Electron-phonon scattering contributions to metallic resistivity at 0 K

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Incorporating the quantum Boltzmann equation, with shielded electron-ion Coulomb interactions, the component of metallic electrical resistivity due to electron-phonon scattering is evaluated for the noble metals and a restricted class of the alkali metals. In addition to Bloch's T^5 contribution at low temperature and canonical T dependence at high temperature, a component of resistivity stemming from electron-phonon scattering is found to survive in the limit $T\rightarrow 0$. This residual resistivity is attributed to the interplay between Fermi-surface electrons and zero-point ion motion, in the presence of an electric field, as well as to the inelastic nature of electron-phonon scattering. An estimate made of the temperature at which this residual component of resistivity comes into play gives the criterion $T\ll\Theta_D/5$ for the class of metals considered, where Θ_D is the Debye temperature. It is further observed that this residual component of resistivity maintains nonsingular behavior of the Lorentz expansion for the electron distribution function at low temperature. Our expression for residual resistivity is given by (in the cgs system)

$$\rho_0=\left(\frac{3\pi^2}{8}\right)\left[\frac{k_B\Theta_D}{mu^2}\frac{\hbar(\hbar\Omega)^2}{E_F^3}\right]S_1(\lambda),$$

where $S_1(\lambda)$ is a positive monotonic function of λ. In the preceding expression, λ varies as $(n/Z^2)^{1/6}$, Ω is the ion plasma frequency, and n is the electron number density. The phonon speed and Fermi energy are written u and E_F, respectively. It is noted that ρ_0 scales as $(Z^{1/6}/nM^{1/2})S_1(\lambda)$, where M and Z are the ion mass and ion valence number respectively. At constant electron and ion number densities, ρ_0 scales as $M^{-1/2}$. At these conditions, in the limit that $M\rightarrow\infty$, $\Omega\rightarrow 0$ and, consistently, $\rho_0\rightarrow 0$. A log-log plot of the expression derived for resistivity, at various values of λ, clearly exhibits the three temperature intervals described above. [S0163-1829(96)03948-3]

I. INTRODUCTION

It is generally recognized that low-temperature metallic resistivity $\rho(T)$ has four basic contributions:[1–6]

$$\rho(T)=\rho_i+AT^5+BT^p\exp(-\hbar\omega_0/k_BT)+CT^2. \qquad (1)$$

The residual, temperature-independent component ρ_i is attributed to electron scattering from impurities and crystal imperfections. The widely cited T^5 contribution stems from electron-phonon scattering and was first evaluated by Bloch.[7] Electron-phonon umklapp scattering contributes the exponential term, where ω_0 is the frequency of the phonon with minimum wave vector that allows electrons to scatter through an unklapp process and p is an empirical parameter. The T^2 contribution corresponds to electron-electron scattering.

In the present model, we return to the electron-phonon scattering contribution to resistivity. Our starting equation is the quantum Boltzmann equation,[8] which includes inelastic electron-phonon screened Coulomb interactions with ion zero-point motion included in electron-phonon scattering matrix elements. It is noted that apart from the Bose-Einstein phonon distribution, no other contribution of ion dynamics is included in the Bloch analysis[7(b)] or variational techniques applied to this problem.[9–13]

The solution for the electron distribution is obtained from a Lorentz expansion[8] about the Fermi-Dirac distribution. In addition to the canonical T^5 contribution at $T\ll\Theta_D$ and the T dependence at $T\gg\Theta_D$, the formalism obtains a residual component of resistivity independent of temperature. This residual resistivity is attributed to the interplay of Fermi-surface electrons with zero-point ion motion, in the presence of an electric field, as well as to the inelastic nature of electron-phonon scattering. Consistent with the present model, it is found that phonon emission in scattering events persists at $T=0$ K, whereas phonon absorption vanishes at this temperature. An additional consistency property is that this residual resistivity vanishes in the limit of large ion mass. We note further that in Bloch's analysis[7(b)] a principal assumption is that the perturbed electron distribution differs from the energy derivative of the unperturbed (Fermi-Dirac) distribution by a constant factor. This assumption is contradicted in the present work.

An expression for impurity resistivity is obtained and compared to the presently derived expression for resistivity due to inelastic scattering. With this comparison at hand, a means of experimentally confirming the present results is suggested.

Stemming from the expression derived for residual resistivity, an estimate is made of the temperature at which this

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component is of the same order as the Bloch contribution. For the sample of metals considered it is found that this transition occurs at a temperature appreciably below the Debye temperature. Furthermore, this residual resistivity scales as $Z^{1/6}/nM^{1/2}$, which is noted to be strongly dependent on ion parameters.

II. ASSUMPTIONS

It is assumed that with no electric field turned on, conduction electrons in the metal are in the Fermi-Dirac distribution

$$f_0(k)=\frac{1}{1+\exp[(E-E_F)/k_BT]}, \tag{2}$$

with normalization

$$\int f_0(k)\,\frac{d\mathbf{k}}{(2\pi)^3}=\frac{N}{V}=n, \tag{3}$$

where E_F is the Fermi energy and N represents the total number of free electrons in the sample, which is of volume V.

A spherical conduction energy surface is assumed. Among the alkali metals (Li, Na, K, Rb, and Cs) it is reasonably certain that potassium and rubidium have spherical Fermi surfaces.[14] The noble metals (Cu, Ag, and Au) have a nearly spherical Fermi surface. For such metals with a spherical and near spherical energy surface we may write

$$d\mathbf{k}=4\pi k^2dk=\frac{2\pi\sqrt{E}dE}{(\hbar^2/2m)^{3/2}} \tag{4}$$

and (3) may be rewritten

$$\int f_0(E)\sqrt{E}\,dE=n(2\pi)^2(\hbar^2/2m)^{3/2}. \tag{5}$$

With ω written for the phonon frequency, the following relations are assumed:

$$\hbar\omega<\hbar\omega_D\ll E_F\simeq E, \tag{6}$$

where ω_D is the Debye frequency. Furthermore, as electron wave vectors lie predominantly on the Fermi surface, we also conclude that $q\ll k_F$ and the electron scattering is predominantly small angle or, equivalently, $\mathbf{k}\cdot\mathbf{q}\ll kq$, where $\mathbf{q}$ denotes phonon wave vector.

