Nonlinear theory of the current instability in a ballistic field-effect transistor

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We study the nonlinear evolution of the current instability in a ballistic field-effect transistor. We show that in the case of small increments, instability results in the establishment of stationary nonlinear oscillations. The amplitude of the oscillations is calculated. We predict peculiarities in the current-voltage characteristic of the transistor. In particular, the differential resistance at zero frequency should show a large step at the instability threshold. [S0163-1829(96)02740-3]

I. INTRODUCTION

Recently, Dyakonov and Shur[1] discovered that a relatively low dc current in a ballistic field-effect transistor may be unstable. This instability is a result of plasma wave amplification due to the reflection from the device boundaries. This effect provides a mechanism for the generation of tunable electromagnetic radiation in the terahertz region. They also showed that the electron fluid in the transistor may be described by equations that are analogous to the hydrodynamics equations for shallow water, the plasma waves playing the role of shallow water waves.

In Ref. 1 the instability criterion was found in the linear approximation. Gelmont[2] considered the nonlinear evolution of the instability in the special case when the electron fluid velocity was close to the velocity of plasma waves. In addition, his consideration was restricted to the idealized situation when the viscosity of the electron fluid and the external friction were absent. For this case, he showed that discontinuities of the distributions of electron concentration and velocity may be formed in the transistor channel, which are analogous to the hydraulic jumps or shock waves. However, consequences of the instability remain unclear in the real situation when the electron fluid velocity is much less than the plasma waves velocity and the viscosity and the external friction are present.

The purpose of this paper is to study the nonlinear evolution of the instability in the ballistic field-effect transistor in the case of relatively low electron fluid velocity with inclusion of the viscosity of the electron fluid (caused by electron-electron scattering) and the external friction (related to electron scattering by phonons and impurities). We show that, in contrast to the case considered in Ref. 2, for small increment, the instability should lead to the establishment of small-amplitude stationary oscillations. This is the case when, in the linear regime, only one mode of plasma oscillations is unstable while other modes are damped out by the viscosity and the external friction. The amplitude of stationary oscillations is calculated. It is shown that the amplitude is proportional to the square root of the increment near the instability threshold. The consequences of the instability for higher values of the increment are discussed.

We predict that the current-voltage characteristic of the transistor should show a peculiarity at the instability threshold. The differential resistance of the transistor at zero frequency should have a large step at the threshold.

II. BASIC EQUATIONS

In the hydrodynamic approximation, the electron fluid may be described by the following equations, which coincide with those for shallow water:

$$\frac{\partial U}{\partial t}+\frac{\partial (VU)}{\partial x}=0, \tag{1}$$

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{e}{m}\frac{\partial U}{\partial x}=-\frac{V}{\tau_p}+\frac{K}{U}\frac{\partial^2 V}{\partial x^2}. \tag{2}$$

Here $U(x,t)=U_{GC}(x,t)-U_T$, U_{GC} is the gate-to-channel voltage, U_T is the threshold voltage, $V(x,t)$ is the local electron velocity, m is the electron effective mass, τ_p is the momentum relaxation time related to the external friction (due to electron scattering by phonons or impurities), and K is the coefficient determining the viscosity related to electron-electron scattering. Equation (1) is the continuity equation in which the gradual channel approximation equation[3]

$$n_S=\frac{CU}{e}$$

is taken into account. Here C is the surface gate capacitance and n_S is the electron surface concentration. Equation (2) is analogous to the hydrodynamic Navier-Stokes equation, with U corresponding to the shallow water level and V corresponding to the local velocity of the water flow.

Dyakonov and Shur[1] studied these equations without taking into account the viscosity ($K=0$) and the external friction ($\tau_p=\infty$). They have shown that, under the boundary conditions

$$U(0,t)=U_0,\quad U(L,t)V(L,t)=\frac{j}{C}=U_0V_0, \tag{3}$$

the stationary electron flow may be unstable. Equations (3) show that the potential U is fixed at the source ($x=0$) and the current density j is fixed at the drain ($x=L$). In the linear approximation and for $V_0\ll s=(eU_0/m)^{1/2}$, it was shown that the instability leads to excitation of plasma waves with the frequencies

$$\Omega_n' = \frac{\pi}{2}\,(2n-1)\,\frac{s}{L}, \quad n=1,2,3,\ldots \tag{4}$$

and the increment

$$\Omega_n'' = \frac{V_0}{L}. \tag{5}$$

Here

$$s = \left(\frac{eU_0}{m}\right)^{1/2}$$

is the plasma wave velocity.

