

Spectral flow, Magnus force, and mutual friction via the geometric optics limit of Andreev reflection

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The notion of spectral flow has given new insight into the motion of vortices in superfluids and superconductors. For a BCS superconductor the spectrum of low energy vortex core states is largely determined by the geometric optics limit of Andreev reflection. We use this to follow the evolution of the states when a stationary vortex is immersed in a transport supercurrent. If the core spectrum were continuous, spectral flow would convert the momentum flowing into the core via the Magnus effect into unbound quasiparticles—thus allowing the vortex to remain stationary without a pinning potential or other sink for the inflowing momentum. The discrete nature of the states, however, leads to Bloch oscillations which thwart the spectral flow. The momentum can escape only via relaxation processes. Taking these into account permits a physically transparent derivation of the mutual friction coefficients. [S0163-1829(96)08842-X]

I. INTRODUCTION

Imagine a two-dimensional superfluid, initially in its ground state, confined to the surface of a torus. Suppose now that a vortex-antivortex pair is created at some point on the surface and the vortex is moved slowly round one of the generators of the torus before being allowed to annihilate with its antivortex partner. One effect of this process is to give the superfluid order-parameter phase a unit winding number around the generator perpendicular to the motion of the vortex. The associated phase gradient implies that a supercurrent has been established in this direction. If no other momentum-carrying excitations were created along with the supercurrent, the system as a whole has acquired momentum perpendicular to the vortex motion. Moving the vortex therefore requires us to supply this momentum from an external source. This is the Magnus effect.¹

For a Bose superfluid this is all there is to the story: If we wish to move a vortex with respect to the background fluid we must (at least at low temperatures when there is no normal fluid component) place a wire or other object in the vortex core to supply the transverse momentum to the fluid. The reaction force the fluid exerts on the wire is the Magnus, or Kutta-Joukowski, lift force.²

For a fermionic *S*-wave superfluid (we consider a neutral condensate for simplicity; the principal effect of the magnetic field in an Abrikosov vortex is to transfer the momentum supplied by the vortex to the positive ion lattice, thus ensuring that no superflow is induced beyond the penetration depth; nothing significant changes in the core) the situation is subtler because the vortex has low-energy bound states^{3,4} whose role in the momentum balance equation has been studied for many years.⁵⁻⁷ Recently Volovik⁸ has cast a new light on this subject by showing that motion of the vortex with respect to the stationary condensate induces a spectral flow among these states. In a cartoon version of his theory this spectral flow generates, even with adiabatic motion of the vortex, a stream of unbound quasiparticles which carry off momentum equal and opposite to that of the induced

superflow. Therefore the vortex can apparently be moved without any external source of transverse momentum. In this sense the spectral flow “cancels” the Magnus effect. The discrete nature of the bound state spectrum, however, complicates the picture. As observed in Ref. 8 and modeled in Ref. 9, a nonzero temperature is required to broaden the closely spaced levels so that they may behave as if the spectrum were continuous. In the hydrodynamic limit a “cancellation” of sorts still takes place, but only in the sense that all the incoming momentum is immediately transferred to the positive ion lattice.

Despite the complicating necessity of level broadening, the spectral flow mechanism provides a very physical picture of the processes occurring in the vortex core. Several questions immediately arise, however. For example: What happens when a vortex is held stationary in a transport supercurrent? By Galilean invariance, this situation is physically equivalent to a moving vortex and a stationary superfluid, yet—with no time dependence in the Bogoliubov–de Gennes equations—it is not immediately clear what drives the spectral flow. A second question is whether the spectral flow picture requires a modification of the conventional theories of momentum balance and Hall angle. The aim of the present paper is to discuss these issues within a simple model for the core states.