III. STARTING EQUATIONS

Our starting equation is the quantum Boltzmann equation[11,12,15–17]

$$\frac{\partial f}{\partial t}+\frac{e\mathscr{E}}{\hbar}\cdot\frac{\partial f}{\partial\mathbf{k}}=\hat{J}(f), \tag{7a}$$

$$\hat{J}(f)=\sum_\alpha\int\frac{d\mathbf{k}'}{(2\pi)^3}\,[f'(1-f)S^{(\alpha)}_{\mathbf{k}'\mathbf{k}}-f(1-f')S^{(\alpha)}_{\mathbf{k}\mathbf{k}'}], \tag{7b}$$

where the sum over $\alpha=\pm1$ corresponds to emission and absorption of a phonon, $\mathscr{E}$ denotes the electric field, and

$$f'\equiv f(\mathbf{k}',t), \tag{7c}$$

represents the electron distribution function, where $\mathbf{k}'$ corresponds to ‘‘after’’ the collision. Electron-phonon scattering rates are written $S^{(\alpha)}_{\mathbf{k}'\mathbf{k}}$ (with dimensions and inverse time) and are given by

$$S^{(\alpha)}_{\mathbf{k}\mathbf{k}'}=|\langle\mathbf{k}',\mathbf{n}'|H_{\text{int}}|\mathbf{k},\mathbf{n}\rangle|^2\,\frac{2\pi}{\hbar}\,\delta(\Delta E), \tag{8a}$$

$$\delta(\Delta E)=\delta(E'-E-\alpha\hbar\omega). \tag{8b}$$

The equality

$$S^{(\alpha)}_{\mathbf{k}\mathbf{k}'}=S^{(-\alpha)}_{\mathbf{k}'\mathbf{k}} \tag{8c}$$

corresponds to the symmetry of the electron-phonon interaction under time reversal.[18] Momentum conservation in a collision is given by

$$\mathbf{k}'=\mathbf{k}+\alpha\mathbf{q}, \tag{9}$$

where $\hbar\mathbf{q}$ is the phonon momentum.

In the relation (8a), $|\mathbf{n}\rangle$ denotes the many-phonon state

$$|\mathbf{n}\rangle=|n_q,n_{q'},\ldots\rangle, \tag{10}$$

where

$$n_q=\frac{1}{e^{\hbar\omega/k_BT}-1} \tag{11}$$

is the Bose-Einstein distribution. For the dispersion relation for phonons we write

$$\omega=uq. \tag{12}$$

An estimate of the phonon speed, u is given by the Bohm-Staver relation[19]

$$u^2=\frac{2}{3}\,\frac{ZE_F}{M}, \tag{12$'$}$$

where M is the ion mass and Z is the atomic valence. Fast relaxation of phonons to the distribution (11) is assumed in the analysis (the so-called Bloch condition).[20]

For metals, the matrix elements (8a) have the value[21]

$$S^{(\alpha)}_{\mathbf{k}\mathbf{k}'}=|C_q|^2\left(n_q+\frac{1}{2}-\alpha\,\frac{1}{2}\right)\frac{2\pi}{\hbar}\,\delta(\Delta E), \tag{13}$$

where (see the Appendix)

$$|C_q|^2=RG(q), \tag{14a}$$

$$G(q)\equiv\frac{q/V}{(q^2+q_{\text{TF}}^2)^2}, \tag{14b}$$

$$R\equiv\frac{\hbar(M\Omega^2)^2}{2\rho_MuZ^2}. \tag{14c}$$

Note that R has dimensions of (energy)^2, G is dimensionless, and C_q has the units of energy. In the preceding, ρ_M is the crystal mass density, Ω is the ion plasma frequency,

$$\Omega^2 = \frac{4\pi n_M (Ze)^2}{M}, \tag{15a}$$

$$q_{\mathrm{TF}}^2 = \left(\frac{4}{a_0}\right)\left(\frac{3n}{\pi}\right)^{1/3} \tag{15b}$$

is the Thomas-Fermi wave number, and

$$n_M = \frac{n}{Z}, \quad \rho_M = M n_M. \tag{15c}$$

The Bohr radius is written a_0 and n_M represents the ion number density. The quantity $\hbar\Omega/2$ may be identified with ion zero-point energy.

With (11) we see that phonon occupation numbers $n_q = 0$ at $T = 0$ K. Nevertheless, from (13) we note that the phonon emission ($\alpha = -1$) matrix element persists at this temperature. Thus inelastic electron-phonon scattering maintains at $T = 0$ K. The fact that phonon absorption matrix elements vanish in this limit is consistent with the 0-K limit. The low-temperature analysis is returned to after general relations are obtained.

IV. COLLISION INTEGRALS

Substituting (13) into the collision integral (7b) we obtain

$$\hat{J}(f) = \sum_\alpha \hat{I}_\alpha[f'(1-f)(n_q + \tfrac{1}{2} + \alpha\tfrac{1}{2}) - f(1-f') \times (n_q + \tfrac{1}{2} - \alpha\tfrac{1}{2})], \tag{16}$$

$$\hat{I}_\alpha[\varphi_\alpha(q)] \equiv V \int \frac{d\mathbf{q}}{(2\pi)^3} |C_q|^2 \delta(\Delta E) \varphi_\alpha(q). \tag{17}$$

In this expression, with (9), we have set $d\mathbf{k}' = d\mathbf{q}$. Furthermore, with (9) we write

$$f'(\mathbf{k}) \equiv f(\mathbf{k}') = f(\mathbf{k} + \alpha\mathbf{q}) \tag{17$'$}$$

so that $f'(\mathbf{k})$ is α dependent. In the following sequence of reduction of integrals, it is noted that the interaction component $G(q)$ maintains its form throughout the evaluation.

Lorentz expansion

To account for anisotropy of the distribution function due to the imposed $\mathscr{E}$ field, we employ the Lorentz expansion[8]

$$f(\mathbf{k}) = f_0(k) + \mu f_1(k) + \cdots, \tag{18a}$$

$$\mu = \hat{\mathbf{k}} \cdot \mathscr{E} = \cos\theta, \tag{18b}$$

$$f(\mathbf{k}') = f_0(\mathbf{k}') + \mu' f_1(\mathbf{k}') + \cdots, \tag{18c}$$

$$\mu' = \hat{\mathbf{k}}' \cdot \mathscr{E}, \tag{18d}$$

where variables with a caret are unit vectors.

Keeping terms to $O(\mu)$ in (18) and substituting the resulting form into the collision integral (16), we obtain

$$\hat{J}[f(\mathbf{k})] = \hat{J}_0(f_0) + \hat{J}_1(f_0, f_1), \tag{19}$$

where

$$\hat{J}_0(f_0) = \sum_\alpha \hat{I}_\alpha[(f_0' - f_0)(n_q + \tfrac{1}{2}) + \alpha\tfrac{1}{2}(f_0' + f_0) - \alpha f_0' f_0], \tag{20}$$

$$\hat{J}_1(f_0, f_1) = \mu \sum_\alpha I_\alpha\left[\left(\frac{\mu'}{\mu} f_1' - f_1\right)(n_q + \tfrac{1}{2}) + \alpha\tfrac{1}{2}\left(\frac{\mu'}{\mu} f_1' + f_1\right) - \alpha\left(\frac{\mu'}{\mu} f_1' f_0 + f_0' f_1\right)\right]. \tag{21}$$

Note that

$$f'(E) \equiv f(E') = f\left(E + \alpha\frac{\hbar\omega}{k_B T}\right). \tag{21$'$}$$

Substituting these expressions into (7) and passing to the steady-state limit, with the orthogonality of Legendre polynomials, we obtain the two equations

$$\frac{2e\mathscr{E}}{3\hbar k}\frac{\partial}{\partial E}(E f_1) = \hat{J}_0(f_0), \tag{22a}$$

$$\mu\frac{2e\mathscr{E}}{\hbar k} E \frac{\partial}{\partial E} f_0 = J_1(f_0, f_1). \tag{22b}$$

V. REDUCTION OF INTEGRALS

It is assumed that $f_0(E)$ is the Fermi-Dirac distribution. In that $\hat{J}_0(f_0)$ vanishes for this choice of f_0, (22a) corroborates the fact that the Fermi-Dirac distribution is relevant to the zero-field situation $\mathscr{E} = 0$. The relation (22b) suffices to determine the correction f_1.

