Length scaling of conductance distribution for random fractal lattices

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One can evaluate the Kubo-Greenwood conductance sum in closed form for regular fractal structures. At a Cantor set of energies, the conductance is independent of lattice size L. Here we study scaling with L of the conductance distribution $f(g)$ near such special energies in the presence of random bond disorder. A scaling theory may apply to the average or median value of ln$g$ for which there is a transition from weak to strong localization as the lattice size L exceeds a critical value L_c that depends on disorder. Of more interest is the form of $f(g)$ with random disorder. We discuss the behavior of $f(g)$ in the weak ($L<L_c$) and strong ($L_c<L$) localization limits as well as in the critical case ($L\sim L_c$) where the conducting paths involve a set of states with fractal dimension different from that of the lattice. We are able to describe the curves in terms of two parameters which do not depend on details of the underlying model. The resulting shape function describes the critical distribution as well as the weak and strong localization limits. [S0163-1829(96)07745-4]

I. INTRODUCTION

A conductance distribution $f(g,L)$ is developed in this paper for an ensemble of fractal lattices with Anderson-type bond disorder. The ensemble is characterized by lattice size L and disorder parameter w. In particular we focus on properties of the distribution at weak disorder for $E=-1$ which is a transmission resonance in the ordered lattice. The shape of $f(g,L)$ and its scaling behavior are of interest in light of the scaling theory of conductance for homogeneous random structures and the superlocalization theory for fractals.

Anderson[1] demonstrated that, in contrast to the semiclassical prediction, when there is sufficient disorder the conductance can drop at low temperatures due to electron localization. In the original theory, localization is a one-electron effect caused by coherent superposition of waves scattered with incommensurate phase from static lattice imperfections. While phonon and many-electron effects play a central role in most observed localization phenomena, the original one-electron mechanism described by Anderson continues to generate interest. This is due partly to the theoretical simplicity of models, such as the Anderson tight-binding Hamiltonian, which embody a clear physical picture and are relatively tractable.

Two things contributed to the increased interest in localization during the past decade. One was the arrival of good empirical data on two-dimensional systems, e.g., on silicon inversion layers, and the other was the scaling theory of Abrahams, Anderson, Licciardello, and Ramakrishnan (AALR).[2]

In AALR theory the scaling of dimensionless conductance g with the size L of the sample is controlled by a function $\beta(g)$ such that $d\ln g/d\ln L=\beta(g)$. A characteristic of the theory is that the Euclidean dimension d plays an important role. For classical resistive material with the conductivity independent of sample size, conductance scales as L^{d-2}, and so $\beta\sim d-2$. When states are strongly localized $\beta\sim\ln g$. Assuming the scaling function $\beta(g)$ is monotonic and smooth in g, then whenever the dimension d exceeds 2 there can exist a transition, as a function of disorder or Fermi level, from localized to delocalized conductance scaling behavior. The marginal case $d=2$ is especially interesting. For $d<2$ there can still be a sort of transition between strong- and weak-localization scaling, corresponding to an elbow in the $\beta(g)$ versus ln$g$ curve.

When random disorder is homogeneous, i.e., distributed independently of position, one-parameter scaling theory has had a fair degree of success. AALR theory does not address several interesting aspects of Anderson localization, however, even in the case of homogeneous disorder. Two of these, both having to do with conductance fluctuations, are of special interest in the work reported below.

First, for a given realization of the random structure there is a set of Fermi energies at which the transmission probability is near unity. The set of such Azbel resonances[3] may be uncountable, though of measure zero with respect to randomly selected energies, in the large-L limit. Pendry[4] showed rigorously that the eigenstates contributing to conduction in these narrow transmission resonances have fractal support. In fact the number of sites participating in current transport in a disordered chain increases as $\sqrt{L}$.

To treat random disorder one considers a structure ensemble. While Azbel resonances produce conductance fluctuations in a single structure as a function of energy, the second fluctuation phenomenon of interest in the current work has to do with variation of conductance between samples at a fixed energy. In rather general circumstances these between-sample conductance fluctuations appear to have universal character.[5,6] The most obvious way to analyze them is in terms of a conductance distribution function $f(g)$ for the ensemble. Of course $f(g)$ depends on energy E and parameters of the structure distribution. AALR theory must apply to a statistic of $f(g)$, for example, the mean value $\langle g\rangle$.[5,6] Since tails of the distribution dominate the moments, it is more practical to characterize scaling in terms of the median rather than mean. However, the fluctuations become larger than the variation of interest in scaling especially in the critical region. Thus a question arises.[7] Under what conditions does the scaling behavior of a single statistic such as $\langle g\rangle$ give a useful indication of the scaling of conductance

g of a typical structure drawn at random from the ensemble?