We will introduce a quasiclassical picture, based on the geometric optics limit of Andreev scattering,¹⁰ for the evolution of the states in a stationary vortex core. This allows us to show that the discreteness of the spectrum leads to effects analogous to those in the one-dimensional Wannier-Stark ladder. Stark ladder resonances occur when a uniform electric field is applied to a Bloch electron in a periodic potential.¹¹ The electron initially accelerates but, in the absence of dissipation, is eventually Bragg reflected from the periodic lattice potential resulting in an oscillatory motion in a localized region. No net current flows—except that arising from the exponentially small interband Zener tunneling. If we describe this process in a gauge where $A_0=0$, $A_x=-Et$ we have a time-dependent Hamiltonian and explicit spectral flow. In a gauge where $A_0=-Ex$, $A_x=0$ there is no explicit

time dependence but the same physics results. In both gauges dissipation via inelastic collisions allows the electron to avoid Bragg reflection and so permits a finite current. Analogously, relaxation processes in the vortex core allows some spectral evolution. By keeping track of the resultant momentum flux we find the consequences of the spectral flow for the vortex dynamics. These turn out to be the well-known mutual friction that couples the superflow to the normal flow via the vortex motion. The traditional Green function formalism of Refs. 6 and 7 must therefore tacitly take the spectral flow into account.

The organization of this paper is as follows. First, in Sec. II, we exhibit a simple version of spectral flow and show how momentum entering the vortex core is recycled as quasiparticle momentum. In Sec. III we will review the theory of the core states and its connection with Andreev reflection. In Sec. IV we interpret the core state spectrum in terms of the failure of exact Andreev retroreflection and, in Sec. V, armed with the insight gained from this interpretation, we show how the spectral flow is mapped onto the Stark-Wannier problem. Finally, also in Sec. V, we account for the momentum flux to the normal component.

II. ONE-DIMENSIONAL SPECTRAL FLOW

As a simple model of a vortex core consider the following one-dimensional Bogoliubov–de Gennes eigenvalue problem in the Andreev approximation:

$$\begin{bmatrix} -iv_f\partial_x & \Delta(x)e^{i\theta(x)} \\ \Delta(x)e^{-i\theta(x)} & iv_f\partial_x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon \begin{bmatrix} u \\ v \end{bmatrix}. \quad (2.1)$$

Here $\Delta(x)=0$ for $0 < x < L$ (the ‘‘core’’) while $\Delta(x)=\Delta = \text{const}$ elsewhere. We take the phase of the order parameter to be $\theta(x)=\theta_L$ for $x < 0$ and $\theta(x)=\theta_R$ for $x > L$. We will use m to denote the fermion mass so that $k_f = mv_f$ is the Fermi momentum and $E_f = \frac{1}{2}mv_f^2$ is the Fermi energy.

The bound-state solutions, $\Psi = \begin{bmatrix} u \\ v \end{bmatrix}$, with $\epsilon < \Delta$ are easily found. The wave functions are of the form

$$\begin{aligned} \Psi(x) &= \begin{bmatrix} \epsilon + ikv_f \\ \Delta e^{-i\theta_R} \end{bmatrix} e^{-k(x-L)}, & x > L, \\ &= \begin{bmatrix} ae^{i\epsilon x/v_f} \\ be^{-i\epsilon x/v_f} \end{bmatrix}, & 0 < x < L, \\ &= \begin{bmatrix} \epsilon - ikv_f \\ \Delta e^{-i\theta_L} \end{bmatrix} e^{kx}, & x < 0, \end{aligned} \quad (2.2)$$

with $\epsilon^2 + (v_f k)^2 = \Delta^2$. Matching the solutions at $x=0$, L fixes the ratio a/b and requires the eigenvalue ϵ_n to obey

$$\epsilon_n = \frac{v_f}{2L} ((\theta_R - \theta_L) + 2\pi n + 2 \cos^{-1}(\epsilon_n/\Delta)). \quad (2.3)$$

For states deep in the gap, $\epsilon \ll \Delta$, this simplifies to

$$\epsilon_n = \frac{v_f}{2L} ((\theta_R - \theta_L) + 2\pi(n + \frac{1}{2})). \quad (2.4)$$

We see that if we gradually increase the phase difference across the core, $\Delta\theta = \theta_R - \theta_L$, the entire spectrum moves up in energy. By the time $\Delta\theta$ has increased by 2π each state has been replaced by the one below it. This is the spectral flow.

