for ground states on the torus, which are like those in Sec. V. We expect that these zero-energy states are the complete set, except in the Pfaffian limit. This is consistent with the continuity of the spectrum as a function of **d** *in a finite-size system*, which requires that the energy levels found in the 331 state must change continuously with the Hamiltonian. We expect that the larger degeneracy appears only at the Pfaffian point, because there the pairs have the special property of being composed of one spin (e) only, unlike the general states $\chi(\mathbf{d})$. This implies that the energy of some excited states decreases as this point is approached, so the gap goes to zero. Therefore Ho's Hamiltonian is pathological at the Pfaffian point.

In spite of this pathology at the Pfaffian point, it is still of interest, given that the degeneracies of the quasiholes and torus ground states are the same in Ho's model (except for the Pfaffian point) as in the 331 state, to ask whether the statistics properties are the same. In the 331 state, the structure of these properties is described in terms of a U(1) [or SO(2)] quantum number, which in the case of 331 is S_z. Also, excitations with opposite values of this quantum number are degenerate, because of the symmetry operation of interchanging the layers, which generates a $\mathbf{Z}_2$ group. These two symmetries combine to make up the symmetry group O(2), which is the semidirect product of $\mathbf{Z}_2$ and SO(2). Ho's Hamiltonian breaks the conservation of S_z, but not the $\mathbf{Z}_2$ symmetry, if we consider only the family $\mathbf{d}=\hat{\mathbf{x}}_z\cos\theta-i\hat{\mathbf{x}}_y\sin\theta$, $0\leqslant\theta\leqslant\pi/4$ as proposed by Ho; these correspond to the spin state $\cos(\theta-\pi/4)e_ie_j+\sin(\theta-\pi/4)o_io_j$ for the pairs, suggested by Halperin,[15] and $\theta=0$ is the 331 case and $\theta=\pi/4$ the Pfaffian. Consequently, one might think the U(1) quantum number is lost. However, the degeneracies of the quasiholes and torus ground states are consistent with the presence of this quantum number, which would be a ''hidden'' U(1) symmetry. One might expect this to be, in some sense, the symmetry of rotation in spin space about the axis of **d**, however, since **d** is complex we must be careful. Under rotations of spin space, **d** is rotated by the action of a (real) orthogonal 3×3 matrix in O(3). In general, there is no rotation that leaves **d** invariant (up to multiplication by a phase), except when **d** is of certain special forms of which the **d** vectors for 331 and the Pfaffian happen to be examples. Remarkably, even though O(2) symmetry is broken by Ho's Hamiltonian, it seems to be reappearing in the low-energy properties. Indeed, in terms of a conformal field theory (CFT) description of the edge states on the cylinder, which gives detailed information about the structure of the states,[8] the low-energy edge states, obtained as zero-energy eigenstates of Ho's model have the same structure as in Ref. 8 for all **d**. If we assume that a CFT description must involve a U(1) theory for charge, together with some unitary $c=1$ theory, combined by the $\mathbf{Z}_2$ orbifold construction of Ref. 8, then the theory described in that reference seems to be the only possibility. However, we should be cautious about concluding on the basis of these observations that the universality class of Ho's model is the same as that of the 331 state, except at the Pfaffian point. Below we will see an example in which the full properties of this class do not emerge, and the degeneracy can be broken by a perturbation even in the thermodynamic limit. This example emerges from further analysis of the Pfaffian point of Ho's model, to which we now turn.

At the Pfaffian point of Ho's model, in the $q=2$ case, it is a two-body projection-operator Hamiltonian which gives positive energy to any two particles which either have opposite spin and relative angular momentum zero, or if one or both of them is o and they have relative angular momentum 1. It is known in general how to find the zero-energy states of such pseudopotential Hamiltonians; this was already discussed at the beginning of Sec. IV. To be zero-energy eigenstates, wave functions must contain the $mm'n$ wave function, Eq. (4.1), as a factor, in which for the particular class (but for general q) considered here, we have $m=q-1$, $m'=q+1$, $n=q$,

$$\tilde{\Psi}_{q-1,q+1,q}(z_1^e,\ldots,z_{N_e}^e,z_1^o,\ldots,z_{N_o}^o)$$

$$=\prod_{i<j}(z_i^e-z_j^e)^{q-1}\prod_{k<l}(z_k^o-z_l^o)^{q+1}\prod_{rs}(z_r^e-z_s^o)^q. \quad (6.3)$$

For functions on the sphere, the number of fluxes seen by e and by o particles must be the same, but this is not the case for function (6.3) as it stands, unless N_o is zero. If $N_o>0$, it is necessary to multiply the function by additional factors, and the space of these factors may be parametrized by viewing them as quasihole factors $U_\sigma(w^\sigma)=\Pi_i(z_i^\sigma-w^\sigma)$, where $\sigma=e$ or o, acting on particles of either spin. On multiplying in n_e factors of U_e, n_o factors of U_o, one finds, for the flux seen by e and o, respectively,

$$N_\phi=(q-1)(N_e-1)+qN_o+n_e \quad (6.4)$$

$$=(q+1)(N_o-1)+qN_e+n_o, \quad (6.5)$$

since the flux must be the same for both; the second line, which is the flux seen by the o particles, applies only if $N_o>0$.