Using the dimensionless variables

$$z=\frac{x}{L}, \quad \tau=\frac{ts}{L}, \quad u=\frac{U}{U_0}-1, \quad v=\frac{V-V_0}{s},$$

one may rewrite Eqs. (1) and (2) and the boundary conditions (3) in the form

$$\frac{\partial u}{\partial \tau}+\frac{\partial v}{\partial z}+\alpha\,\frac{\partial u}{\partial z}+\frac{\partial(uv)}{\partial z}=0, \tag{6}$$

$$\frac{\partial v}{\partial \tau}+\alpha\,\frac{\partial v}{\partial z}+v\,\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}=-\gamma(v+\alpha)+\frac{\kappa}{1+u}\,\frac{\partial^2 v}{\partial z^2}, \tag{7}$$

$$u=0 \quad \text{at } z=0, \quad v=-\frac{\alpha u}{1+u} \quad \text{at } z=1, \tag{8}$$

where

$$\alpha=\frac{V_0}{s}, \quad \gamma=\frac{L}{s\tau_p}, \quad \kappa=\frac{K}{LsU_0}.$$

Note that the parameter α corresponds to the Mach number in hydrodynamics. From Eqs. (4) and (5), we obtain the expression for the dimensionless frequencies ω_n' and increments ω_n'',

$$\omega_n'=\frac{\Omega_n' L}{s}=\frac{\pi}{2}\,(2n-1), \quad \omega_n''=\frac{\Omega_n'' L}{s}=\alpha. \tag{9}$$

III. INCREMENT OF PLASMA OSCILLATIONS WITH CONSIDERATION FOR THE VISCOSITY AND THE EXTERNAL FRICTION

Let us estimate the order of magnitude of the parameters γ and κ, determining the external friction and viscosity. They can be represented in the forms $\gamma=L/s\tau_p$ and $\kappa=\lambda_{ee}V_F/Ls$. Here V_F is the Fermi electron velocity and λ_{ee} is the mean free path for electron-electron collisions. For the typical values $U_0\sim 0.5$ V, $n_S\sim 10^{12}$ cm^{-2}, and the temperature 77 K, we have[1] $\tau_p\sim 10^{-11}$ s, $s\sim 10^8$ cm/s, and $\lambda_{ee}V_F\sim \hbar/m$. Then, for $L\sim 0.2$ μm, we find $\gamma\approx 10^{-2}$ and $\kappa\approx 10^{-2}$. Therefore we shall assume below that $\gamma,\kappa \ll 1$. We shall also restrict ourselves to the most realistic case $\alpha \ll 1$.

The inclusion of the external friction presents no difficulties and leads to the reduction of the increment by the quantity $\gamma/2$:

$$\omega_n''=\alpha-\frac{\gamma}{2}.$$

In contrast, allowance for the viscosity of the electron fluid cannot be made in an easy way. For $\kappa\neq 0$, the differential equations (6) and (7) are of higher order and, consequently, require an additional boundary condition besides the conditions given by Eq. (3). This additional condition should be determined by the properties of the source and drain contacts, which have not been thoroughly studied.

We shall see below that, for $\kappa\neq 0$, a thin boundary layer occurs in the transistor channel as is often the case in problems with a small parameter before the highest derivative.[4] In our case, the boundary layer is located at the source and is of the width $\sim\alpha\kappa$. Beyond the layer, the term accounting for the viscous force in Eq. (7) is small and can be allowed for in the terms of the perturbation theory using the expansion in powers of small parameter κ. Within the layer, the solution cannot be represented as a series in powers of κ and essentially depends on the additional boundary condition. The boundary layer, however, is not important in calculations of the increment for a large body of additional boundary conditions. In particular, this is the case when an additional boundary condition is given at the source contact and allows uniform distributions of u and v in the stationary regime. This boundary condition may be written in the form

$$F\left(\frac{\partial U}{\partial x},\,\frac{\partial V}{\partial x},U,V\right)\bigg|_{x=0}=0, \tag{10}$$

where F is a function determined by the properties of the source contact. The uniformity of stationary solutions $U(x)=U_0$ and $V(x)=V_0$ imposes the following restriction on the F function:

$$F(0,0,U_0,V_0)|_{x=0}=0. \tag{11}$$

This equation is supposed to be valid for any U_0 and V_0.