The physical interpretation of the spectral flow depends on the context. If (2.1) were describing a charge density wave (CDW) system, the upper and lower components of Ψ would be the amplitude of left- and right-going particles. Then a summation over occupied states gives the local charge density and current

$$\begin{aligned} \langle \Psi^\dagger(x) \Psi(x) \rangle &= \langle \psi_R^\dagger(x) \psi_R(x) + \psi_L^\dagger(x) \psi_L(x) \rangle = \langle \rho(x) \rangle, \\ v_f \langle \Psi^\dagger(x) \sigma_3 \Psi(x) \rangle &= v_f \langle (\psi_R^\dagger(x) \psi_R(x) - \psi_L^\dagger(x) \psi_L(x)) \rangle \\ &= \langle j(x) \rangle. \end{aligned} \quad (2.5)$$

In a CDW a time rate of change of the phase of the order parameter induces a current $\langle j \rangle \approx (1/2\pi) \dot{\theta}$, with the corrections being small when $\dot{\theta}$ is small. Consequently the slow twisting of θ_R relative to θ_L tells us that charge is flowing into the the gapless region $0 < x < L$. Since the time-dependent version of (2.1) implies that ρ and j obey the conservation law

$$\partial_t \rho(x) + \partial_x j(x) = 0, \quad (2.6)$$

the inflowing charge must be accumulating in the gapless region.¹² Each time the relative twist increases by 2π , a unit charge will have accumulated. In the same interval one of the (occupied) negative energy bound state levels has adiabatically crossed the zero energy level and taken the place of a positive energy state. The occupation number of the positive energy bound states has therefore increased by unity, consistent with the accumulation of unit charge. Eventually, with more twisting (the amount depending on L), the filled levels will reach the top of the gap and merge with the upper continuum. After this point each new unit of charge that flows in will appear as a low energy quasiparticle.

In a superconductor the upper and lower components of Ψ are ψ_R , ψ_L^\dagger , respectively. In this case the expressions for the current and charge density are interchanged

$$\begin{aligned} v_f \langle \Psi^\dagger(x) \Psi(x) \rangle &= v_f \langle \psi_R^\dagger(x) \psi_R(x) + \psi_L(x) \psi_L^\dagger(x) \rangle \\ &= v_f \langle \psi_R^\dagger(x) \psi_R(x) - \psi_L^\dagger(x) \psi_L(x) \rangle = \langle j(x) \rangle, \end{aligned} \quad (2.7a)$$

$$\begin{aligned} \Psi^\dagger(x) \sigma_3 \Psi(x) &= \langle \psi_R^\dagger(x) \psi_R(x) - \psi_L(x) \psi_L^\dagger(x) \rangle \\ &= \langle \psi_R^\dagger(x) \psi_R(x) + \psi_L^\dagger(x) \psi_L(x) \rangle = \langle \rho(x) \rangle. \end{aligned} \quad (2.7b)$$

The conservation law

$$\begin{aligned} \partial_t (\psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L) + v_f \partial_x (\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L) \\ = \partial_t \Psi^\dagger \Psi + v_f \partial_x \Psi^\dagger \sigma_3 \Psi = 0, \end{aligned} \quad (2.8)$$

also changes its physical interpretation as, on multiplication by k_f , it becomes the equation of momentum conservation.¹³ Now, instead of charge, each occupied bound state carries momentum $+k_f$. The relative twisting of the phases on the two sides of the core represents an inflow of momentum from the condensate, and the spectral flow leads to its recycling as the momentum of low energy quasiparticles. (This

one-dimensional spectral flow has consequences for the flow of vortices in Josephson junctions.^{15,16)}