Shapiro[8] studied $f(g,L)$ for homogeneously random disorder in dimension $d=2+\epsilon$. The computation consists of an approximate renormalization using transverse-longitudinal decoupling. When $\epsilon>0$, scaling theory predicts a transition from metallic to localized scaling behavior as $\langle g\rangle$ at small L passes (as a function of E or disorder) from above to below the value g_c defined by $\beta(g_c)=0$. The distribution tends to one of three different limits as $L\to\infty$ depending on whether the scaling of $\langle g\rangle$ is metallic, insulating, or critical. These can be defined as the regions where $\langle g\rangle$ at the shortest length scale is greater than, less than, or equal to g_c. Moments of the conductance distribution are controlled by the tails of the total $f(g,L)$ and are completely different from the moments of the limit shape. In the insulating and metallic cases the shape of the main body of the limit $f(g)$ depends only upon $\langle g\rangle$, though the shape in the insulating case is different form that in the metallic case. Thus in either of these two regions, the scaling of $\langle g\rangle$ would determine that of $f(g)$. In the critical region Shapiro found that $f(g)$ depends only on dimension d.

Al'tshuler *et al.*[7] addressed the question of the meaningfulness of scaling theory in the presence of universal fluctuations, especially in the near critical region for $d=2$ where $\langle g\rangle$ is close to 0. The object of study is the conductance distribution $f(g,L)$ for finite lattice size L for homogeneous disorder in $d=2+\epsilon$ concentrating on the critical cases $\epsilon=0$ and $\langle g\rangle\sim g_c$ for $\epsilon>0$. The nonlinear-σ model is used. At $d=2$ one expects a transition from weak to strong localization as a function of increasing L for a given amount of disorder. In other words there should exist a localization length L_c which depends on the amount of disorder. Conductance should scale as in weak localization, i.e., logarithmically in L, until L exceeds L_c. For larger L, the conductance should decrease exponentially. The localization length L_c goes to ∞ as the disorder goes to zero. Adding positive ϵ to d sharpens the transition from the weak-to-strong localization type to a metal-insulator transition. For ϵ small but positive the scaling of $f(g,L)$ with increasing L depends on whether L or L_c goes to ∞ the fastest.

In the metallic region where $L_c>L$ the distribution looks nearly Gaussian, except that it may have long tails as stressed in Ref. 8. Even in the metallic region the tails begin to look log normally distributed, which would result from a product of independent transmission factors. The relative importance of the tails in the metallic region vanishes as $L\to\infty$. In the nonmetallic region $L>L_c$ the tails become more important with increasing L, until finally $f(g,L)$ crosses over to a fully log normal distribution. A similar description applies at the critical dimension $d=2$. A question of some interest is whether or not there is a universal shape for $f(g,L)$ which holds throughout the crossover. It is concluded in Ref. 7 that, in contrast to the conclusions of Ref. 8, for the homogeneous random models at $d=2$ there is no such universal function.

Most work on localization and conductance scaling has been on homogeneously random models. However, an alternative approach is to replace random disorder by lattice aperiodicity having quasiperiodic or inflation symmetry. Such models include hierarchical structures[9] which bear some similarity to random models[10,11] although they are regular.

Normal modes or eigenstates can often be found on a regular fractal lattice by real space renormalization.[12,13] Anderson localization occurs at most energies on fractals for the same reason it does in random lattices. There is variability in the bonding geometry in the neighborhoods of the lattice sites. This causes the incommensurate reflections necessary to suppress conductance. For these and other reasons[10,11] regular fractals like the Sierpinski gasket are sometimes suggested as tractable substitutes for disordered structures such as aerogels, percolation clusters, and so on.

The Kubo-Greenwood conductance sum can be evaluated in closed form[14] on many regular fractal lattices.[15,16] Ignoring narrow resonances, the conductance scaling on fractals, both regular and random, appears to be described well by the superlocalization theory.[17–19] At most eigenstate energies, the eigenfunctions are expected to have the superexponential form $\psi\sim\exp[-kr^{\alpha}]$, where r is the Euclidean distance from some center of localization, and the exponent α depends on the structure. In fact r^{α} is often the scaling of the minimum chemical distance, measured on the lattice, between two points separated by the Euclidean distance r. From this one can conclude that in general the conductance at large L should also decrease superexponentially in general. The structure of the Sierpinski lattice is such that $\alpha=1$, and so superlocalized conductance scaling is exponential. Because superlocalization theory still applies, we refer to conductance scaling in this case as superlocalized even though the functional form is not superexponential.