Linearizing Eq. (10) with respect to small deviations from U_0 and V_0 and using Eq. (11), we have

$$C_1\,\frac{\partial u}{\partial z}+C_2\,\frac{\partial v}{\partial z}=0 \quad \text{at } z=0, \tag{12}$$

where C_1 and C_2 are functions of U_0 and V_0. On the other hand, it follows from Eqs. (6) and (8) that

$$\frac{\partial u}{\partial z}+\alpha\,\frac{\partial v}{\partial z}=0 \quad \text{at } z=0 \tag{13}$$

in the linear approximation. Then we find from Eqs. (12) and (13) that

$$\frac{\partial v}{\partial z}\bigg|_{z=0}=0 \tag{14}$$

at $C_2/C_1\neq\alpha$. The boundary condition (10) is thus reduced to Eq. (14) if Eq. (10) allows uniform stationary solutions.

The linear solutions of Eqs. (6) and (7) with boundary conditions (8) and (14) have the form

$$u=A\ \mathrm{Re}\left\{\left[(1-i\alpha\kappa\omega_n')\exp(ik_1 z)-(1+i\alpha\kappa\omega_n')\exp(ik_2 z)\right.\right.$$
$$\left.\left.+2i\alpha\kappa\omega_n'\exp\left(-\frac{z}{\alpha\kappa}\right)\right]\exp(-i\omega_n' t+\omega_n'' t)\right\}, \tag{15}$$

$$v = A\ \mathrm{Re}\left\{\left[\left(1-\frac{i\kappa\omega_n'}{2}-\frac{i\gamma}{2\omega_n'}\right)[\exp(ik_1z)+\exp(ik_2z)]\right.\right.$$
$$\left.\left.-2i\alpha^2\kappa\omega_n'\exp\left(-\frac{z}{\alpha\kappa}\right)\right]\exp(-i\omega_n't+\omega_n''t)\right\}, \qquad (16)$$

where A is the amplitude of plasma oscillations,

$$k_1=\omega_n'(1-\alpha)+\frac{i}{2}(\gamma+\kappa\omega_n'^2),$$
$$k_2=-\omega_n'(1+\alpha)-\frac{i}{2}(\gamma+\kappa\omega_n'^2), \qquad (17)$$

$$\omega_n'=\frac{\pi}{2}(2n-1), \qquad (18)$$

$$\omega_n''=\alpha-\frac{\gamma}{2}-\frac{\kappa\pi^2}{8}(2n-1)^2. \qquad (19)$$

In the smooth terms of Eqs. (15) and (16) containing $\exp(ik_1z)$ and $\exp(ik_2z)$, we have kept the main corrections with respect to powers of α, γ, and κ. The terms with $\exp(-z/\kappa\alpha)$, decreasing drastically along the channel, describe the boundary layer. Equation (19) gives the increment of plasma oscillations taking into account the viscosity-related damping. It is seen that instability takes place when $\alpha>\alpha_c$, where

$$\alpha_c=\frac{\gamma}{2}+\kappa\frac{\pi^2}{8}. \qquad (20)$$

It should be noted that Eq. (19) can also be obtained using the perturbation theory with respect to small parameter κ without additional boundary condition, the solution of Eqs. (6)–(8) with $\kappa=0$ being used as a first approximation. This is due to the fact that, as can be seen from the analysis of Eqs. (15) and (16), the viscosity-related energy dissipation is distributed over the entire channel length. The dissipation in the boundary layer is negligible and leads to the correction to the increment of the order of $\alpha\kappa^2$.