We can make a simplistic model of the consequences of two-dimensional vortex motion by assuming that (to a first approximation) the process described in the Introduction, the passage of a vortex around the L_y generator of an $L_x \times L_y$ torus, can be mimicked by the breaking and reconnection after a 2π phase twist of the order parameter in a collection of one-dimensional superfluids, one for each allowed $k_y = 2\pi n/L_y$. From the discussion following (2.8) we see that a single twist accumulates a momentum $k_x = \sqrt{|k_f|^2 - k_y^2}$ for each of the one-dimensional systems. This is the same amount of momentum we would get by translating the k_x value of each particle on the torus by $\delta k_x = \pi/L_x$. If the total number of electrons is N , the net momentum accumulated in a passage of a vortex around the L_y generator is therefore

$$\Delta P_x = \frac{\pi N}{L_x} = \frac{\pi \rho}{m} L_y, \quad (2.9)$$

where $\rho = mN/(L_x L_y)$ is the mass density.

This implies a rate of momentum accumulation in the core of

$$\frac{dP_x}{dt} = \frac{\pi}{m} \rho v_y.$$

Since the circulation in a BCS vortex is $\kappa = \pi/m$ we see that momentum is accumulating, or being recycled, at a rate equal to the Magnus force on the vortex, $F_x = \kappa \rho v_y$.

This cartoon version of the process is of course overly simplistic. The bound states in the two-dimensional vortex core are not those of the one-dimensional equation (although we will soon see that they are closely related), and the variation of the order parameter in the y direction will couple the

k_y momenta so that they cannot be dealt with individually. In the next section we will begin to deal with these deficiencies.

III. TWO-DIMENSIONAL BOUND STATES

In this section we will review the classical results of Refs. 3 and 4 on bound states in the core of a two-dimensional vortex. Our aim is to show that the physics of Andreev scattering reduces the full problem to a collection of one-dimensional problems of the form considered in the previous section.

We wish to solve the Bogoliubov–de Gennes equation

$$\begin{bmatrix} -\frac{1}{2m} \nabla^2 - E_f & \Delta(r) e^{i\theta} \\ \Delta(r) e^{-i\theta} & \frac{1}{2m} \nabla^2 + E_f \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \epsilon \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \quad (3.1)$$

Here r and θ are polar coordinates with origin at the vortex center. For the moment we will leave the gap profile $\Delta(r)$ unspecified, but the angular dependence of the order parameter is such that the superflow is *anticlockwise* with a single quantum of circulation.

We now separate the radial and angular parts of the wave function. To do this we must first appreciate that u, v are invariant only under 4π rotations. (The same is true for the one-dimensional problem in Sec. II. There we saw that a 2π twist in the order parameter shifts the particle momenta by $\delta k_x = \pi/L_x$. If there were no quasiparticle created along with the twist this would lead to a double valued many-body wave function. Fortunately the x dependence of the quasiparticle serves to restore the single valuedness to the total wave function.) Therefore we seek solutions in the form

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} u(r, \theta) e^{i\theta/2} \\ v(r, \theta) e^{-i\theta/2} \end{bmatrix} = \begin{bmatrix} u_l(r) e^{i\theta/2} \\ v_l(r) e^{-i\theta/2} \end{bmatrix} e^{il\theta}, \quad l \in \mathbf{Z}. \quad (3.2)$$

We find that $u_l(r), v_l(r)$ obey

$$\begin{bmatrix} -\frac{1}{2m} \left(\partial_{rr}^2 + \frac{1}{r} \partial_r - \frac{(l+1/2)^2}{r^2} + k_f^2 \right) & \Delta \\ \Delta & \frac{1}{2m} \left(\partial_{rr}^2 + \frac{1}{r} \partial_r - \frac{(l-1/2)^2}{r^2} + k_f^2 \right) \end{bmatrix} \begin{bmatrix} u_l \\ v_l \end{bmatrix} = \epsilon \begin{bmatrix} u_l \\ v_l \end{bmatrix}. \quad (3.3)$$