Exact evaluation of g at fixed energies on regular fractals stems from the self similarity or inflation symmetry. This gives rise to a Lie group which commutes with the real space renormalization recursions for lattice Green functions, which makes the recursions decouple completely.[20,21] There often exists a Cantor set of energies at which the conductance g becomes independent of lattice size L.[16] We will identify these with Azbel resonances. They are outside the scope of superlocalization theory in the same sense in which Azbel resonances are outside the scope of scaling theory. Since the resonances occur on regular lattices, it is fair to ask whether they owe their existence to special symmetries. The first question is, when the lattice symmetry is perturbed even slightly by disorder, can one determine whether there still exists a set of transmission resonances? It is difficult to answer this negatively since failure to find resonances, i.e., numerically, does not tell whether they no longer exist or whether the energies, a set of measure zero, have become lost. One example is known[22] of a regular fractal lattice with a distortion that removes most obvious symmetries but for which the Lie group can still be found. In this case resonances still exist, but scaling of conductance is different. At the resonance energies g decays as a power of L rather than exponentially, but there are no energies where g becomes L independent. Hood and Southern conclude[23] that breaking symmetry in such models may cause molecular localized states to delocalize and form special continua.

In the work reported here we study the conductance distribution $f(g,L)$ at fixed energy for an ensemble of Sierpinski lattices with random bond disorder. With a slight change in the initial conditions, the same analysis can apply to a rather general, finitely ramified, random fractal structure.

The cases where the energy E is at the spectral maximum (ground state) and where E is an Azbel resonance energy are considered separately. We discuss the nature of Azbel resonances and the form of the distribution as L and localization length L_c both become large in the three cases $L<L_c$, $L>L_c$, and $L\sim L_c$.

II. RENORMALIZATION OF CONDUCTANCE DISTRIBUTION

A. Discrete Schrödinger equation for a general graph

We take the usual approach of transcribing the relevant differential equation onto a fractal lattice which can be viewed as a network of quadrature points. The Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V\right]\psi=E\psi \text{ or } \nabla^2\psi=\epsilon\psi, \tag{2.1}$$

with $\epsilon=(2m/\hbar^2)(V-E)$, is discretized by the formula

$$\nabla^2\psi_l \rightarrow \sum_{n@l}(\psi_n-\psi_l), \tag{2.2}$$

where $n@l$ reads for all n adjacent to l. One can write

$$\sum_n (A_{ln}-p_l\delta_{ln})\psi_n=\epsilon\psi_l, \tag{2.3}$$

where A is the adjacency matrix such that $A_{ij}=1$ if the sites i and j are connected by a bond on the quadrature grid, and otherwise 0, and $p_l=\Sigma_n A_{nl}$ is the number of nearest neighbors at the site l. By replacing ∇^2 by the sum in Eq. (2.2) we introduced a generalized tight-binding approximation. On regular lattices it is convenient to shift the energy zero by the constant value p of site coordination so that $\epsilon=0$ at the ground state or band edge. The resolvent operator or Greenian for Eq. (2.3) is defined as usual,

$$G=[zI-A]^{-1}, \tag{2.4}$$

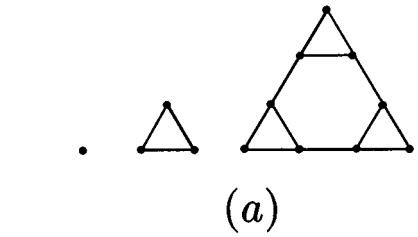
where z is a complex energy parameter. A lattice is constructed in steps or generations. Let n be the generation number. A generation $n+1$ lattice is built by connecting three generation n lattices together via three new bonds [see Fig. 1(a)]. The length L is 2^n-1, the number of sites is 3^n.

The method of computing G numerically as a function of z for an entire structure is a standard rennormalization procedure using Dyson's equation $G=G_0+G_0VG$ where G_0 is the Greenian for generation n and G for generation $n+1$.

B. Conductance sum

To compute Kubo-Greenwood conductance g one-dimensional chains[24] are attached as leads to provide a current path to and from infinity. One needs the four one-electron Green functions $x_1=G_{pp}$, $x_2=G_{qq}$, $y=G_{pq}$, and $\gamma=G_{rr}$ where p and q are two sites on opposite corners of the lattice and r is at the end of a semi-infinite chain [see Fig. 1(b)]. The conductance formula given by Lee and Fisher[14] simplifies further to

$$g=4[\mathrm{Im}(\gamma)]^2\left|\frac{y}{\Delta}\right|^2, \tag{2.5}$$



(a)

(b)

FIG. 1. Construction of an ordered S lattice. (a) A generation-$(n+1)$ lattice is built by connecting three generation-n lattices together via three new bonds. The length L is 2^n-1; the number of sites is 3^n. Generations $n=0$, 1, and 2 are shown. (b) Semi-infinite one-dimensional chains are attached to arbitrary graph $\mathcal{G}$ as leads to provide a current path to and from infinity. Letters s, p, and r indicate site indices.