As another example let us consider the more general condition when the F function in Eq. (10) depends also on the drain current j. Using reasoning similar to that used in the derivation of Eq. (14), we obtain the additional boundary condition in the form

$$\left.\left(\frac{\partial v}{\partial z}+av\right)\right|_{z=0}=0, \qquad (21)$$

where a is a coefficient depending on α and U_0. Using Eqs. (6)–(8) and (21), one can show that there is a correction $-a\kappa$ to the increment in Eq. (19) related to the dissipation in the boundary layer. For $a\ll 1$, this dissipation is negligible. Then the increment is given by Eq. (19) and can be found by perturbation theory. This is, presumably, the case if α is small enough. Indeed, having solved Eqs. (6) and (7) with the boundary conditions (8) and (21), one can see that the electric field at the source contact $(\partial u/\partial z)|_{z=0}$ in the linear approximation is proportional to a/α. For low current ($\alpha\to 0$), this field tends to infinity if the coefficient a remains finite. This is apparently possible only for a special design of the source contact. We shall not consider here this special case and shall assume that $a\to 0$ if $\alpha\to 0$. Then the state of affairs under the conditions given by Eq. (21) does not differ essentially from the previous case when $\partial v/\partial z=0$ at $z=0$. Thus, for both of the cases considered, the dissipation in the boundary layer is negligible and the viscosity-related corrections to the increment come from the entire channel length. One can show that this is the case for a number of more general additional boundary conditions. The physical meaning of this fact can be understood from the following reasoning. For arbitrary additional boundary conditions, the dissipation in the boundary layer may be estimated as $j_{\omega_n'}(0)\Delta u_{\omega_n'}(0)$, where $j_{\omega_n'}(0)$ and $\Delta u_{\omega_n'}(0)$ are the current through the source contact and the voltage drop across the boundary layer at the plasma oscillation frequency ω_n', respectively. For $\Delta u_{\omega_n'}(0)$ we have $\Delta u_{\omega_n'}(0)\sim E_{\omega_n'}(0)\kappa\alpha$, where $\alpha\kappa$ is the boundary layer width and $E_{\omega_n'}(0)$ is the electric field at the source ($z=0$). The dissipation in the boundary layer is thus proportional to the small parameter $\alpha\kappa$ under arbitrary additional boundary condition if the field $E_{\omega_n'}(0)$ is finite.

In the subsequent calculations of the amplitude of stationary nonlinear oscillations we shall neglect the dissipation in the boundary layer and use the perturbation theory assuming the viscosity-related term in Eq. (7) to be small. As the first approximation, we shall use the solution of Eqs. (6) and (7) at $\kappa=\gamma=\alpha=0$.

IV. PHENOMENOLOGY OF THE NONLINEAR PROBLEM

In this section we consider phenomenologically the evolution of the instability for small enough values of the increment $\omega_1''=\alpha-\alpha_c$. We shall show that the instability leads to an establishment of small-amplitude stationary oscillations. We shall assume that the following condition is fulfilled:

$$\frac{\gamma}{2}+\frac{\kappa\pi^2}{8}=\alpha_c<\alpha<\frac{\gamma}{2}+\frac{9\kappa\pi^2}{8}. \qquad (22)$$

In this case, only the first mode with the frequency $\omega_1'=\pi/2$ is amplified in linear regime, while other modes are damped out. Then the first mode should also dominate in the spectrum of stationary nonlinear oscillations. Using the line of reasoning similar to that used in the theory of weak turbulence,[5] one can write the phenomenological equation

$$\frac{dA^2}{dt}=2(\alpha-\alpha_c)A^2-\beta A^4, \qquad (23)$$

which describes the evolution of the dimensionless amplitude A of the first mode. It is assumed here that α slightly exceeds the critical value α_c: $(\alpha-\alpha_c)/\alpha_c\ll 1$. The parameter β entering Eq. (23) is a coefficient depending on α, κ, and γ. For similar problems, this coefficient was calculated in a number of papers (see, for example, Ref. 6). We shall calculate β in Sec. V.

It is easy to see that β should be an odd function of α, κ, and γ. This follows from the fact that Eqs. (6)–(8) are invariant with respect to the time inversion and substitution $\alpha\to-\alpha$, $\gamma\to-\gamma$, and $\kappa\to-\kappa$. Then the first terms in the expansion of β in powers of small parameters α, γ, and κ

should be linear in α, γ, and κ. Taking into account that $\alpha\approx\alpha_c$ and using Eq. (20), we can write

$$\beta=\beta_1\gamma+\beta_2\kappa, \tag{24}$$

where β_1 and β_2 are unknown coefficients. Generally speaking, the sign of β can be arbitrary. However, as we shall see in the next section, β is positive in our case. Then, in accordance with Eq. (23), the amplitude of oscillations tends to a stationary value determined by the equation

$$2(\alpha-\alpha_c)A^2-\beta A^4=0. \tag{25}$$

Hence the amplitude of stationary oscillations is proportional to the square root of $\alpha-\alpha_c$.