We wish to analyze the situation $N_\phi\geqslant q(N-1)-1$, that corresponds to the Pfaffian ground state or the same plus quasiholes. If we write

$$\Delta N_\phi=N_\phi-[q(N-1)-1], \quad (6.6)$$

which was denoted n in Sec. II, then we find the q-independent equations

$$n_e=N_e-2+\Delta N_\phi, \quad (6.7)$$

$$n_o+N_o=\Delta N_\phi, \quad (6.8)$$

where again the second equation does not apply if $N_o=0$. Now we see that if $\Delta N_\phi\leqslant0$, we must have $N_o=0$ since $n_o\geqslant0$. So in this region, all zero-energy states contain only e particles. In particular, this includes the Pfaffian state and all states degenerate with it at $\Delta N_\phi=0$; for $q=1$, this space of states includes all states in which all the bosons have spin e, and the same is true for the fermions at $q=2$, as remarked above, since antisymmetry requires all states to include the factor $\Pi_{i<j}(z_i^e-z_j^e)$ as a factor. The nondegenerate ground state of the model occurs at $N_\phi=(q-1)(N-1)$ [i.e., at $\Delta N_\phi=-(N-1)+1$], and for $q=2$ is the LLL filled with e particles. For $\Delta N_\phi>0$, the maximum number of o particles possible in the zero-energy states is $N_o\leqslant\Delta N_\phi$. The numbers of zero-energy states, and their angular-momentum decomposition, for each N, N_ϕ for this model can now be obtained, as in other cases (see especially the case of the Laughlin state, and the 331 state in bosonic language).

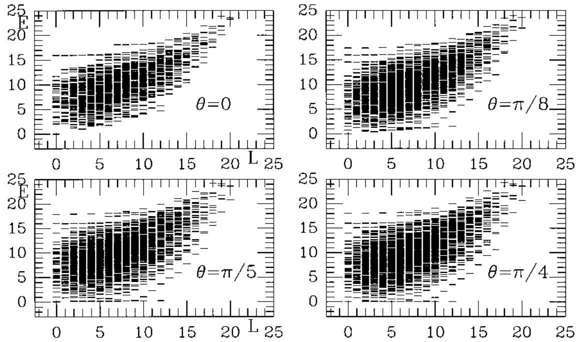

FIG. 9. Spectrum of the Ho model, for four values of θ which parametrize the **d** vector, all for $N=6$ and $N_\phi=9$; that, is $\nu=\frac{1}{2}$ and no quasiholes.

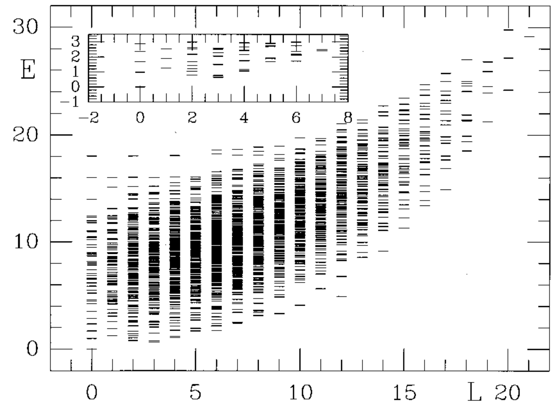

FIG. 11. Spectrum of the Ho model plus the three-body interaction at the Pfaffian point $\theta=\pi/4$, again for $N=6$ and $N_\phi=9$. The inset enlarges the low-lying levels.

Numerical study of Ho's model confirms the above discussion. Figures 9 and 10 show representative spectra with no quasiholes, for various values of θ, including the 331 point $\theta=0$, and for $N=6$ and $q=2$. We chose $V_0=1$ and $V_1=0.5$. For **d** not at the Pfaffian point, the zero-energy ground state at $N_\phi=q(N-1)-1$, is nondegenerate. As the Pfaffian point is approached, a set of states approaches zero energy, and at the Pfaffian point, the degeneracies of the zero-energy states are exactly those of the $mm'n$ system that we have just analyzed. Already at $\theta=\pi/8$, one can identify most of the states that reach zero energy at the Pfaffian point. Calculations not shown in the figures also confirm that states with quasiholes have the degeneracies, and the angular momenta, of those for the 331 state. We notice that, in addition to the zero-energy states at the Pfaffian point, which are fully explained by the above analysis, there are also some very low-energy excited states, for which at present we do not have a detailed explanation.