where

$$\Delta=(1-\gamma x_1)(1-\gamma x_2)-\gamma^2y^2 \tag{2.6}$$

and

$$\gamma=1-\frac{z}{\sqrt{z^2-4}}. \tag{2.7}$$

This formula is discussed in more detail elsewhere.[16,25,26]

C. Resonances on ordered lattices

The ordered lattice is the three-simplex introduced by Dhar.[27] The recursive construction is shown in Fig. 1(a). Recursion relations for the two pivotal Green functions $x=G_{11}$ and $y=G_{21}$, where sites 1 and 2 are at corners, are derived easily using a standard transfer matrix method:[28,29]

$$X=x+\frac{2y^2(x-x^2+y^2)}{\Delta}, \quad Y=\frac{y^2(1-x+y)}{\Delta}, \tag{2.8}$$

with

$$\Delta=(1-x-y)(1-x^2+y+y^2), \tag{2.9}$$

where (x,y) and (X,Y) correspond to generations n and $n+1$ of the lattice construction, respectively. The recursions are exact. One finds Green functions (x,y) for complex energy z at generation n by iterating Eqs. (2.8) numerically n times, starting from $x=y=1/z$. In new variables

$$b=\frac{(1+x-y)(1-x-2y)}{y}, \quad t=\frac{1-x+y}{1+x-y}, \tag{2.10}$$

the recursions reduce to

$$B=b(5+b), \quad T=\frac{(3+b)}{(5+b)}t. \tag{2.11}$$

In terms of these variables,

$$x(b,t)=\frac{1-t}{1+t}+y(b,t), \quad y(b,t)=\frac{4t}{(1+t)[6+b(1+t)]}. \tag{2.12}$$

The reduction of order evident in Eqs. (2.11) happens because the recursions commute with the Lie group $t\rightarrow\alpha t$ representing a continuous symmetry related to inflation. Because the b recursion is completely decoupled, one has an infinite hierarchy of closed form solutions for (b,t), and hence for (x,y) in terms of generation number n (hence of L) at fixed energy. These occur at a Cantor set of energies each of which corresponds to a fixed cycle of the b recursion. Physically, these solutions are the Azbel resonances. For example, $b=-4$ is fixed, corresponding to $E=-1$. Substituting $b=-4$ into the t recursion we find that $t=(-1)^n$ at this energy. Thus the conductance for even n becomes $g=1$ independent of L. At any finite fixed cycle of b, the conductance decays no faster than a power of L. All resonance energies and power law conductance exponents can be obtained exactly. At randomly chosen z, however, the conductance decays exponentially in accordance with superlocalization. The complete analysis of these resonances of the perfect lattice is given elsewhere.[15,16]

The amplitude pattern for the transmission resonance at $E=-1$ is illustrated in Fig. 2, which shows the $n=2$ structure with three leads attached. One can visualize a scattering problem or a T connection in a wave guide. Stationary states on the leads at $E=-1$ can have amplitude patterns $\ldots,0,1,-1,0,1,-1,\ldots$. A Bloch state consists of a phased combination of such states. Figures 2(a)–2(c) show eigenstates of the lattice with leads which add to form current-carrying states. Refer to these three stationary states as ϕ_1, ϕ_2, and ϕ_3 respectively. If $\omega=(1+i\sqrt{3})/2$, then $\psi_k=\phi_1+\omega\phi_2+\omega^2\phi_3$ carries current through the lattice with zero reflection. A resonance pattern of this form can only develop for even generations of the structure. It is degenerate with ψ_{-k} where ω is replaced by $1/\omega$ as well as with many standing wave states. The latter include for example patterns $\ldots,0,1,-1,0,1,-1,\ldots$ around the border of any of the internal holes with length larger than 3. Regarding construction of such molecular states, see Refs. 12 and 13 as well as the discussion by Montgomery.[30]

One finds for a perfectly regular fractal, at $E=-1+\epsilon$ rather than on resonance, that the conductance remains close to 1 out to a cutoff size $L(\epsilon)$ and beyond that falls off exponentially. The cutoff size and the exponent depend on ϵ. Clearly ϵ determines the initial distance from an unstable fixed point in the renormalization. Asymptotic scaling as $L\rightarrow\infty$ depends on the initial data. However, conductance fluctuations due to fluctuations in E on the regular lattice do not shed much light on the disordered case.

D. Random lattice and structure ensemble

A conductance distribution function $f(g,L)$ is obtained by simulation. We define an ensemble of approximately 6000 random structures and then compute a histogram for $f(g,L)$ for fixed L and for fixed Fermi energy E. Conductance is computed exactly for each realization of the random structure using the telescoped form of the Kubo-Greenwood conductance sum, Eq. (2.6). Necessary Green functions are computed by exact recursions similar to Eq. (2.8) for the regular case, but involving six pivotal Green functions per lattice per generation in place of (x,y). Only even generation numbers n are used for studying resonance transmission, since no resonance exists at $E=-1$ for odd generations. Numerical limitations permit a maximum $n=12$, or $L=2^{12}=4096$, or the number of sites $N=3^{12}$.