V. CALCULATION OF THE AMPLITUDE OF STATIONARY OSCILLATIONS

We shall consider the situation when α exceeds its threshold value α_c by a small margin and the amplitude of the stationary oscillations

$$A=\left(\frac{2(\alpha-\alpha_c)}{\beta}\right)^{1/2}$$

is much less than unity. Therefore, we can regard A, along with α, γ, and κ, as a small parameter. While in the nonlinear problem the first mode generates oscillations of multiple frequencies, their amplitudes are small in powers of A. This allows us to take into account only a limited number of multiple frequencies and write a closed system of equations for their amplitudes. The amplitude A of oscillations at the fundamental frequency can be found by solving this system with the use of an expansion in powers of A. This treatment requires, however, cumbersome calculations. We show that it can be considerably simplified by using the relationship

$$\alpha[\langle v^2(0,\tau)\rangle-\langle u^3(1,\tau)\rangle+\tfrac{3}{4}\langle u^4(1,\tau)\rangle]$$
$$=\gamma\int_0^1 dz(\langle v^2\rangle-2\langle v^2u\rangle)+\kappa\int_0^1 dz\left(-\left\langle v\,\frac{\partial^2 v}{\partial z^2}\right\rangle\right.$$
$$\left.+3\left\langle vu\,\frac{\partial^2 v}{\partial z^2}\right\rangle-3\left\langle vu^2\,\frac{\partial^2 v}{\partial z^2}\right\rangle\right), \tag{26}$$

which is valid in the case of stationary oscillations and may be derived from Eqs. (6)–(8) (see the Appendix). Here the angular brackets denote averaging over the period of oscillations. In deriving this relationship we kept only the terms linear in small parameters α, γ, and κ.

Since all the terms in Eq. (26) are proportional to α, γ, or κ, one should put there u and v found from Eqs. (6)–(8) at $\alpha=\gamma=\kappa=0$. In so doing, finding u and v requires no additional boundary condition. These u and v are close to the solutions obtained for small nonzero κ, α, and γ. At the same time, in the boundary layer, the quantity $\partial^2 v/\partial z^2$ entering in Eq. (26) is large and differs from its value found at $\alpha=\kappa=\gamma=0$. It is, however, easy to show that the contribution of this layer to Eq. (26) is negligible. Indeed, putting Eqs. (15) and (16) (for $n=1$) at the generation threshold $\omega_1''=0$ ($\alpha=\alpha_c$) in Eq. (26), one can see that the contribution of the boundary layer is of the order of $\alpha^2\kappa$. Allowance for the nonlinear correction to Eqs. (15) and (16) leads to negligible terms containing higher powers of α, κ, and γ. In order to find the amplitude of the stationary oscillation we have to expand all the terms in Eq. (26) in powers of A. Then we obtain an equation for A that is equivalent to the phenomenological Eq. (25) and find the coefficient β.

In the linear approximation and for $\alpha=\gamma=\kappa=0$, we have for the first mode

$$v=A\cos\left(\frac{\pi}{2}z\right)\sin(\omega\tau),$$
$$u=-A\sin\left(\frac{\pi}{2}z\right)\cos(\omega\tau), \tag{27}$$
$$\omega=\omega_1'=\frac{\pi}{2}.$$

Putting this solution in Eq. (26) and keeping only the terms square in the amplitude, we obtain

$$(\alpha-\alpha_c)A^2=0.$$

This means that, as expected, in the linear approximation, the stationary oscillations are possible only at $\alpha=\alpha_c$.