There is a simple way to remove the ''excess'' degeneracy at the Pfaffian point, without destroying the 331-like behavior of zero-energy states elsewhere. The Pfaffian, with all particles e, would be selected by the three-body Hamiltonian used earlier in the spinless case, if it acted on the e particles. We also observe, from the structure of the wave functions (6.2), that in these wave functions no three particles have total angular momentum $3N_\phi/2-3(q-1)$. Therefore, the three-body operator which projects each set of three particles onto angular momentum $3N_\phi/2-3(q-1)$ and spin $\frac{3}{2}$ (or its analog on the torus) annihilates the Ho states, and the quasihole states and ground states on the torus derived from it *for all* **d** (the quantum numbers are most easily derived by considering spin-$\frac{1}{2}$ bosons for $q=1$; in these Ho states, three bosons are never found at the same point—if they were, they would be in a symmetric spin state). If it is added to the Ho Hamiltonian, all the degeneracies of zero-energy states will be maintained, away from the Pfaffian point. At the Pfaffian point, the total Hamiltonian now selects, at $N_\phi=q(N-1)-1$, the Pfaffian ground state as the unique zero-energy state, in which all spins are e. For smaller N_ϕ there are no zero-energy states, and for larger N_ϕ the zero-energy states are just those of the Ho form (6.2) and its generalizations. In these states, the paired fermions are all e, but the unpaired ones can be either e or o. Consequently, the degeneracies are again those of the 331 state, *not those of the Pfaffian*. This was of course inevitable by continuity, given that no states now come down to zero energy at the Pfaffian point. Numerical spectra confirm these predictions, as shown in Fig. 11 for zero quasiholes, and in Fig. 12 for four quasiholes. In these figures, the coefficient of the three-body projection operator is 1.

We have now arrived at a Hamiltonian which has the degeneracies of the 331 state at all **d** vectors. Yet the Ho model was supposed to represent the effect of tunneling, which should raise the o particles to high energy, and it was expected that the ground state for strong tunneling would be

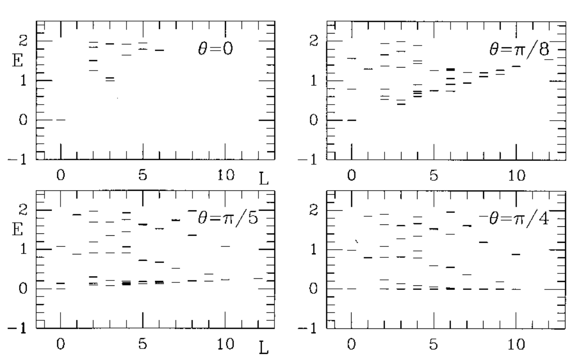

FIG. 10. Same as Fig. 9, but enlarged to show low-lying levels.

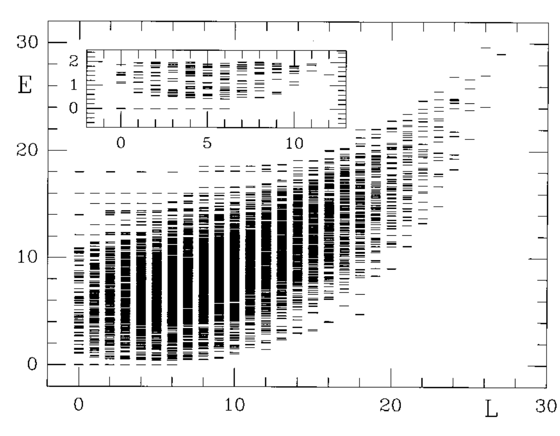

FIG. 12. As in Fig. 11, but with $N=6$ and $N_\phi=11$; that is, four quasiholes.

the Pfaffian. While the ground state we find at the Pfaffian point is the Pfaffian, the problem is that unpaired particles of either spin can be in the zero modes. But this can now be easily (and exactly) remedied by adding to the Hamiltonian the tunneling term $-t\Sigma_i\sigma_{x,i}$. Since the tunneling term is diagonal in the e-o basis we are using, it simply splits the states we have found. Clearly, all states containing only e particles now have energy $-Nt$, and these have the degeneracies of the Pfaffian state for any number of quasiholes, or for edge states, or on the torus. States containing o particles are higher in energy. If we consider the same model, as a function of **d**, and let t vary with **d**, such that t is zero at the 331 point and of order 1 at the Pfaffian point, then we can say that its lowest-energy states are known exactly at the 331 and Pfaffian points, but unfortunately not in between. There must be one or more phase transitions as the Hamiltonian is varied between these limits. Notice that there is a surprising effect at the Pfaffian point, which has edge states including unpaired fermions of either spin when t is zero, but the o states obtain a gap when $t>0$.

When studying more general Hamiltonians that include tunneling, Ho's state would in general be a better choice of trial state than the original 331 state, and **d** can be used as a variational parameter. We wish to emphasize that a large overlap of the ground state of a Hamiltonian with a particular trial state, say, the 331 state (as in Ref. 18) does not prove that the state is in the corresponding (say, 331) universality class; to show that, its properties must be calculated, and the thermodynamic limit taken, and this is a more difficult task numerically. Conversely, a low overlap would not prove it is not in that class.