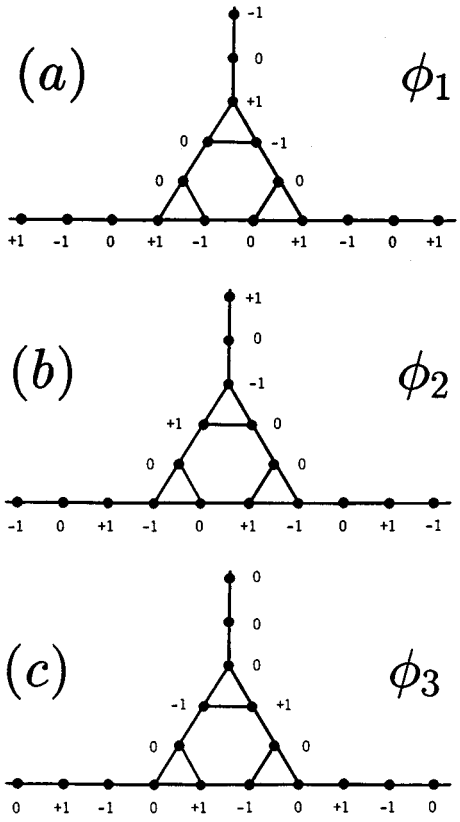


FIG. 2. Amplitude pattern for transmission resonance at $E=-1$. The figure shows the $n=2$ structure with three leads attached. Stationary states on the leads have amplitude patterns $\ldots,0,1,-1,0,1,-1,\ldots$. (a)–(c) show eigenstates ϕ_1, ϕ_2, and ϕ_3 of the lattice with leads which add to form current-carrying states.

We begin by defining the structure ensemble. At a given stage n in the iteration, suppose an ensemble of $M=6000$ structures is already defined and we need to construct the next generation. Each individual structure of ensemble $n+1$ is defined by drawing three generation-n building blocks at random and connecting them together with bonds also drawn at random according to a prescribed bond distribution function. The actual choices are made using pseudorandom numbers generated sequentially. This is done M times to create the ensemble at generation $n+1$. Since each lattice requires three blocks, the blocks are drawn with replacement; i.e., each can be redrawn as many times as necessary. It is essential that the pseudorandom numbers can be

restarted so that the same structure ensemble can be prepared again, for example, at a different energy, simply by restoring the seed.

The numerical quantities stored in the computer are six double complex Green functions for each of M current and M previous generation structures. A conductance distribution is built up from the Green functions as a final step. Thus $f(g,L)$ is represented as a histogram. No approximations are made in computing the histogram.

III. SHAPE OF CONDUCTANCE DISTRIBUTION

As in AALR theory, one can begin with the behavior of median ln$g$ as a function of L. For the Azbel resonance (and not in the nonresonant case) a plot of this behavior shows crossover between two regions which we consider analogous to weak and strong localization. A localization length L_c can be defined by extrapolating a straight line fitted at large L, where conductance decreases exponentially, to its intersection with the line where ln$g=0$. For example, $L_c \approx 100$ when $w=0.01$. The value of L_c obtained in this way is defined only within an order of magnitude due to statistical fluctuations. Plotting lnL_c vs ln$w$ one can determine a critical exponent ν such that $L_c \sim w^{\nu}$. This procedure provides an extremely rough estimate of ν. A much more precise method of determining ν will be suggested below, but this is not part of the purpose of the current study.

Histograms of ln$g$ for an ensemble with $w=0.01$ are shown for resonance ($E=-1$) and at the band edge ($E=3$) for four different lattice sizes in Fig. 3. The numbers on the abscissas are the ln$g$ values. The ranges shown are chosen in order to demonstrate the similarity in shape.

For all values of L, the $E=3$ histograms look like log normal distributions. The median value of ln$g$ drops linearly as a function of increasing L, consistent with superlocalization. There is no transition either in scaling of the median conductance with L or in the shape of the histogram for the nonresonant case.

The resonant case is more interesting. The lattice size L begins to exceed L_c around $n=8$. There is a clear change in the histogram in this region from ramp shape for $L \leqslant L_c$ to bell shaped for $L \gg L_c$. For the latter range the median value of ln$g$ drops linearly in L as in the nonresonant case. For the small-L region, the decrease of median conductance with increasing L becomes sublinear. We refer to the region with $L<L_c$ as weak localization. The ramp shape of the distribution looks exponential, but in fact it deviates slightly. At $n=8$, near the transition, the tail of the distribution is acquiring more weight and there is a slight relative enhancement of the residual perfect-conductance fraction. Since at $n=8$ the size $L=256$ probably exceeds L_c, the increasing weight of the distribution tails takes place between $L=L_c$ and the appearance of the broad hump.