In the nonlinear case, we shall seek the solution of Eqs. (6)–(8) at $\kappa=\alpha=\gamma=0$ in the form of the Fourier series

$$v=\tfrac{1}{2}[v_0(z)+v_1(z)e^{i\omega\tau}+v_2(z)e^{2i\omega\tau}+\cdots+\text{c.c.}],$$
$$u=\tfrac{1}{2}[u_0(z)+u_1(z)e^{i\omega\tau}+u_2(z)e^{2i\omega\tau}+\cdots+\text{c.c.}]. \tag{28}$$

Putting Eq. (28) in Eqs. (6)–(8), one can easily obtain the infinite system of equations for the unknown functions $v_i(z)$, $u_i(z)$, and the frequency ω. One should use the linear solution (27) as a first approximation and expand $v_i(z)$, $u_i(z)$, and ω in powers of the amplitude A. In so doing, only the leading terms, proportional to A^2, should be kept in $v_0(z)$, $u_0(z)$, $v_2(z)$, $u_2(z)$, and in the frequency shift $\omega-\omega_1'$, while in $v_1(z)$ and $u_1(z)$ one should keep the cubic in A corrections to Eq. (27). Taking into account the higher-order corrections and the higher terms of the Fourier series (28) is not required since this would lead to terms containing the sixth or higher powers of A in Eq. (25). On this basis, it is easy to obtain the closed system of equations, which allows us to find functions $v_0(z)$, $u_0(z)$, $v_1(z)$, $u_1(z)$, $v_2(z)$, and $u_2(z)$ at $\alpha=\gamma=\kappa=0$:

$$\frac{\partial v_0}{\partial z}=-\frac{1}{4}\frac{\partial}{\partial z}(v_1u_1^*+v_1^*u_1),$$
$$\frac{\partial u_0}{\partial z}=-\frac{1}{4}\frac{\partial}{\partial z}(v_1v_1^*), \tag{29}$$
$$u_0(0)=v_0(1)=0;$$

$$\frac{\partial u_1}{\partial z}+i\omega v_1=-\frac{\partial}{\partial z}\left(v_0v_1+\frac{v_2v_1^*}{2}\right),$$
$$\frac{\partial v_1}{\partial z}+i\omega u_1=-\frac{\partial}{\partial z}\left(v_0u_1+u_0v_1+\frac{v_2u_1^*+u_2v_1^*}{2}\right), \tag{30}$$

$$u_1(0)=v_1(1)=0;$$

$$\frac{\partial u_2}{\partial z}+2i\omega v_2=-\frac{1}{4}\frac{\partial}{\partial z}(v_1^2),$$

$$\frac{\partial v_2}{\partial z}+2i\omega u_2=-\frac{1}{4}\frac{\partial}{\partial z}(v_1 u_1), \tag{31}$$

$$u_2(0)=v_2(1)=0.$$

Their solution can be found by the successive approximations using the expansion in powers of A. In doing so, one should hold only those terms which make contributions to Eq. (26) containing A in powers not higher than the fourth. As a first approximation, one should use again the linear solutions (27). Then we find

$$v_0(z)=0, \quad u_0(z)=\frac{A^2}{4}\sin^2\left(\frac{\pi z}{2}\right);$$

$$v_2(z)=-i\frac{A^2}{16}[\sin(\pi z)-3\pi(1-z)\cos(\pi z)],$$

$$u_2(z)=\frac{3\pi A^2}{16}(1-z)\sin(\pi z);$$

$$\begin{aligned} v_1(z)=&-iA\cos\left(\frac{\pi z}{2}\right)\\ &+iA^3\left[z\sin\left(\frac{\pi z}{2}\right)\mathrm{Im}\int_0^1 e^{i(\pi z'/2)}\frac{\partial f}{\partial z'}dz'\right.\\ &\left.-\mathrm{Re}\int_0^z e^{i(\pi/2)(z'-z)}\frac{\partial f}{\partial z'}dz'\right], \end{aligned} \tag{32}$$

$$\begin{aligned} u_1(z)=&-A\sin\left(\frac{\pi z}{2}\right)-A^3\left[\cos\left(\frac{\pi z}{2}\right)\mathrm{Im}\int_0^1 e^{i(\pi z'/2)}\frac{\partial f}{\partial z'}dz'\right.\\ &\left.-\mathrm{Im}\int_0^z e^{i(\pi/2)(z'-z)}\frac{\partial f}{\partial z'}dz'\right]; \end{aligned}$$

$$\begin{aligned} f=\frac{1}{8}\Bigg\{&\frac{3\pi}{4}(1-z)\left[\sin\left(\frac{\pi z}{2}\right)+i\cos(\pi z)\cos\left(\frac{\pi z}{2}\right)\right]\\ &-\frac{1}{4}\sin(\pi z)\left[3\sin\left(\frac{\pi z}{2}\right)-i\cos\left(\frac{\pi z}{2}\right)\right]\Bigg\}. \end{aligned}$$