To reduce the integral (21) occurring on the right-hand side of (22b), we first note that

$$\int d\mathbf{q} = \int_0^{q_D} dq\, q^2 \int_0^{2\pi} d\beta \int_{-1}^{1} d\cos\gamma, \tag{23a}$$

where

$$\hat{\mathbf{q}} \cdot \hat{\mathbf{k}} = \cos\gamma. \tag{23b}$$

Furthermore, we note that

$$\delta(E' - E - \alpha\hbar\omega) = \frac{k}{2qE}\,\delta(\cos\gamma - [-\alpha(q/2k) + \varepsilon(k/2q)]), \tag{24}$$

where, with (6), we have set

$$\varepsilon \equiv \frac{\hbar\omega}{E}. \tag{24$'$}$$

The δ function in $\hat{I}_\alpha$ restricts the domain of integration. With (24) we write

$$\cos\gamma = -\alpha\frac{q}{2k} + \varepsilon\frac{k}{2q}. \tag{25}$$

With

$$-1 < \cos\gamma < +1$$

and small ε, we obtain the upper (U) and lower (L) limits on the q integration:

$$q_U = 2k + \alpha \frac{\varepsilon k}{2} + O(\varepsilon^2), \qquad q_L = 0 + O(\varepsilon^2). \tag{26}$$

Thus (17) reduces to

$$\hat{I}_\alpha(\varphi_\alpha) = \frac{Vk}{4\pi\hbar E} \int_0^{q_D} dq\; q |C_q|^2 \varphi_\alpha, \tag{27}$$

where q_D (corresponding to q_U) represents the Debye wave number

$$q_D = \frac{\omega_D}{u} = [6\pi^2 n_M]^{1/3} \tag{27$'$}$$

and n_M is the ion number density.

Inserting the expression (27′) into (21) and summing over α, we obtain

$$\begin{aligned}\hat{J}_1(f_0, f_1) = \hat{I} n_q \{ & \langle \mu' f_1^{(+)} [e^y(1-f_0) + f_0] \\ & - \mu f_1 [e^y f_0^{(+)} + (1 - f_0^{(+)})] \rangle_+ \\ & + \langle \mu' f_1^{(-)} [e^y f_0 + (1-f_0)] \\ & - \mu f_1 [e^y (1 - f_0^{(-)}) + f_0^{(-)}] \rangle_- \},\end{aligned} \tag{28}$$

where $\langle\,\rangle_\pm$ correspond to $\alpha = \pm 1$ and

$$y \equiv \frac{\hbar\omega}{k_B T} = \frac{\hbar u q}{k_B T} = \frac{q}{Q}, \tag{28a}$$

$$Q \equiv \frac{k_B T}{\hbar u}, \tag{28b}$$

and, with (21a),

$$f^{(\pm)} = f(E \pm \hbar\omega). \tag{28c}$$

Note further that Q has the dimension of wave number.

In (28) we introduced

$$\hat{I} \equiv \frac{Vk}{4\pi\hbar E} \int_0^{q_D} dq\; q |C_q|^2. \tag{29}$$

With (14), (29) is written

$$\hat{I} = \frac{kRV}{4\pi\hbar E} \int_0^{q_D} dq\; q\; G(q). \tag{30}$$

Since most of the electron scattering occurs on the Fermi surface, we may write $k' \simeq k$. With (9) we then obtain

$$\mu' \approx \mu\left(1 - \frac{q^2}{2k^2}\right) = \mu\left[1 - \frac{Q^2 y^2}{2k^2}\right]. \tag{31}$$

Substituting this expression into (28) permits the starting equation (22b) to be written

$$e\mathscr{E} \frac{\partial f_0}{\partial E} = \frac{VRmQ^2}{4\pi E\hbar^2} \int_0^{y_D} \frac{dy\; y\; G(y)}{e^y - 1} \left(L_1 - \frac{Q^2 y^2}{2k^2} L_2\right), \tag{32}$$

where

$$\begin{aligned}L_1 \equiv{} & f_1^{(+)}[e^y(1-f_0)+f_0] - f_1[e^y(1-f_0^{(-)}) + f_0^{(-)}] + f_1^{(-)} \\ & \times [e^y f_0 + (1-f_0)] - f_1[e^y f_0^{(+)} + (1 - f_0^{(+)})],\end{aligned} \tag{33a}$$

$$L_2 \equiv f_1^{(+)}[e^y(1-f_0)+f_0] + f_1^{(-)}[e^y f_0 + (1-f_0)]. \tag{33b}$$

VI. PERTURBATION DISTRIBUTION

The relations (32) and (33) comprise a self-contained integro-difference equation for the perturbation distribution $f_1(E)$. When written in terms of nondimensional variables (x,y), the definition (28c) is given by

$$f^{(\pm)}(x) = f(x \pm y), \qquad x = E/k_B T, \quad y = \hbar\omega/k_B T, \tag{33c}$$

whereas with (6) we note

$$x \gg y. \tag{33d}$$

Thus, in this same limit, (33c) becomes

$$f^{(\pm)}(x) \simeq f(x) \tag{34}$$

(both for f_0 and f_1). [Note that whereas both x and y grow large at low temperature, the inequality (33d) maintains.]

Substituting (34) into (33) reduces L_1 to zero, whereas in this same limit

$$L_2 = f_1(e^y + 1). \tag{35}$$

Defining

$$F(E) \equiv -\frac{f_1(E)/E}{\partial f_0/\partial E} \tag{36}$$

and

$$B \equiv \frac{Rm}{4\pi\hbar^2}, \tag{37}$$

(32) becomes

$$\frac{e\mathscr{E}}{B} = \frac{QW}{2k^2} F(E) \tag{38}$$

where W is the dimensionless integral

$$W(T) \equiv Q^3 V \int_0^{y_D} dy\; y^3 G(y) \left(\frac{e^y+1}{e^y-1}\right) \tag{39}$$

or, equivalently,

$$W(T) = \int_0^{y_D} \frac{dy\; y^4}{(y^2 + y_{\rm TF}^2)^2} \left(\frac{e^y+1}{e^y-1}\right), \tag{40a}$$

$$y_{\rm TF} \equiv \frac{\Theta_{\rm TF}}{T}, \quad y_D \equiv \frac{\Theta_D}{T}, \tag{40b}$$

$$k_B\Theta_{\rm TF}=\hbar u q_{\rm TF}, \quad k_B\Theta_D=\hbar u q_D. \tag{40c}$$

It follows that

$$f_1(E)=\frac{-16\pi e\varepsilon E^2\partial f_0/\partial E}{RK(T)}, \tag{41a}$$

where the temperature-dependent term

$$K(T)\equiv QW(T)=\overline{Q}TW(T), \tag{41b}$$

$$\overline{Q}\equiv k_B/\hbar u. \tag{41c}$$

Note that $K(T)$ has dimensions of wave number. The relation (41a) contradicts Bloch's principal assumption $f_1(E)=\alpha\partial f_0(E)/\partial E$, where α is a constant.[20] With (2) and (18), the expression (41a) gives the corrected electron distribution to the given order in μ.