To develop a qualitative understanding of the shape of the main body of the conductance distribution one might reason as follows. For a given structure in the ensemble g has a particular value. Thus g is a function of each random bond fluctuation. The fluctuations comprise a very large number of independent, identically distributed random numbers with known distribution. The distribution is chosen to be Gaussian with zero mean and width parameter w. Partly because of the

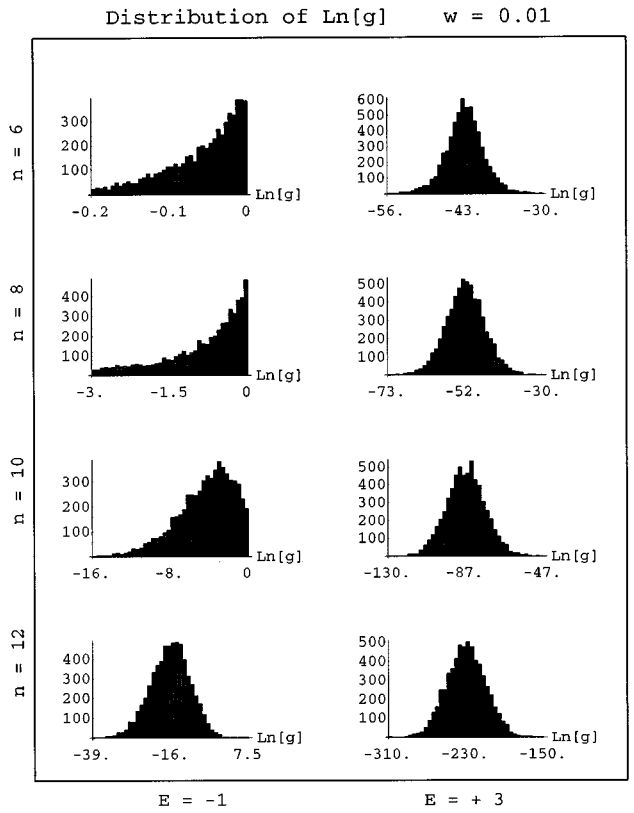


FIG. 3. Histograms of ln$g$ for ensemble with $w=0.01$ for resonance ($E=-1$) and at the band edge ($E=3$) for lattice size, $n=6$, 8, 10, and 12, where $L=2^n$. Numbers on abscissas are ln$g$ values. Transition from weak to strong localization occurs around $n=8$ where the shape of distribution changes.

bandpass of the one-dimensional leads,[24] the maximum value of g for a given structure is 1.

Let us consider first the case of weak disorder at resonance, since the distribution off resonance will be at the strong-localization limit for any disorder. Suppose E is fixed at -1, a resonance energy of the perfect lattice at which $g=1$ independent of L. Thus ln$g$ has a narrow quadratic maximum value at the origin in the multidimensional space of bond fluctuations. When there is finite disorder, contributions to the average (or logarithmic average) conductance come mostly from resonant structures. The origin is only one such structure. We will assume $\ln g = -a(x-x_0)^2 + \cdots$, where x is a generic Gaussian random variable. Parameters a and x_0 must represent physical properties controlling the width and maximum of the conductance distribution. The use of a single variable x corresponds to an assumption of invariance of the g distribution with respect to rotation of the multidimensional fluctuation space. Let a, which contains an average over directions, be scaled so that x has the same distribution as each of the bond fluctuations. Thus one may expect $a \sim a_0 N$.

Therefore suppose $\ln g = y(x)$ has a quadratic maximum in generic variable x which is distributed as

$$P(x) = \frac{1}{w\sqrt{2\pi}} \exp\left[-\frac{x^2}{2w^2}\right]. \tag{3.1}$$

Then using $y \sim -a(x-x_0)^2 + \cdots$ and assuming $a>0$,

$$Q(y) = \int_{-\infty}^{\infty} P(x)\,\delta[y + a(x-x_0)^2]dx \tag{3.2}$$

or

$$Q(y) = \frac{\Theta(-y)}{\sqrt{2\pi(-y)}\xi} \exp\left[\frac{-\eta^2 - y}{2\xi^2}\right] \cosh\left[\frac{\eta}{\xi^2}\sqrt{-y}\right], \tag{3.3}$$

where

$$\xi = \sqrt{a}w \quad \text{and} \quad \eta = \sqrt{a}x_0. \tag{3.4}$$

This yields

$$\langle y \rangle = \int_{-\infty}^{\infty} y Q(y) dy = -(\xi^2 + \eta^2),$$
$$\langle y^2 \rangle = 3(\xi^2 + \eta^2)^2 - 2\eta^4. \tag{3.5}$$

From these equations we relate ξ and η to mean and variances of the ln$g$ distribution.