At the same time, we obtain the following expression for the nonlinear frequency shift of the first mode:

$$\omega-\omega_1'=A^2\,\mathrm{Im}\int_0^1 dz'\,e^{i(\pi/2)z'}\frac{\partial f}{\partial z'}=-\frac{53\pi}{512}A^2. \tag{33}$$

Inserting these expressions into Eq. (26), after laborious calculations, we obtain the following equation, analogous to the phenomenological expression (25):

$$2\left(\alpha-\frac{\gamma}{2}-\frac{\kappa\pi^2}{8}\right)A^2-\frac{A^4}{64}\left[\frac{57\gamma}{8}+\frac{\kappa\pi^2}{4}\left(\frac{9\pi^2}{4}+112\right)\right]=0.$$

From this formula we find the amplitude of stationary nonlinear oscillations

$$A=8\left[\frac{2(\alpha-\alpha_c)}{\frac{57\gamma}{8}+\left[\frac{9\pi^2}{4}+112\right]\frac{\pi^2\kappa}{4}}\right]^{1/2}. \tag{34}$$

Note that the numerical factor before κ is much greater than the one before γ. Therefore, if γ and κ have the same order of magnitude, the amplitude A at a given value of $\alpha-\alpha_c$ is predominantly determined by the viscosity κ. At the same time, the value of α_c itself depends on κ and γ to the same extent.

The presence of stationary oscillations above the instability threshold should result in peculiarities of the voltage-current characteristics of the transistor. Particularly, the differential resistance at zero frequency

$$R_0=\frac{dU_{sd}}{dj}, \quad U_{sd}=U_0[u_0(L)-u_0(0)] \tag{35}$$

should have a step when the drain current density j reaches its threshold value $j_c=CU_0 s\alpha_c$, following from the condition $\alpha=\alpha_c$. Here U_{sd} is the drain-to-source voltage at zero frequency. One can see from Eqs. (32), (34), and (35) that $R_0=0$ when $j<j_c$ and

$$R_0=\frac{1}{Cs}\frac{32}{\frac{57\gamma}{8}+\frac{\kappa\pi^2}{4}\left[\frac{9\pi^2}{4}+112\right]} \tag{36}$$

when $j>j_c$.

Note that zero values for U_{sd} and R_0 below the threshold results from our neglecting the terms proportional to α, γ, and κ in Eqs. (32). Actually, there is a finite differential resistance at $j<j_c$ that is, however, negligibly small compared to the differential resistance over threshold. As one can easily show, the differential resistance under the threshold at $\kappa=0$ is determined by

$$R_0=\frac{\gamma}{Cs}.$$

This resistance is less than the resistance above the threshold by a factor $\gamma^2\sim 10^{-4}$.

Let us discuss the condition of the validity of our results. Usually equations like Eq. (23) hold when $A\ll 1$. However, in our case, the criterion is stronger. Indeed, the nonlinearity caused third harmonic of the first mode is in resonance with the second mode [$n=2$ in Eq. (18)], which is neglected in our calculations. Actually, the amplitude of the second mode, containing the resonant denominator, is of order A^3/κ. Its feedback impact on the fundamental first mode leads to additional terms of order A^5/κ^2. These terms were neglected compared to A^3 terms in Eq. (32). Therefore, our results [Eqs. (34) and (36)] are valid under the condition

$$A\ll\kappa. \tag{37}$$

What happens when $A>\kappa$ is rather complicated. Let us discuss this problem qualitatively. The amplitude of the second mode is small compared to the amplitude of the first mode when

$$A\ll\kappa^{1/2}.$$

This means that there exists an interval of amplitude values

$$\kappa \ll A \ll \kappa^{1/2} \quad (38)$$

in which the first mode predominates in the oscillation spectrum but the resonant influence of the second mode on the first mode is not small. This influence can be phenomenologically taken into account as follows. The right-hand side of Eq. (23) presents an expansion in powers of A^2. The next term in this expansion should be proportional to A^6/κ. The factor κ^{-1} accounts for the resonant nature of the interaction between modes. Then we have

$$\frac{dA^2}{dt} = 2(\alpha - \alpha_c)A^2 - \beta A^4 - \eta \frac{A^6}{\kappa}, \quad (39)$$

where η is an unknown numerical coefficient.