VII. ELECTRICAL RESISTIVITY

The current density is given by

$$\mathbf{j}=\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{e\hbar\mathbf{k}}{m} f(k). \tag{42}$$

Substituting (18) into (42) we obtain

$$\begin{aligned}\mathbf{j}&=\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{e\hbar\mathbf{k}}{m}\hat{\mathbf{k}}\cdot\mathscr{E}f_1(k)\\ &=\frac{e\hbar}{m(2\pi)^3}\mathscr{E}\cdot\int d\mathbf{k}\,\hat{\mathbf{k}}\hat{\mathbf{k}}\,f_1(k)k\\ &=\frac{1}{3}\frac{e\hbar}{m(2\pi)^3}\mathscr{E}\cdot\bar{\bar{\mathbf{I}}}\int d\mathbf{k}\,f_1(k)k,\end{aligned} \tag{43}$$

where $\bar{\bar{\mathbf{I}}}$ is the unit matrix. A double-barred variable represents a dyad. The following results:

$$\mathbf{j}=\frac{1}{3}\frac{e\hbar}{m(2\pi)^3}\mathscr{E}\int d\mathbf{k}\,f_1(k)k. \tag{44}$$

With (4) we write

$$k\,d\mathbf{k}=4\pi k^3dk=2\pi\left(\frac{2m}{\hbar^2}\right)^2E\,dE.$$

It follows that

$$\mathbf{j}=\frac{em}{3\pi^2\hbar^3}\mathscr{E}\int dE\,Ef_1(E). \tag{45}$$

In estimating f_1 it is further assumed that

$$\frac{\partial f_0}{\partial E}=-\delta(E-E_F). \tag{45$'$}$$

Since $k_BT\ll E_F$, f_0 is sharply peaked in the temperature range of interest ($0\text{ K}\leq T\leq 300\text{ K}$) and (45′) remains a good approximation. Substituting (45′) into (41) gives the desired solution for the perturbations distribution $f_1(E)$. When substituted into (45) this solution gives the conductivity- (σ) resistivity (ρ) expression

$$\sigma=\frac{1}{\rho}=\frac{16}{3\pi}\frac{e^2mE_F^3}{\hbar^3RK(T)}=\left(\frac{e^2mE_F^3}{3\pi\hbar^3R}\right)\left(\frac{16\hbar u}{k_BTW(T)}\right), \tag{46}$$

whose temperature dependence is contained entirely in $TW(T)$ [see (41b)]. As will be shown below, the expression (46) for ρ gives both Bloch's T^5 dependence at $(T/\Theta_D)\ll 1$ as well as the canonical T dependence at $(T/\Theta_D)\gg 1$ in addition to a residual resistivity at $T=0$ K.

A. Properties of $W(T)$ and $S_1(\lambda)$

The function $W(T)$ is singular at $T=0$ K. To expose this singularity first we note the relation

$$\frac{e^y+1}{e^y-1}=1+\frac{2}{e^y-1},$$

so that [recall (40)]

$$\begin{aligned}W(T)&=\int_0^{y_D}\frac{dy\;y^4}{(y^2+y_{\rm TF}^2)^2}\left(1+\frac{2}{e^y-1}\right)\\ &\equiv W_1(T)+W_2(T).\end{aligned} \tag{47}$$

The $W_2(T)$ contribution corresponds to the exponential term and is finite at $T=0$ K. The singularity of $W(T)$ lies in $W_1(T)$. To obtain the T dependence of this singularity we introduce the variable

$$z\equiv Ty.$$

The following results (relabeling $z_D\equiv\Theta_D$, etc.):

$$W_1=\frac{1}{T}\int_0^{\Theta_D}\frac{z^4dz}{[z^2+\Theta_{\rm TF}^2]^2}\equiv\frac{\Theta_DS_1}{T}, \tag{48}$$

where S_1 is the implied nondimensional temperature-independent integral. Relation (48) indicates that $W(T)$ has a simple pole at $T=0$ K. Evaluating the integral S_1 gives

$$S_1(\lambda)=1+\frac{1}{2(1+\lambda^2)}-\frac{3}{2\lambda}\tan^{-1}\lambda, \tag{48a}$$

$$\lambda\equiv\frac{\Theta_D}{\Theta_{\rm TF}}=\frac{\omega_D}{\omega_{\rm TF}}=\frac{q_D}{q_{\rm TF}}=\frac{y_D}{y_{\rm TF}}. \tag{48b}$$

The parameter $q_{\rm TF}$ is given by (15b), q_D by (27a), and we have set $\tan^{-1}(0)=0$.

The function $S_1(\lambda)$ is a positive monotonic function with properties

$$\begin{aligned}&S_1(0)=S_1'(0)=0,\\ &S_1(\lambda)\sim 1,\quad S_1'(\lambda)\sim 0,\quad \lambda\gg 1.\end{aligned} \tag{48c}$$

For $\lambda\ll 1$, one obtains

$$S_1(\lambda)=\frac{\lambda^4}{5}+O(\lambda^6). \tag{48d}$$

Values of $S_1(\lambda)$ pertinent to the problem at hand are obtained as follows. First we note that

$$\lambda^2=\left(\frac{3\pi^5}{16}\right)^{1/3}a_0\left(\frac{n}{Z^2}\right)^{1/3} \tag{49a}$$

or, equivalently (with $Z=1$),

$$\lambda=1.43\times10^{-4}n^{1/6}, \tag{49b}$$

where n is electron density in cm^{-3}.

Among the alkali and noble metals, n is maximum for Cu, for which we obtain $\lambda_{\mathrm{Cu}}=0.96$. In this group n is minimum for Cs, for which we obtain $\lambda_{\mathrm{Cs}}=0.66$. We may conclude that the expansion (48d) is appropriate to the metals addressed in this analysis. A more accurate description of $S_1(\lambda)$ is obtained by fitting the curve of this function to a parabola in the λ domain of interest. The following results:

$$S_1(\lambda)=-0.042\lambda+0.11\lambda^2, \quad 0.60\leqslant\lambda\leqslant1.00. \tag{49c}$$

Thus

$$\begin{aligned} S_1(\lambda_{\mathrm{Cs}})&\simeq0.020,\\ S_1(\lambda_{\mathrm{Cu}})&\simeq0.061. \end{aligned} \tag{49d}$$

Combining (49b) and (49c) we obtain

$$10^4S_1(\lambda)=-0.06n^{1/6}+0.16\times10^{-4}n^{1/3}, \tag{49e}$$

whose values are seen to agree with (49d).