To separate the rapidly varying degrees of freedom from the slow, we follow Ref. 3 and write

$$\begin{bmatrix} u_l \\ v_l \end{bmatrix} = \begin{bmatrix} f_l \\ g_l \end{bmatrix} H_m^{(1)}(k_f r) + \text{c.c.}, \quad (3.4)$$

where $H_m^{(1)}(k_f r)$ with $m = \sqrt{l^2 + 1/4}$ is a Hankel function, and f_l, g_l are slowly varying envelope functions. We may approximate the Hankel function by the WKB form

$$H_m^{(1)}(k_f r) \sim \frac{\text{const}}{(r^2 - b^2)^{1/4}} \exp \left\{ ik_f \int_b^r \frac{\sqrt{r^2 - b^2}}{r} dr \right\}, \quad (3.5)$$

which is valid for r larger than the WKB turning point $r = b$. This turning point is related to k_f and l by $k_f b = l$.

Substituting (3.5) into (3.3) and keeping only the large terms we find

$$\begin{bmatrix} -iv_f \frac{\sqrt{r^2 - b^2}}{r} \partial_r & \Delta \\ \Delta & iv_f \frac{\sqrt{r^2 - b^2}}{r} \partial_r \end{bmatrix} \begin{bmatrix} f_l \\ g_l \end{bmatrix} = \left(\epsilon - \frac{l}{2mr^2} \right) \begin{bmatrix} f_l \\ g_l \end{bmatrix}. \quad (3.6)$$

We can make (3.6) more intelligible by trading r for a variable x defined by $r^2 = x^2 + b^2$.⁴ We find

$$\begin{bmatrix} -iv_f \partial_x & \Delta \\ \Delta & iv_f \partial_x \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \left(\epsilon - \frac{l}{2m(x^2 + b^2)} \right) \begin{bmatrix} f \\ g \end{bmatrix}. \quad (3.7)$$

This substitution has a geometric meaning: For any Bessel function $J_l(kr)$ the turning point distance, b , is the impact parameter of a particle moving past the origin in a straight line with momentum k and angular momentum l . The quantity x is the distance of the particle from the point of closest approach to the origin. Along with x we introduce a polar angle “ θ ” by $x = b \tan \theta$. This angle differs only by a constant from the true polar angle θ , so we temporarily abuse notation and make no distinction between the new angle and the original. We then notice that

$$\frac{l}{2m(x^2 + b^2)} = \frac{1}{2} v_f \frac{d\theta}{dx} \quad (3.8)$$

so that, defining,

$$\begin{bmatrix} \tilde{f}_l \\ \tilde{g}_l \end{bmatrix} = \begin{bmatrix} f_l e^{i\theta/2} \\ g_l e^{-i\theta/2} \end{bmatrix} \quad (3.9)$$

to undo the transformation that removed the angle dependence from the order parameter in (3.1), we get

$$\begin{bmatrix} -iv_f \partial_x & \Delta e^{i\theta} \\ \Delta e^{-i\theta} & iv_f \partial_x \end{bmatrix} \begin{bmatrix} \tilde{f}_l \\ \tilde{g}_l \end{bmatrix} = \epsilon \begin{bmatrix} \tilde{f}_l \\ \tilde{g}_l \end{bmatrix}. \quad (3.10)$$