Our discussion still leaves the question of whether, for large enough t, there is a transition to the Pfaffian universality class, or to some other class. Because of the similarity in the ground-state structure of the 331 and Pfaffian states emphasized in Refs. 6,15 and 20, a second order transition seems to be a possibility. Another possibility is that the Halperin class of paired states is involved, those for which a trial state can be constructed by first pairing the particles into charge-2 bosons, then forming a $\nu_b=1/4q=\frac{1}{8}$ state of the bosons.[2,6] This is clearly another Abelian quantum Hall state, for which the edge excitations would consist only of the U(1) density-fluctuation chiral scalar boson modes, in different charge sectors; it would lack the gapless fermion excitations characteristic of the paired states discussed here.[8]

While we cannot rule out the Halperin type of paired state, and all of these states might describe the universality classes of various Hamiltonians, even within the LLL, we can point out an experimental signature that will distinguish the former from the others. First we recall that, for the Laughlin states, the Luttinger liquid at the edge leads to the local density of states $N(\omega)\sim|\omega|^{q-1}$ for the filling factor $1/q$ (see Ref. 31 for a review). The corresponding tunneling differential conductance for a point contact at zero temperature will be $dI/dV\sim V^{q-1}$. The exponents are simply related to the scaling dimension Δ of the operator that creates a single electron in the low-energy theory of the edge, as $dI/dV\sim V^{2\Delta-1}$, and also to the statistics θ of the excitation by $\theta/2\pi\equiv\Delta \pmod 1$. In the theory for the Laughlin states, the charge n/q operators at a single edge have dimensions $n^2/2q$ (n must be an integer); the nonintegral charge operators must be combined with similar operators at the other edge (see Refs. 31 or 8). Thus $\Delta=q/2$ for the electron. In the charge-2 boson system, the charged operators that create charges at the edge within the low-energy theory (subject to some straightforward conditions discussed in Ref. 8) have scaling dimension $n^2/8q$ when they create charge $\pm n/2q$ (in electron units) at the edge at filling factor $1/q$, again for n integral. As in the old argument of Tao and Wu,[32] in either theory the operator of charge 1 has $\theta=2\pi\Delta=\pi q$, which is Bose statistics for q even, and cannot represent an electron (similarly, for q odd, it cannot represent a charge-1 boson). In addition, such an operator cannot be used at a single edge in the charge-2 boson theory, but must be combined with another charged operator at the other edge, or with some other operator at the same edge. Only operators with an even-integer charge can act on one edge. Therefore, at low energies (that is, bias voltages), tunneling into the edge from outside will be impossible at these filling factors $\nu=1/q$ for this charge-2 boson universality class. To make an electron on one edge, the charged operator must be combined with an operator making an unpaired fermion (or BCS quasiparticle), which in the present case would exist as excitations, but would have a finite energy gap, even at the edge; thus the tunneling current will be zero below a threshold voltage. On the other hand, in paired states like the Pfaffian, HR, and 331 states, the fermion excitations are gapless at the edge, and a nonzero tunneling current with a power-law dependence at small bias voltage should be observable (the power law can be calculated from the theories in Ref. 8; Δ has an extra contribution from the fermions, to yield V^q, for both the Pfaffian and 331 cases, and V^{q+1} for HR). These arguments are for leading order in the tunneling. It is possible that some sort of higher-order tunneling process could transfer two electrons into the edge, through a virtual transition to one or more higher-energy states; as this does not require any fermions to be created except virtually, this would give a current at arbitrarily low bias voltage. However, the exponent would be related to the scaling dimension (and statistics) for the charge-2 operator in the low-energy edge field theory of the charge-2 boson state, and so the power law would be V^{4q-1}; thus for $q=2$ the current would be much lower than for Pfaffian, 331, and HR states, and still clearly distinguishable. For another discussion of the experimental consequences of the Halperin-type charge-2 boson Laughlin state, in connection with even-integer filling factors, see Ref. 33; our formulas for the charge-2 boson universality class also apply for the conductance in this case, with $q=\frac{1}{2}$, to give $dI/dV\sim V$.

VII. CONCLUSION

In summary, we obtained a full description of the quasihole states of several paired FQHE states, for the Hamiltonians for which the exact ground states are known. The degeneracies found in the Pfaffian and HR cases are as required for non-Abelian statistics. For the 331 states, the statistics are Abelian, and the degeneracies are due to the layer index. Ho's model was found to be pathological at its Pfaffian point, but the pathology was removed by adding a three-body term to Ho's Hamiltonian. With tunneling also added, the Pfaffian state was recovered, but the model was no longer exactly soluble for the low-energy states at intermediate points in the parameter space between the 331 and Pfaffian points. In Appendix B, the permanent state was also considered, which is another candidate for non-Abelian statistics, but should probably be rejected because of its proximity to a ferromagnetic ground state, and its correspondingly gapless nature. It remains to be seen whether the other non-Abelian states, though not apparently close to an obviously gapless state, are in fact stable against small generic perturbations. This is an important outstanding issue, to which we hope to return elsewhere. It probably requires an analytical, field-theoretic technique to settle it in general, which should be a theory that describes the paired condensate, and not just a Chern-Simons theory of the low-energy sector containing non-Abelian statistics. In the meantime, we pointed out in Sec. VI how the different paired states, the Halperin-type state of charge-2 bosons, and the Pfaffian and HR types with non-Abelian statistics, can be distinguished in a point-contact tunneling experiment. As for an actual demonstration that adiabatic transport of quasiparticles does produce non-Abelian statistics in some systems, that also will have to be left for treatment elsewhere.