We note that W_1 as given by (47), with (41b) gives

$$K_1(T)=\overline{Q\Theta}_DS_1=\frac{k_B\Theta_D}{\hbar u}S_1(\lambda). \tag{49f}$$

which is independent of temperature. The temperature dependence of the distribution f_1 resides entirely in W_1.

B. The W_2 integral

To examine the finite integral W_2, we revert to y dependence and write

$$W_2=\int_0^{\Theta_D/T}\frac{2\,dy\;y^4}{[y^2+(\Theta_{\mathrm{TF}}/T)^2]^2}\frac{1}{(e^y-1)}. \tag{49g}$$

With these results at hand, we consider first the high-temperature limit.

1. Case (a): $T\gg\Theta_D$

In this limit, expanding the integral (49g) about $y_D=0$, we obtain

$$W_2=-\frac{\lambda^2}{1+\lambda^2}+\ln(1+\lambda^2)+O(y_D). \tag{49h}$$

In the limit of $\lambda\ll1$,

$$W_2\rightarrow\tfrac{1}{2}\lambda^4+O(\lambda^6). \tag{49i}$$

With (41b) and (49i), in the said limit, (46) gives the result

$$\rho=\frac{3\pi}{16}\left(\frac{\hbar^3R}{e^2mE_F^3}\right)\left(\frac{k_BT}{\hbar u}\right)\left(\frac{\lambda^4}{2}+\frac{\Theta_DS_1}{T}\right)\rightarrow\frac{\pi\lambda^4}{32}\left(\frac{\hbar^3R}{e^2mE_F^3}\right)\times\left(\frac{k_BT}{\hbar u}\right), \tag{50}$$

which is noted to have the canonical form[11] $\rho\propto T$.

2. Case (b) $T\ll\Theta_D$

In this limit we obtain

$$W_2=\left(\frac{T}{\Theta_{\mathrm{TF}}}\right)^4\int_0^\infty\frac{2\,dy\;y^4}{e^y-1}\equiv\left(\frac{T}{\Theta_{\mathrm{TF}}}\right)^4S_2 \tag{51}$$

where S_2 is the implied nondimensional, temperature-independent integral with the value

$$\frac{1}{2}S_2=\Gamma(5)\zeta(5)=24.886 \tag{52}$$

and Γ and ζ are gamma and zeta functions,[22] respectively.

C. General resistivity expressions

Returning to (41a, 46) we write

$$\rho=\frac{K(T)}{A}=\frac{\overline{QT}W}{A}. \tag{53}$$

Note the relations

$$\begin{aligned} \overline{Q\Theta}_D&=\frac{\omega_D}{u}=q_D,\\ \overline{Q\Theta}_{\mathrm{TF}}&=\frac{\omega_{\mathrm{TF}}}{u}=q_{\mathrm{TF}}, \end{aligned} \tag{54}$$

$$A\equiv\frac{16e^2mE_F^3}{3\pi\hbar^3R}. \tag{55}$$

The parameter A has dimensions of wave number, so that K/A has the correct resistivity dimensions (in the cgs system): time. The parameter R is defined in (14c). Collecting results we write

$$K(T)=Q\left[\frac{\Theta_DS_1}{T}+\left(\frac{T}{\Theta_{\mathrm{TF}}}\right)^4S_2\right], \tag{56a}$$

$$K(T)\equiv K_0+K_B(T), \tag{56b}$$

where K_0 is independent of T and $K_B(T)$ leads to the Bloch result. Inserting this finding into (53) gives

$$\rho=\frac{\overline{Q}}{A}\left[\Theta_DS_1+\Theta_{\mathrm{TF}}\left(\frac{T}{\Theta_{\mathrm{TF}}}\right)^5S_2\right] \tag{57a}$$

$$\equiv\rho_0+\rho_B(T), \tag{57b}$$

where

$$\rho_0=\frac{q_DS_1(\lambda)}{A} \tag{57c}$$

is the component of resistivity due to electron-phonon scattering that survives at 0 K and ρ_B is the Bloch contribution. We note that ρ_B may be written in the more canonical form[23]

$$\rho_B=\frac{\omega_{\mathrm{TF}}}{uA}\left(\frac{\Theta_D}{\Theta_{\mathrm{TF}}}\right)^5\left(\frac{T}{\Theta_D}\right)^5, \tag{57d}$$

where, with (15b) and (27′), one notes that

$$(\vartheta_D/\vartheta_{TF})^6 = \left(\frac{q_D}{q_{TF}}\right)^6 = (3\pi^5/16)(a_0^3 n_M^2/n). \quad (57e)$$

The relations (57) indicate that ρ_0 dominates over ρ_B for temperatures

$$\left(\frac{T}{\Theta_{TF}}\right)^5 \ll \frac{\lambda S_1(\lambda)}{S_2} \simeq \frac{\lambda S_1(\lambda)}{250} \quad (58a)$$

or, equivalently,

$$\frac{T}{\Theta_D} \ll \left[\frac{S_1(\lambda)}{250\lambda^4}\right]^{1/5} \equiv \tau(\lambda). \quad (58b)$$

For Cs we find $\tau = 0.25$. For Cu we find $\tau = 0.20$. Thus one expects ρ_0 to come into play at

$$y_D \gg 5 \quad (58c)$$

for the class of metals considered.

VIII. PHYSICAL PROPERTIES OF ρ_0

A. Scale parameters

We wish to obtain the manner in which ρ_0 scales with basic metallic parameters. To this end we write

$$\rho_0 = \overline{\rho_0} S_1(\lambda). \quad (59)$$

First consider the $\overline{\rho_0}$ factor. With (57a) we write

$$\overline{\rho_0} = \frac{3\pi^2}{8} \frac{k_B \Theta_D}{mu^2} \frac{\hbar(\hbar\Omega)^2}{E_F^3}. \quad (60a)$$

To find the manner in which $\overline{\rho_0}$ scales with metallic parameters, in (60) we set all parameters that are constant with respect to change of metallic samples (e.g., e, m, and k_B) equal to one. The following results:

$$\overline{\rho_0} \propto \frac{n_M \Theta_D Z}{E_F^4}. \quad (60b)$$

To further reduce this relation we recall (15c) and note that

$$E_F \propto n^{2/3},$$

$$\Theta_D \propto n^{2/3} Z^{1/6}/M^{1/2}.$$

It follows that

$$\overline{\rho_0} \propto Z^{1/6}/nM^{1/2}. \quad (60c)$$

B. Large-mass consistency limit

As the ion mass grows large, $\Omega \to 0$ and electrons do not interact with the lattice. It follows that in this limit one should find that $\rho_0 \to 0$. To explore this situation we examine the limit $M \to \infty$ at otherwise fixed ion parameters n_M and Z. With these constraints we note that

$$\overline{\rho_0} \propto \frac{\Theta_D}{u^2} \Omega^2 = \frac{q_D^2 \Omega^2}{u}. \quad (61a)$$