A rigorous treatment of the influence of the second mode requires keeping the terms $v_3(z)e^{3i\omega\tau}$ and $u_3(z)e^{3i\omega\tau}$ in the series (28), which were neglected in our calculations. However, as one can show, the terms oscillating at higher frequencies (the third, fourth, and higher modes) still are negligible at $A \ll \kappa^{1/2}$. The coefficient η can be found by solving the system of equations that are similar to Eqs. (29)–(31) and contain the functions $v_3(z), u_3(z)$. This calculation is, however, extremely cumbersome and is beyond the scope of this paper. Assuming that η is positive, we obtain the following expression for the amplitude of stationary oscillations:

$$A = \left[\frac{2(\alpha - \alpha_c)\kappa}{\eta}\right]^{1/4}. \quad (40)$$

This expression is valid in the interval given by inequalities (38). Using Eq. (40), these inequalities may be rewritten in the form

$$\kappa^3 \ll \alpha - \alpha_c \ll \kappa. \quad (41)$$

Thus the increase of the amplitude A with the current slows down because of the energy dissipation in the second mode.

VI. CONCLUSION

We have studied the consequences of the current instability in a ballistic field-effect transistor. We have shown that, when the current exceeds its threshold value by a small enough margin (when $\alpha - \alpha_c < \kappa$), the instability leads to the establishment of stationary nonlinear oscillations. The amplitude of oscillations has been calculated. For $\alpha - \alpha_c < \kappa^3$, the amplitude is proportional to the square root of the increment: $A \sim (\alpha - \alpha_c)^{1/2}$ [see Eq. (34)]. In the interval $\kappa^3 \ll \alpha - \alpha_c \ll \kappa$, the dependence of the amplitude on the current slows down: $A \sim (\alpha - \alpha_c)^{1/4}$ [see Eq. (40)]. For higher current values ($\alpha - \alpha_c \gg \kappa$), the instability may lead to the formation of shock waves with steplike distributions of the electron concentration and velocity.

We have predicted that the instability should result in peculiarities in the current-voltage characteristic of the transistor. Particularly, the differential resistance at zero frequency should have a large step at the instability threshold. The resistance value above the threshold is larger than its value under the threshold by a factor of $\gamma^{-2} \sim 10^4$.

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APPENDIX

Adding Eq. (6) multiplied by $3\ln(1+u) + v^2 - 2u + \alpha v$ and Eq. (7) multiplied by $v - 2uv + \alpha u$ and neglecting terms containing α, γ and κ in powers higher than the first, we obtain

$$\frac{\partial}{\partial\tau}\left[\frac{v^2}{2} - 3u + 3(1+u)\ln(1+u) - uv^2 - u^2 + \alpha uv\right] + \frac{\partial}{\partial z}\left[-3\alpha u - uv^3 - 2uv - \frac{\alpha u^2}{2} - 2u^2 v + \alpha v^2 + 3(\alpha + v)(1+u)\ln(1+u)\right] = \kappa \frac{(1-2u)v}{1+u}\frac{\partial^2 v}{\partial x^2} + \gamma v^2(2u-1). \quad (A1)$$

After averaging this equation over the time of the oscillation period, the first term (with $\partial/\partial\tau$) vanishes. Then we integrate this expression over the transistor channel length (from $z=0$ to 1) using boundary conditions (8) and expanding $\ln(1+u)$ and $(1+u)^{-1}$ in powers of u. Keeping there the terms proportional to A in powers not higher than the fourth and neglecting terms, containing α, γ, and κ in powers higher than the first, we obtain Eq. (26).

[1] M. I. Dyakonov and M. S. Shur, Phys. Rev. Lett. **71**, 2465 (1993).
[2] B. L. Gelmont (unpublished).
[3] M. S. Shur, *Physics of Semiconductor Devices* (Prentice-Hall, Englewood Cliffs, NJ, 1990).
[4] Julian D. Cole, *Perturbation Methods in Applied Mathematics* (Blaisdell, London, 1986).
[5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1966).
[6] M. Golubitsky and W. F. Langford, J. Diff. Eq. **41**, 375 (1981).