This looks very much like the one-dimensional eigenvalue problem solved in Sec. II. It is not yet identical, however, In (3.10) the coordinate $x = \sqrt{r^2 - b^2}$ is restricted to positive values. Furthermore boundary conditions have to be imposed on \tilde{f}_l, \tilde{g}_l at $x=0$ to ensure that the Hankel functions can combine to give the $J_{l \pm 1/2}(kr) = 1/2(H_{l \pm 1/2}^{(1)}(kr) + H_{l \pm 1/2}^{(2)}(kr))$ which are finite at the origin. We may nonetheless extend x to negative values by regarding the part of (3.4) with the incoming Hankel function $H_m^{(2)}(k_f r)$ as living on the negative x axis, and the outgoing part $H_m^{(1)}(k_f r)$ on the positive x axis. With this interpretation the boundary conditions at $x=0$ translate into the requirement of continuity of \tilde{f}_l, \tilde{g}_l there.

Physically the transformation of the two-dimensional eigenvalue problem (3.1) into the one-dimensional (3.10) occurs because each bound quasiparticle is bouncing back and forth along a straight line, its direction of motion being repeatedly reversed by Andreev scattering off the increasing value of the gap.

The $\Delta(r)$ profile found from a self-consistent solution of the Bogoliubov–de Gennes and gap equations has scale $R \approx v_f/\Delta$.¹⁴ In order to have analytic expressions for the bound state eigenvalues, we will not use such a self-consistent $\Delta(r)$, but take instead a step function: $\Delta(r)=0, r < R$ and $\Delta(r)=\Delta$ for $r > R$. We will also assume R to be somewhat larger than v_f/Δ so that we can ignore, in (3.10), the variation of θ in the regions where the wave function is evanescent. With these assumptions we can directly apply (2.3) and (2.4).

Each quasiparticle trajectory now coincides with a chord of a circle of radius R . If this chord subtends an angle of 2χ at the center of the circle it has length $L=2R \sin\chi$ (see Fig. 1). Furthermore the quantity $\Delta\theta = \theta_L - \theta_R$, the difference in

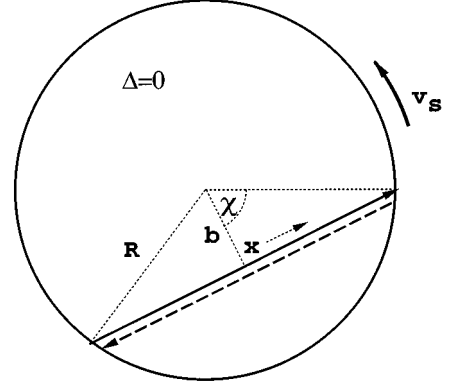


FIG. 1. A bound state with an electron (solid arrow) being Andreev reflected as a hole (dashed arrow).

order parameter phase at the two ends, is equal to 2χ . For states with energy $\epsilon \ll \Delta$ the energy eigenvalues are therefore

$$\epsilon_n(l) = \frac{v_f}{4R \sin\chi} (2\chi + 2\pi(n + \frac{1}{2})). \quad (3.11)$$

Here $l = bk_f = Rk_f \cos\chi$. For small l and $n = -1$, Eq. (3.11) reduces to

$$\epsilon_{-1}(l) = -\omega_0 l, \quad \omega_0 = \frac{1}{2mR^2}. \quad (3.12)$$

This is the topological branch of low energy excitations which is important for the spectral flow. Note that $\omega_0 = (2mR^2)^{-1}$ is the angular velocity of the superflow at the boundary of the core.

The maximum value of l occurs where the chord length becomes zero and $\epsilon \rightarrow \Delta$. This maximum value is $l_{\max} = Rk_f \approx v_f k_f / \Delta = 2E_f / \Delta \approx 10^3 - 10^4$. This is large enough that l can almost be regarded as a continuous parameter. We will often use this observation to write expressions such as $\partial\epsilon/\partial l$ without further comment.