We recall that $u \propto \sqrt{Z/M}$, $q_D^2 \propto n_M^{2/3}$, $\Omega^2 \propto n_M Z^2/M$, and

$$\lambda = \frac{\Theta_D}{\Theta_{TF}} = \frac{q_D}{q_{TF}} \propto n_M^{1/3}, \quad (61b)$$

which together with $S_1(\lambda)$ are constant under the said constraints. The following results:

$$\overline{\rho_0} \propto \frac{\text{const}}{M^{1/2}} \to 0. \quad (61c)$$

This property agrees with the preceding observation that in the given limit, electrons do not interact with the lattice so that $\rho_0 \to 0$. In this same limit, $\rho_B \to 0$ as $T \to 0$, providing $T/M < 1$ [as follows from (57)].

C. Relative resistivity

Experimental readings of resistivity are often presented as the ratio of low- to high-temperature readings at a fixed high-temperature value. Combining the preceding results for high- and low-temperature resistivity limits we write

$$r \equiv \frac{\rho(T \ll \Theta_D)}{\rho(T \gg \Theta_D)} \approx \frac{1}{2}\left(\frac{\Theta_D}{T}\right) S_1(\lambda), \quad (62a)$$

where T denotes the high-temperature value. With (48d) we write $S_1(\lambda) = \lambda^4/5$, so that at constant T, we obtain

$$r \propto \Theta_D \left(\frac{\Theta_D}{\Theta_{TF}}\right)^4 = \Theta_D \left(\frac{3\pi^5}{16} \frac{a_0^3 n}{Z^2}\right)^{2/3} \propto \frac{n}{Z^{5/6} M^{1/2}}. \quad (62b)$$

D. Comparison with impurity resistivity

An elementary model of impurity resistivity is given by the following.[24,25] First we recall the Drude result

$$\rho_i = \frac{m}{ne^2} \frac{1}{\tau}. \quad (63)$$

For the relaxation time τ we write

$$l \simeq v_F \tau, \quad (63')$$

where l represents the mean free path of electron-impurity scattering and v_F is the Fermi velocity,

$$m v_F = \hbar k_F,$$

$$k_F^3 = 3\pi^2 n.$$

Introducing the total cross section Σ we write

$$l \simeq 1/n\Sigma,$$

where n_i is the impurity number density. We obtain

$$\frac{1}{\tau} = v_F n_i \Sigma. \quad (64)$$

The following results:

$$\rho_i = \left(\frac{n_i}{n}\right)\left(\frac{\hbar k_F}{e^2}\right)\Sigma. \quad (65)$$

In the Born approximation

$$\Sigma=\int_{4\pi}|f(\theta)|^2 d\Omega, \tag{66}$$

where Ω is a solid angle. For the shielded Coulomb interaction, the scattering amplitude is given by

$$f(\theta)=\frac{2\mu Ze^2}{\hbar^2}\left(\frac{1}{\kappa^2+q_{\rm TF}^2}\right),$$
$$\kappa\equiv 2k_F\sin\left(\frac{\theta}{2}\right), \tag{67a}$$

where $q_{\rm TF}$ is the Thomas-Fermi shielding wave number (15b), defined with respect to the impurity Bohr radius

$$a_z=\frac{\hbar^2}{\mu Ze^2}, \tag{67b}$$

where μ is reduced mass. Performing the integral (66) gives

$$\Sigma=\frac{4\pi(2/a_z)^2}{q_{\rm TF}^2(4k_F^2+q_{\rm TF}^2)}. \tag{68a}$$

With $q_{\rm TF}^2=4k_F/a_z$, (68a) may be written in the more concise form

$$\Sigma=\frac{\pi(4a_z)^2}{(4k_Fa_z)(k_Fa_z+1)}. \tag{68b}$$

Combining this expression with (65), we obtain

$$\rho_i=\left(\frac{n_i}{n}\right)\left(\frac{4\pi a_z\hbar}{e^2}\right)\frac{1}{4a_zk_F(a_zk_F+1)}. \tag{69a}$$

A comparison of the preceding result with the electron-phonon scattering residual resistivity ρ_0 (57) and (62) indicates that ρ_0 is more sharply dependent on ion parameters than ρ_i. To further delineate between impurity and electron-phonon scattering contributions to residual resistivity we note the ratio

$$\frac{\rho_i}{\rho_0}=\frac{1}{3\pi}\frac{1}{e^2k_F}\frac{mu^2}{k_B\Theta_D}\frac{n_i}{n}\left(\frac{E_F^3}{(\hbar\Omega)^2}\right)\left[\frac{k_F^2\Sigma}{S_1(\lambda)}\right]. \tag{69b}$$

Note that $k_F^2\Sigma$ is the dimensionless ratio

$$k_F^2\Sigma=\frac{4\pi k_Fa_Z}{(k_Fa_Z+1)}. \tag{69c}$$

As $k_F\approx 10^8$ cm^{-1} for most metals and $a_Z\approx a_0/Z$, it follows that $k_F^2\Sigma\approx 4\pi/(1+Z)$.

IX. DISTRIBUTION FUNCTION

Returning to the expansion (18) and inserting the solution (41) gives the electron distribution

$$f(E,\mu)=f_0(E)+\frac{D\mu}{K(T)}E^2\frac{\partial f_0}{\partial E}, \tag{70a}$$

$$D\equiv\frac{32e\mathscr{E}}{R}, \tag{70b}$$

where $f_0(E)$ is the Fermi-Dirac distribution (2). Consider the function $K(T)$ as given by (56). Let us suppose that there is no residual term and set $K_0=0$. Then as $T\to 0$ K, $K(T)\to 0$ and the perturbation term in (56) becomes singular at all E, thereby violating the Lorentz expansion (18). At $T=0$ K, $\partial f_0/\partial E$ is zero except at $E=E_F$. However, with $K_0=0$, this zero is divided by $K_B(0)=0$ and the distribution (56) is indeterminate. For the case $K_0>0$, as found in the present analysis, this pathological behavior of $f(E)$ is circumvented and, save for the singular point $E=E_F$ at $T=0$ K, a well-defined distribution results for all E.