To examine the case of small ξ, we must rescale. Rescaling leads to a shape function that depends on a single parameter p. Let $\eta = p\xi$, $y = -v\xi^2$. Then the distribution for v is

$$\widetilde{Q}(v) = \frac{e^{-(p-\sqrt{v})^2/2} + e^{-(p+\sqrt{v}2/2}}{2\sqrt{2\pi v}} \tag{3.6}$$

or

$$\widetilde{Q}(v) = e^{-(p^2+v)/2} \frac{\cosh(p\sqrt{v})}{\sqrt{2\pi v}}. \tag{3.7}$$

The statistics are

$$\langle v \rangle = 1 + p^2,$$
$$\langle v^2 \rangle = 3 + 6p^2 + p^4, \tag{3.8}$$
$$\Delta v^2 = 2(1 + 2p^2).$$

As p increases, the shape changes from monotonically decreasing with exponential tail to log normal with a tiny residuum near perfect conductance $y=0$. The critical value p_c at which a maximum and minimum separate is found by solving simultaneously the inflection equations

$$\frac{\partial \widetilde{Q}}{\partial v} = 0 \quad \text{and} \quad \frac{\partial^2 \widetilde{Q}}{\partial v^2} = 0. \tag{3.9}$$

These two conditions are true simultaneously when p satisfies

$$e^{-2p\sqrt{p^2-3}} = \frac{p\sqrt{p^2-3} - p^2 + 2}{p\sqrt{p^2-3} + p^2 + 2}. \tag{3.10}$$

The root $p_c = 2.053 > \sqrt{3}$ can be used to redefine a transition from weak to strong localization. The corresponding value of n defines $w(L_c)$ via $L_c = 2^n$ and the critical inflection $v_c = 1.2165$. The approach taken from here on is to identify

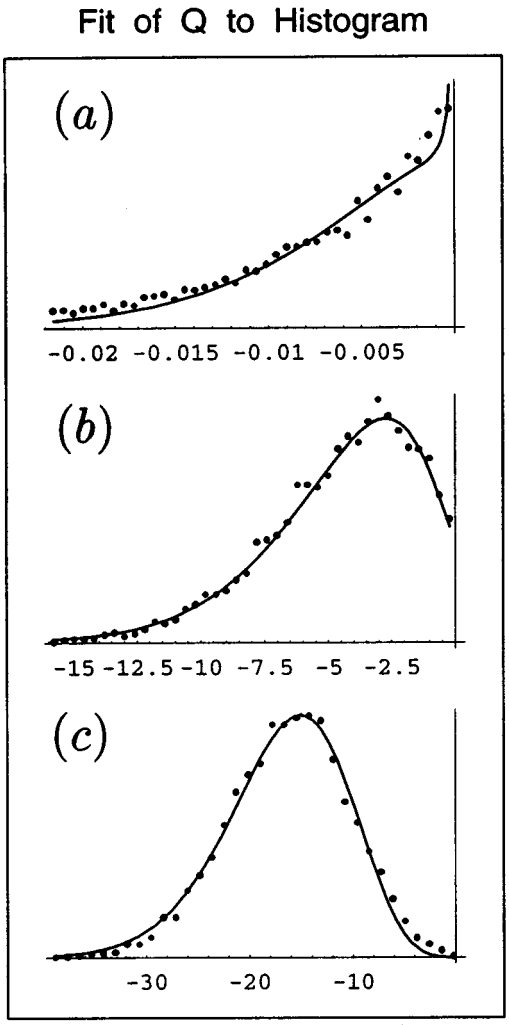


FIG. 4. Conductance histograms for resonant case fitted to the shape function $\widetilde{Q}(v)$. The fit is actually performed using two parameters ξ and η in $Q(y)$ of Eq. (3.3). (a), (b), and (c) correspond to $n=4$, 10, and 12 with p values 1.82, 2.65, and 5.39, respectively. Disorder is $w=0.01$ and crossover occurs around $n=8$. Distributions for $L<L_c$, $L\sim L_c$, and $L>L_c$ fit shape functions for $p<2$, $p\sim 2$, and $p>2$.

$p<p_c$, $p\sim p_c$, and $p>p_c$ with $L<L_c$, $L\sim L_c$, and $L>L_c$. This leads to a redefinition of L_c.

Shapes for $n=6$ and 8 with $E=-1$ in Fig. 3 agree reasonably well with the small p limit of Eq. (3.7), indicating an asymptotic power law for $f(g,L)$ as a function of g for L below L_c. With increasing lattice size, the median conductance decreases, thus the rate of exponential decrease in the two histograms, and hence the power-law exponent for these two cases are different. Nevertheless, by scaling the abscissas with respect to the median value, the curves appear similar.