ACKNOWLEDGMENTS

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APPENDIX A: ELLIPTIC FUNCTIONS

In general, ϑ functions with characteristics are defined as

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau)=\sum_n e^{i\pi\tau(n+a)^2+2\pi i(n+a)(z+b)}, \tag{A1}$$

where the n sum is over all the integers and a and b are real. From the definition we obtain

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+1|\tau)=e^{2\pi i a}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau),$$

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+\tau|\tau)=e^{-i\pi\tau-2\pi i(z+b)}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau). \tag{A2}$$

Consequently we can restrict a and b to lie between 0 and 1.

The standard Jacobi theta functions[29] are particular cases of those above. There are four of them:

$$\vartheta_1(z|\tau)=\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \vartheta_2=\vartheta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$\vartheta_3=\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \vartheta_4=\vartheta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.$$

Of these, ϑ_1 is of particular importance, since it is odd under $z\to -z$, and so has a zero at the origin. In general, all $\vartheta[{a \atop b}]$ can be related to each other by shifts of z.

The identity used to rewrite the $F_{\rm cm}$ found in Ref. 23 in the form given in Sec. V can be obtained, by shifts of z, from the simplest version

$$\prod_{r=1}^{M}\vartheta_3(z-r/M|\tau)=\vartheta_3(Mz+(M-1)/2|M\tau)\times\eta^M(\tau)/\eta(M\tau), \tag{A3}$$

which is obtained by writing the ϑ functions on the left-hand side in the product form[29]

$$\vartheta_3(z|\tau)=\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n})(1+2e^{2\pi i\tau(2n-1)}\cos 2\pi z+e^{4\pi i\tau(2n-1)}), \tag{A4}$$

and doing the r product first. Here $\eta(\tau)$ is the Dedekind η function,

$$\eta(\tau)=e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n}). \tag{A5}$$

The Weierstrass elliptic function can be defined by[29]

$$\wp(z|\tau)=\frac{1}{z^2}+\sum_{m,n}{}'\left\{\frac{1}{(z-m-n\tau)^2}-\frac{1}{(m+n\tau)^2}\right\}, \tag{A6}$$

where the prime indicates that $m=n=0$ is to be omitted from the sum. It can be shown[29] that $\wp$ is invariant under $z\to z+1$ and $z\to z+\tau$. To obtain functions that have a double pole at the origin, like the Weierstrass function, but are antiperiodic, we may use the definitions

$$\wp_{a,b}(z|\tau)=\sum_{m,n}(-1)^{ma+nb}\frac{1}{(z-m-n\tau)^2}, \tag{A7}$$

where a and b are integers. $\wp_{a,b}$ depends on a and b only modulo 2. For a and b both even, the sum is not convergent, which is why $\wp$ was not defined this way. For the other three cases, the series converges, and it is clear that the functions have the periodicity properties that we denoted $+-$, $-+$, and $--$ in the text.

APPENDIX B: PERMANENT STATE

In this appendix, we will introduce a Hamiltonian for which a certain state containing a permanent[5] (other such states can be found in Ref. 4) is the unique zero-energy eigenstate of maximum density, and summarize results for the quasihole and edge excitations, and for the ground states on the torus. We also describe a relation with fully spin-polarized states and their Skyrmion excitations, and argue that the permanent state is at a phase transition from ferromagnet to paramagnet.

The permanent state is a spin-singlet ground state of spin-$\frac{1}{2}$ fermions for q odd, and of spin-$\frac{1}{2}$ bosons for q even. It can be viewed as spin-singlet p-wave pairing of composite bosons.[5] The Hamiltonian for the simplest case $q=1$ is a three-body Hamiltonian, which penalizes the closest possible approach of three spin-$\frac{1}{2}$ fermions. On the sphere, three particles are at their closest, consistent with Fermi statistics, when the total orbital angular momentum for the three is $3N_\phi/2-1$, and the total spin is $\frac{1}{2}$. Our three-body Hamiltonian is therefore taken to be a positive number times the projection operator onto this multiplet of states for three particles, summed over all triples of particles:

$$H=\sum_{i<j<k} VP_{ijk}(3N_\phi/2-1,1/2). \tag{B1}$$

We have verified numerically that this does produce a unique many-particle state at zero energy at the N_ϕ value that corresponds to the permanent state at $\nu=1$. As the permanent state is a zero-energy state for this Hamiltonian by inspection (and is nonzero), it must be the state obtained numerically. As for the Pfaffian state in Sec. II, a suitable Hamiltonian, consisting of a combination of spin-independent two-body projection operators onto angular momenta N_ϕ, $N_\phi-1$, $\ldots$, $N_\phi-q+2$, and a certain three-body projection onto angular momentum $3N_\phi/2-3(q-1)-1$ and spin $\frac{1}{2}$, can be constructed for which the generalizations of these states to $q>1$ are again the complete set of zero-energy states.