X. GRAPHICAL RESULTS

We note that the reciprocal of the right-hand side of (46) may be written

$$\rho=H(\Theta_D,\Theta_{\rm TF})\frac{W(y_D,\lambda)}{y_D}, \tag{71a}$$

where the coefficient $H(\Theta_D,\Theta_{\rm TF})$ is as defined, the ''reduced'' resistivity is given by

$$\tilde{\rho}\equiv\frac{\rho}{H(\Theta_D,\Theta_{\rm TF})}=\frac{W(y_D,\lambda)}{y_D}=\frac{W_1(y_D,\lambda)}{y_D}+\frac{W_2(y_D,\lambda)}{y_D}, \tag{71b}$$

and we have recalled that

$$(\Theta_{\rm TF}/T)^2=y_D^2/\lambda^2.$$

log-log plots of the components of $\tilde{\rho}$ vs y_D at $\lambda=0.1$, 0.3, and 1.0 are shown in Figs. 1(a), 1(b), and 1(c), respectively. In all cases the canonical Bloch temperature dependence at $y_D\gg 1$ and the linear temperature dependence at $y_D\ll 1$ are clearly indicated. In addition, one observes the constancy of a residual component that dominates at very low temperature. It is noted that the temperature at which this residual component comes into play is in accord with the criterion (58c).

XI. CONCLUSION

Incorporating the quantum Boltzmann equation, with shielded electron-ion Coulomb interactions, metallic electrical resistivity due to electron-phonon scattering is evaluated for the noble metals and a component of the alkali metals. In addition to Bloch's T^5 contribution, a component of resistivity is found to survive in the limit $T\to 0$. This residual resistivity is attributed to interplay between Fermi-surface electrons and zero-point ion motion, in the presence of an electric field, as well as to the inelastic nature of electron-phonon scattering. A consistency calculation was made on our analysis in which it was found that at fixed ion number density and valence number, residual resistivity vanishes as ion mass grows large. It is further consistently observed that whereas absorption scattering matrix elements vanish at $T=0$ K, emission scattering matrix elements survive. An estimate made of the temperature at which this residual component of resistivity comes into play gives the criterion $T\ll\Theta_D$ for the class of metals considered. It is further observed that this residual component of resistivity maintains nonsingular behavior of the Lorentz expansion for the elec-

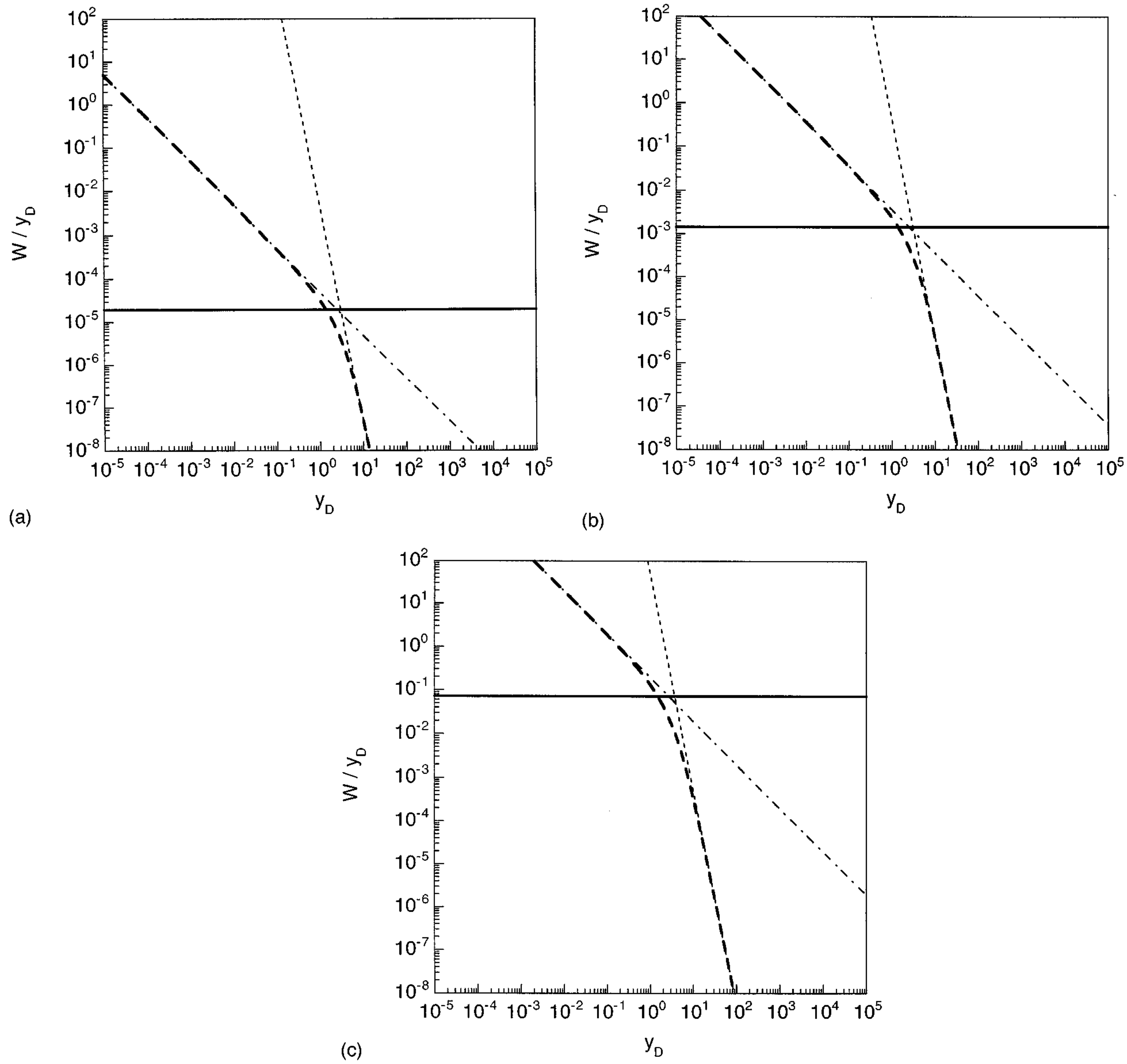

FIG. 1. Log-log graphs of low- and high-temperature components of reduced resistivity $\tilde{\rho}$ vs y_D and asymptotic dependences. (a) $\lambda=0.1$, (b) $\lambda=0.3$, and (c) $\lambda=1.0$. $W_1/y_D=S_1$, ———; W_2/y_D, – – –; $W_2/y_D, y_D \gg 1$, - - -; $W_2/y_D, y_D \ll 1$, – - – -.

tron distribution function at low temperature. It is noted that ρ_0 scales roughly as $Z^3/M^2E_F^4$. It is suggested that the distinctive ion-parameter dependence of this scaling may offer a means of experimentally comparing this mode of residual resistivity with impurity residual resistivity. log-log plots of our expression for resistivity vs inverse temperature at various values of λ returned canonical expressions at high and low temperature in addition to residual resistivity in the vicinity of 0 K. It is noted that experimental detection of this electron-phonon residual resistivity requires metal samples free of impurities, dislocations, or ''frozen-in'' defects as such properties alter resistivity.