Although the current study does not purport to shed any light on Anderson localization in nonfractal structures, it is reasonable to compare conductance scaling at resonance with the critical case of the homogeneous lattices, $d=2+\epsilon$ with either when $\epsilon=0$ or else the case when the initial conductance is at the mobility edge. For the latter, Shapiro's ap-

proximation yields an exactly power-law form for $f(g)$. The exponent for the critical distribution in Shapiro's calculation for ϵ small but positive depends on the initial, small-L, conductance. In the resonant case on the fractal, the exponent depends on a scaling combination of L and the disorder w. Numerical results from the histogram support the conclusion that for weak disorder the median of ln$g$ is $\propto L^{-d\overline{w}^2}$ (with $d=\ln 3/\ln 2$) as suggested by Eqs. (3.3) and (3.4).

Figure 4 shows conductance histograms for the resonant case fitted to the shape function $\widetilde{Q}(v)$. The fit is actually performed using two parameters ξ and η in $Q(y)$ of Eq. (3.3). Figures 4(a), 4(b), and 4(c) correspond to $n=4$, 10, and 12 with p values 1.82, 2.65, and 5.39, respectively. The disorder is $w=0.01$, and so the crossover occurs around $n=8$. The distributions $L<L_c$, $L\sim L_c$, and $L>L_c$ fit the shape functions for $p<2$, $p\sim 2$, and $p>2$. One can see from Eq. (3.6) that for large p the distribution should not be exactly log normal, but a Gaussian in the variable $p-\sqrt{v}$. Expanding about the peak center shows this to be an excellent approximation to log normal for a narrow peak, consistent with the central limit theorem. Notice in Fig. 4(a) that the shape of the low-conductance tail is not given correctly by the simple phenomenological distribution $\widetilde{Q}(v)$. Nevertheless, $\widetilde{Q}(v)$ interpolates well between strong- and weak-localization distributions.

IV. COMMENTS

To study a transition similar to the transition from weak to strong localization one has to look at energies where g does not depend on L. The perfectly ordered Sierpinski structure has L-independent conductance only at a Cantor set of energies. These we compare to Azbel resonances of the homogeneous random models. For $E=-1$, which is a transmission resonance in the perfect lattice, we have illustrated a conductance scaling behavior similar to a weak- to strong-localization transition as a function of increasing L at finite w or increasing w at fixed L.

Results suggest that $\widetilde{Q}(v)$ of Eq. (3.7) describes the main body of the conductance distribution reasonably well for the model studied. For weak localization, as defined either by $L<<L_c$ or by $p<<2$, the conductance distribution drops monotonically starting from $g=1$. At $p\sim 2$ an inflection occurs so that a broad maximum develops for $p>2$. The area of the residual peak at ln$g=0$ with divergence $\sim v^{-1/2}$ representing near-perfect conductance tends rapidly to zero with increasing p. For $p>>2$, $\widetilde{Q}(v)$ becomes qualitatively log normal (normal in $p-\sqrt{v}$). The inflection point at $p=p_c\sim 2$ suggests a more robust way to study scaling of the transition as a function of L and w by means of fitting histograms to $\widetilde{Q}(v)$ and identifying $L=L_c$ with $p=p_c$. In this way the determination is made using the body of the distribution rather than by statistics more sensitive to the tails.

The method used to develop conductance histograms applies to any finitely ramified fractal. Experience with related lattices indicates that for the case of continuously distributed bond disorder the same qualitative theory applies in general, the difference between lattice structures being reflected in different scaling of $\langle \ln g\rangle$. Similar calculations with random removal of bonds rather than continuous fluctuations (i.e., quantum percolation) have also been performed, but these give distributions with much more complex shapes which are not universal and depend on the detailed morphology of the lattices.

The conductance distribution for E at the ground state of the ordered lattice or at a randomly chosen energy is essentially log normal over all ranges of w and L. This is because randomly chosen energy almost always corresponds to superlocalization with medium conductance $g\sim e^{-\kappa L}$. At the ground state or at an infinite set of other resonance energies $g\sim L^{-\beta}$ with β depending on energy. In either of these cases the distribution is qualitatively as expected for strong localization. One finds a negative value for the parameter x_0 which would imply negative L_c.

We conclude that the scaling behavior of conductance as a function of lattice size for a regular fractal with weak Anderson-type bond disorder is in some ways analogous to conductance scaling in homogeneous random models in the presence of Anderson localization. The analogy is strongest when energy E is at a transmission resonance of the ordered fractal. In that case there is a transition from weak to strong localization scaling as a function of lattice size. We have developed a phenomenological expression $\widetilde{Q}(v)$ that interpolates the shape of the body of the logarithmic conductance distribution at resonance between the weak- and strong-localized regimes. The result does not apply to quantum percolation.

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