In our much-abused notation, the permanent state is defined by the wave function[5]

$$\tilde{\Psi}_{\rm perm}(z_1^\uparrow,\ldots,z_{N/2}^\uparrow,z_1^\downarrow,\ldots,z_{N/2}^\downarrow) = \sum_{\sigma\in S_{N/2}} \frac{1}{(z_1^\uparrow-z_{\sigma(1)}^\downarrow)\cdots(z_{N/2}^\uparrow-z_{\sigma(N/2)}^\downarrow)}\prod_{i<j}(z_i-z_j)^q. \tag{B2}$$

The fact that it represents a singlet is most easily seen by viewing it as singlet pairs of composite bosons of spin $\frac{1}{2}$. It is totally antisymmetric for q odd, symmetric for q even. It resembles the 331 state, but with the sign of the permutation omitted. Such a summation over permutations defines the permanent of a matrix, $\mathrm{perm}M=\Sigma_\sigma\Pi_i M_{i,\sigma(i)}$, in which M is an $L\times L$ matrix with elements M_{ij}, and the sum is over all members σ of S_L.

States with $2n$ quasiholes can be written down in the now-familiar manner:

$$\frac{1}{(N_\uparrow-B_\uparrow)!}\sum_{\substack{\sigma\in S_{N_\uparrow}\\ \rho\in S_{N_\downarrow}}}\prod_{k=1}^{B_\uparrow}(z_{\sigma(k)}^\uparrow)^{n_k}\prod_{l=1}^{B_\downarrow}(z_{\rho(l)}^\downarrow)^{m_l}\prod_{r=1}^{N_\uparrow-B_\uparrow} \times\frac{\Phi(z_{\sigma(B_\uparrow+r)}^\uparrow,z_{\rho(B_\downarrow+r)}^\downarrow;w_1,\ldots,w_{2n})}{(z_{\sigma(B_\uparrow+r)}^\uparrow-z_{\rho(B_\downarrow+r)}^\downarrow)}\prod_{i<j}(z_i-z_j)^q, \tag{B3}$$

in which Φ is as in Secs. II–IV, and n_k and m_l must be $\leqslant n-1$. In these states the flux is $N_\phi=q(N-1)-1+n$. It is clear that these are all zero-energy states for the three-body interaction Eq. (B1) at $q=1$, and its generalization to $q>1$. The counting of states is most similar to the 331 state, but because the unpaired particles are here spin-$\frac{1}{2}$ bosons, the Pauli principle restriction on the number of unpaired particles does not apply, and there is no upper limit, except that the number of unpaired particles cannot exceed the total number of particles. For $n\geqslant1$, the number of states for fixed w's, and a fixed number of broken pairs, is that for $B=B_\uparrow+B_\downarrow$ unpaired bosons, in $2n$ orbitals (including spin $\uparrow$ or $\downarrow$), which yields

$$\binom{B+2n-1}{B}. \tag{B4}$$

Including the positional degeneracy of the quasiholes, and summing over B as for the other states, gives

$$\sum_{B,(-1)^B=(-1)^N}\binom{B+2n-1}{B}\binom{(N-B)/2+2n}{2n}. \tag{B5}$$

The sum over B satisfying $(-1)^B=(-1)^N$ diverges as $N\to\infty$. Note that for $n\geqslant1$, it is possible to break all the pairs and make all the spins polarized, still with zero energy. In other words, the quasihole states of the spin-polarized $\nu=1$ state form a subset of the zero-energy states for the three-body Hamiltonian.

As we saw in the main text, the edge states are closely related to the bulk quasihole states. For the permanent state, we will just state that there are $4q$ sectors of edge states involving unpaired bosons, much like the other examples in Ref. 8. In the twisted sector, there are zero modes which can be occupied with arbitrarily many bosons of either spin. Turning to the ground states on the torus, there are again $4q$ sectors for N even. The ground states for $\mathbf{k}\neq\mathbf{0}$ are an obvious generalization of those for the 331 state, containing a permanent instead of a determinant. These states are again spin singlets. For $\mathbf{k}=\mathbf{0}$, the construction that worked for 331 made essential use of antisymmetrization, and does not work here. Instead, the only possibility is to break all the pairs, and put the bosons in the constant single-particle state on the torus. The spin state is then totally symmetric, so we obtain a spin $N/2$ multiplet of states. This construction also works for N odd.

The large degeneracies of states in certain sectors in this system make sense in the interpretation in terms of correlators in CFT. The theory relevant to the permanent is a β-γ ghost system,[5] where β and γ are free bosonic fields of conformal weight $\frac{1}{2}$, so that this theory is nonunitary. It realizes Kac-Moody symmetry at level $k=-\frac{1}{2}$ (Ref. 5) (not to be confused with the vector $\mathbf{k}$ on the torus). The latter theory includes the "spin" fields for the β-γ system, which behave as infinite-dimensional fractional-spin representations of SU(1,1) symmetry, that is related to, but not quite the same as, the SU(2) symmetry in which we are interested. Consequently the infinite degeneracies when quasiholes are present, or in the ground state on the torus in the corresponding sector, are not surprising.