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APPENDIX: ELECTRON-PHONON INTERACTION

The interaction Hamiltonian between electrons and ions in the lattice, in the second quantization, may be written

$$\hat{H}=\int d\mathbf{x}\;\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\sum_{\mathbf{y}}\;\Phi[\mathbf{x}-\mathbf{y}-\hat{\mathbf{z}}(\mathbf{y})], \tag{A1}$$

where Φ is the electron-ion interaction potential, $\hat{\psi}^{\dagger}$ is the electron creation operator, $\mathbf{x}$ denotes the electron position, $\mathbf{y}$

denotes the ion equilibrium position, and $\hat{\mathbf{z}}(\mathbf{y})$ denotes the ion displacement from equilibrium. For small ion displacements we write

$$\hat{H}_{\rm int}=\hat{H}-\hat{H}(\hat{\mathbf{z}}=\mathbf{0})$$
$$=\int d\mathbf{x}\ \hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\sum_{\mathbf{y}}\ \hat{\mathbf{z}}(\mathbf{y})\cdot\frac{\partial}{\partial\mathbf{y}}\Phi(\mathbf{x}-\mathbf{y}). \qquad \text{(A2)}$$

Fourier expanding electron operators

$$\hat{\psi}^{\dagger}(\mathbf{x})=\int\frac{d\mathbf{k}_2}{(2\pi)^3}e^{-i\mathbf{k}_2\cdot\mathbf{x}}\hat{\psi}^{\dagger}(\mathbf{k}_2),$$
$$\hat{\psi}(\mathbf{x})=\int\frac{d\mathbf{k}_1}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}}\hat{\psi}(\mathbf{k}_1) \qquad \text{(A3)}$$

and quantizing ion displacements

$$\hat{\mathbf{z}}(\mathbf{y})=\sum_{\mathbf{q},\mu}\sqrt{\frac{\hbar}{2MN\omega}}[\hat{\mathbf{e}}_{\mu}(\mathbf{q})e^{i(\mathbf{q}\cdot\mathbf{y})}\hat{a}_{\mu}(\mathbf{q})$$
$$+\hat{e}_{\mu}(-\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{y}}\hat{a}^{\dagger}_{\mu}(\mathbf{q})] \qquad \text{(A4)}$$

gives

$$\hat{H}_{\rm int}=\int d\mathbf{x}\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6}\sum_{\mathbf{y}}\sum_{\pm}\sum_{\mathbf{q},\mu}\sqrt{\frac{\hbar}{2MN\omega}}\,\hat{\mathbf{e}}_{\mu}(\pm\mathbf{q})$$
$$\times[\psi^{\dagger}(\mathbf{k}_2)\psi(\mathbf{k}_1)]\begin{bmatrix}\hat{a}_{\mu}(\mathbf{q})\\ \hat{a}^{\dagger}_{\mu}(\mathbf{q})\end{bmatrix}$$
$$\times(\mp i\mathbf{q}\exp i[\mp\mathbf{q}\cdot\mathbf{y}-\mathbf{k}_2\cdot\mathbf{x}+\mathbf{k}_1\cdot\mathbf{x}])\Phi(\mathbf{x}-\mathbf{y}), \qquad \text{(A5)}$$

where $\mathbf{q}$ denotes phonon wave vector and $\omega=uq$.

The column vector notation in (A5) is such that the upper $\hat{a}_{\mu}(\mathbf{q})$ term corresponds to the (+) phonon absorption mode and the lower $\hat{a}^{\dagger}_{\mu}(\mathbf{q})$ term corresponds to the (−) phonon emission mode. In these expressions

$$\rho_M=Mn_M$$

represents the ion mass density, μ is the polarization index, and ω is phonon frequency. Now we note that

$$\sum_{\mathbf{y}}e^{\pm i\mathbf{q}\cdot\mathbf{y}}\Phi(\mathbf{x}-\mathbf{y})=e^{\pm i\mathbf{q}\cdot\mathbf{x}}\sum_{\mathbf{u}}e^{\mp i\mathbf{q}\cdot\mathbf{u}}\Phi(\mathbf{u})$$
$$=\frac{N}{V}e^{\pm i\mathbf{q}\cdot\mathbf{x}}\tilde{\Phi}(\pm\mathbf{q}), \qquad \text{(A6)}$$

where

$$\tilde{\Phi}(\mathbf{q})=\int d\mathbf{x}\ e^{-i\mathbf{q}\cdot\mathbf{x}}\Phi(\mathbf{x}) \qquad \text{(A6}')$$

is the Fourier transform of the potential. The preceding combines with the exponentials in (A5) to give

$$H_{\rm int}=\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6}\sum_{\pm}\sum_{\mathbf{q}}\delta(\pm\mathbf{q}-\mathbf{k}_2+\mathbf{k}_1)\sqrt{\frac{\hbar}{2MN\omega}}$$
$$\times\left\{\mp i\mathbf{q}\cdot\hat{\mathbf{e}}_{\mu}(\pm\mathbf{q})\frac{N}{V}\Phi(\pm\mathbf{q})[\hat{\psi}^{\dagger}(\mathbf{k}_2)\hat{\psi}(\mathbf{k}_1)]\begin{bmatrix}\hat{a}_{\mu}(\mathbf{q})\\ \hat{a}^{\dagger}_{\mu}(\mathbf{q})\end{bmatrix}\right\}. \qquad \text{(A7)}$$

With (A7) we see that only longitudinal modes contribute, so we may set

$$\mp i\mathbf{q}\cdot\hat{\mathbf{e}}_{\mu}(\pm\mathbf{q})=\mp iq.$$

Thus, rewriting (A7) in the form

$$H_{\rm int}=\int\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6}\sum_{\pm}C_q\delta(\pm\mathbf{q}-\mathbf{k}_2+\mathbf{k}_1)\hat{\psi}^{\dagger}(\mathbf{k}_2)\hat{\psi}(\mathbf{k}_1)$$
$$\times\begin{bmatrix}\hat{a}_{\mu}(\mathbf{q})\\ \hat{a}^{\dagger}_{\mu}(\mathbf{q})\end{bmatrix} \qquad \text{(A8)}$$

gives the coefficient

$$C_q=\mp\sqrt{\frac{\hbar}{2MN\omega}}\,iq\,\frac{N}{V}\tilde{\Phi}(\pm\mathbf{q}). \qquad \text{(A9)}$$

For the shielded Coulomb potential one writes

$$\tilde{\Phi}(\mathbf{q})=\frac{4\pi Ze^2}{q^2+q_{\rm TF}^2}, \qquad \text{(A10)}$$

where $q_{\rm TF}$ is the Thomas-Fermi shielding wave number (15b). With $\omega=uq$ we obtain (14) of this paper:

$$|C_q|^2=\frac{\hbar(M\Omega^2)^2}{2Z^2\rho_M u}\frac{q}{(q^2+q_{\rm TF}^2)^2}, \qquad \text{(A11)}$$

where the ion plasma frequency Ω is given by (15a).

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