Another interesting question is the excitation spectrum when the zero-energy states of our Hamiltonian are fully (or even just macroscopically) spin polarized, as occurs in the presence of two quasiholes or on the torus. Since the ground state breaks the spin-rotation symmetry, and the Hamiltonian is short ranged, we expect low-energy spin-wave excitations to exist, in which one or more spins are flipped. For a generic Hamiltonian, one would expect these to occur at low but nonzero energy, with a gapless quadratic dispersion relation as in a quantum ferromagnet. Then the system would not be fully gapped for spin excitations, unlike (we believe) the other systems studied in this paper. It is interesting to note the relation with the $\nu=1$ spin-polarized system that has been much studied recently.[34] For a two-body Hamiltonian consisting of a projection operator onto zero relative angular momentum for two particles of total spin 0, the spin-polarized filled Landau-level state, and quasihole states including reversed spins (Skyrmions) are zero-energy eigenstates (note that the number of quasiholes for this system is identified with the number of flux added to the polarized ground state, which is smaller by 1 than the number of flux added to the permanent ground state). For each number of added flux, these are exactly the same as the states above with no unbroken pairs. There are also the expected spin-wave excitations at nonzero energy. However, in the case of the quasihole states of the Hamiltonian we have studied here, we already know that there are other states with some additional reversed spins at *zero* energy; they are simply the states where not all the pairs are broken. Clearly, these states would not be zero-energy states for the two-body Hamiltonian. It is tempting to identify them with a subset of the spin-wave states.

From the summand in Eq. (B5), we can obtain the number of zero-energy states for each number of unbroken pairs, $(N-B)/2$. It is instructive to begin with the case $n=1$ that corresponds to the sector containing the fully polarized filled Landau-level state. As we increase $(N-B)/2$ from zero, we expect that the total spin must decrease due to the formation of singlet pairs. The first binomial coefficient in Eq. (B5) is equal to $B+1$, which is the degeneracy of a single multiplet of spin $S=B/2$. The second binomial coefficient in Eq. (B5) is the orbital degeneracy of the quasiholes of the permanent, which we here interpret as the number of ways of placing $(N-B)/2$ bosons in $2n+1=3$ orbitals. The bosons are to be viewed as the spin waves. The three orbitals form an $L=1$ multiplet. The spin-wave excitations in general could have angular momenta $0,1,\ldots$; the $L=0$ ones simply represent a global rotation of the spin, and have already been counted in the degeneracy of each spin multiplet. We conclude that, for our Hamiltonian, $L=1$ spin waves have zero energy at $n=1$. The other spin-wave states would have to be obtained by a collective excitation of the condensate of singlet pairs, or spin waves (depending on our point of view), which excites one or more of them to higher angular momentum. A similar picture holds for $n>1$. Based on these arguments, we do expect a gapless branch of low-energy spin waves to be present in the spectrum of elementary excitations of the spin-polarized zero-energy states for our Hamiltonian, on the sphere, and also on the torus.

The three-body Hamiltonian (B1) above has been diagonalized numerically for $q=1$ and N up to 12, with $V=\frac{1}{18}$, for various numbers of quasiholes; results for $n=0$ and 1 are shown in Figs. 13 and 14. For $n=1$, the degeneracies of the zero-energy quasihole states have been confirmed, and in addition (see Fig. 14) there are low-lying states, so the system is not obviously gapped. In fact, by examining the states at $n=1$ with $S_z=N/2-1$, that is one less than the maximum value, which we expect to be the single-spin-wave states of the ferromagnet, we obtain a dispersion relation of the spin waves, which is shown as the lower-right inset in Fig. 14. It has the expected form of a finite-size version of a gapless branch of states, and has zero energy for both $L=0$ and 1;

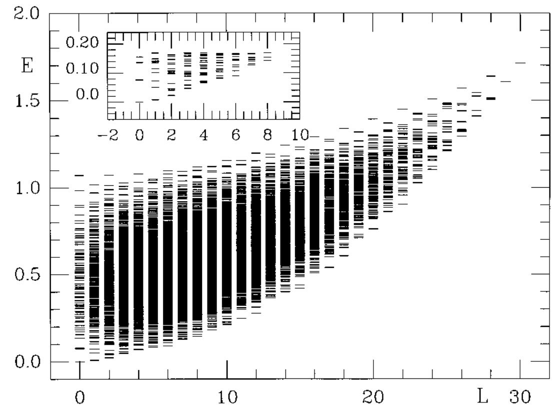

FIG. 13. Spectrum of the three-body interaction for the permanent state of fermions with $\nu=1$, for $N=12$ and $N_\phi=10$; that is, no quasiholes. The inset enlarges the low-lying levels.

the latter property may mean that the dispersion relation has vanishing coefficient of the wave vector squared in the thermodynamic limit (this coefficient is proportional to the spin stiffness in the ferromagnet). We note that these states penetrate quite far into the full spectrum. The apparent slight gap above the zero-energy states for $L\neq 5$ is in fact just a finite-size effect, since at least the $S_z=N/2-1$ states must form a gapless branch as $N\rightarrow\infty$. The low-lying states in the full spectrum should be other multiple-spin-wave states.

For the ground-state sector of the permanent, there is a nondegenerate zero-energy state, as in the other cases studied in this paper, but there is also an apparent gapless branch of states at low L (see Fig. 13). From the point of view of the ferromagnet, this N_ϕ value represents a quasielectron state, which would be an anti-Skyrmion. For the two-body Hamiltonian, the anti-Skyrmions form a set of states with $L=S$, but which do not have zero energy or exactly soluble wave functions. The lowest-energy states in Fig. 13 have $L=S$ for $L=0$, 1, 2, and 3, but $S=1$ for $L=4$, though here an $S=4$ state lies at slightly higher energy. Moreover, the $L=0$ ground state has overlap squared 0.81 with that for the Cou-

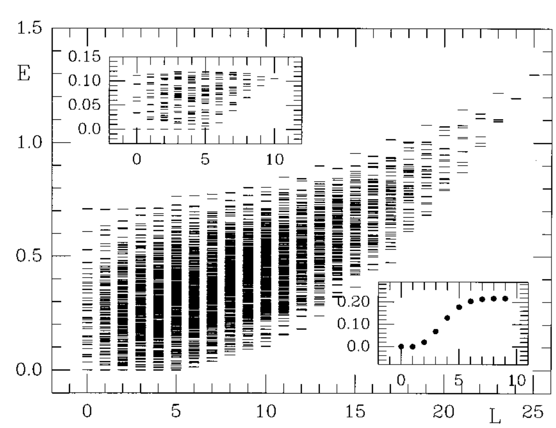

FIG. 14. As in Fig. 13, but with $N=10$ and $N_\phi=9$; that is, two quasiholes. Upper-left inset: low-lying levels. The lower-right inset shows only the states with $S_z=N/2-1$, i.e., single spin flips (or spin waves) of the fully polarized state.

lomb interaction at the same N and N_ϕ. There is thus some evidence that this branch of states represents something similar to an anti-Skyrmion.

The proximity to the ferromagnet, albeit at a different number of flux [$N_\phi = q(N-1)$ for the ferromagnet], suggests that the system is at the transition to the ferromagnet. Indeed, if the two-body Hamiltonian containing V_0 only (for $q=1$, or the usual generalizations for other q) is added to the three-body Hamiltonian, then for $V_0>0$ the ferromagnet and its quasihole excitations (Skyrmions) will be the only remaining zero-energy states, and the $L=1$ spin waves [at $N_\phi = q(N-1)$] will have nonzero energy. On the torus, the $4q$-fold degenerate sectors will be split to leave only the q states with $\mathbf{k}=\mathbf{0}$, which have spin $N/2$. For V_0 negative, we expect the splittings to reverse sign, and the ground states on the torus or on the sphere with quasiholes will presumably be unpolarized; in this region no exact wave functions are available for the lowest-energy eigenstates. In other words, the polarized and unpolarized states will differ in energy density. We expect that the unpolarized states are paired, and indeed the attractive pseudopotential $V_0<0$ should favor the pairs. In view of the higher-energy density of the polarized ground state on the torus, we guess that these states are no longer in the same universality class as the permanent state, but may be a simple Halperin-type state, a Laughlin state of charge-2 boson pairs. This probably would occur because the attractive potential decreases the size of the pairs, and modifies the pairing function from the simple form $(z_i^\uparrow - z_j^\downarrow)^{-1}$ found hitherto. As non-Abelian statistics probably relies upon the long-range character of this part of the wave function, it could disappear under this perturbation. In any case, the ferromagnetic order parameter is constant for $V_0>0$, and will vanish for $V_0<0$, which indicates a first-order phase transition.

The transition also has a simple interpretation in terms of composite bosons. In the permanent state, the bosons are paired, but when one flux quantum is added, there are broken pair states of the same energy. In these states, the unpaired composite bosons occupy the $L=0$ zero mode, and can be viewed as a Bose condensate. Since they carry spin $\frac{1}{2}$, such a condensate is necessarily a ferromagnet, and when more flux is added, the Skyrmion zero-energy states are obtained. Thus the permanent three-body Hamiltonian can have pairs, but the bosons can also unpair and form a condensate. In Bose liquid systems, the condensation of single bosons is the usual occurrence. It seems that the permanent state is on the borderline between a single-particle Bose condensate and a condensate of pairs only. It is possible that, while the ferromagnetic order indicates a first-order transition, the behavior of the pair order parameter (specifically, the size of the pairs), or of some properties on the ferromagnetic side, possibly related to a spin stiffness going to zero, could be characteristic of a second-order transition, with a length that diverges at the transition point.

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