In the next section we will show that the l dependence in (3.11) and (3.12) is a consequence of the failure of the Andreev scattering to be perfectly retroreflective.¹⁷

IV. ANDREEV REFLECTION AND THE BOUND STATE SPECTRUM

The bound states may be thought of as standing waves set up by Andreev reflected quasiparticles repeatedly traversing a chord of a circle. For a bound state of definite angular momentum, l , the orientation of this chord will be indefinite. To produce a state localized on a chord with specified orientation and impact parameter we must take a linear combination of angular momentum states,

$$|\theta\rangle = \sum_l a_l e^{-il\theta} |l\rangle, \quad (4.1)$$

where the coefficients a_l are large only for $l \approx k_f b$. Because of the Rayleigh criterion relating the extent of a plane wave front and its diffraction spread, there will be an uncertainty relation between the sharpness of θ and b .

We may follow the time evolution of these wave packets by using the small l approximation $\epsilon_{-1}(l) = -\omega_0 l$. We find

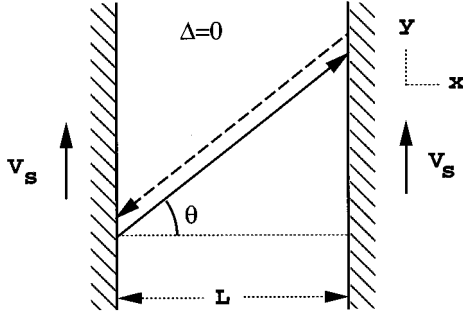


FIG. 2. Bound state in a SNS junction with superflow. The solid arrow is the electron trajectory and the dashed arrow the hole.

$$|\theta, t\rangle = \sum_l a_l e^{-il\theta} e^{+il\omega_0 t} |l\rangle = |(\theta - \omega_0 t), t=0\rangle. \quad (4.2)$$

The chord is thus seen to precess at the angular frequency ω_0 in the sense *opposite* to the superflow. If we use the more general form (3.11) for the energies, we will need to replace $-\omega_0$ by a group angular velocity

$$\omega = \frac{\partial \epsilon(l)}{\partial l}, \quad (4.3)$$

but the same general picture will hold. In this section we will see that this precession can be understood rather precisely as the combined effect of two processes each of which causes the Andreev reflection to fail to be perfectly retroreflective.¹⁷ One of these processes is due to the superflow and we will be able to use this insight to determine the evolution of the bound states when the vortex is immersed in a transport supercurrent.

To begin with we consider a plane superconducting-normal-superconducting (SNS) junction of width L having a supercurrent v_s flowing parallel to the junction (see Fig. 2).

There will be bound states in the junction with momenta close to $\mathbf{k} = (k_x, k_y) = (k_f \cos \theta, k_f \sin \theta)$. These states are found from the Andreev Hamiltonian

$$H = \begin{bmatrix} -i(k_x \partial_x + k_y \partial_y)/m & \Delta(x) e^{2imv_s y} \\ \Delta(x) e^{-2imv_s y} & i(k_x \partial_x + k_y \partial_y)/m \end{bmatrix}, \quad (4.4)$$

where $\Delta(x) = 0$ for $0 < x < L$ and $\Delta(x) = \Delta$ elsewhere. They have the form

$$\psi = \begin{bmatrix} e^{i(\delta k_x x + \delta k_y y)} \\ e^{-i(\delta k_x x + \delta k_y y)} \end{bmatrix}, \quad 0 < x < L. \quad (4.5)$$

Here $\delta k_y = m v_s$ and (for states with $\epsilon \ll \Delta$) $\delta k_x = (1/2L)[2\pi(n + \frac{1}{2})]$. Their energy is

$$\epsilon = v_f (\delta k_x \cos \theta + \delta k_y \sin \theta). \quad (4.6)$$

In the figure we have drawn the trajectories of the electron and hole as if they were at a definite location in the junction. In reality, specifying a y location requires making a wave packet

$$|y\rangle = \int \frac{dk_y}{2\pi} a(k) e^{-ik_y y} |k\rangle, \quad (4.7)$$

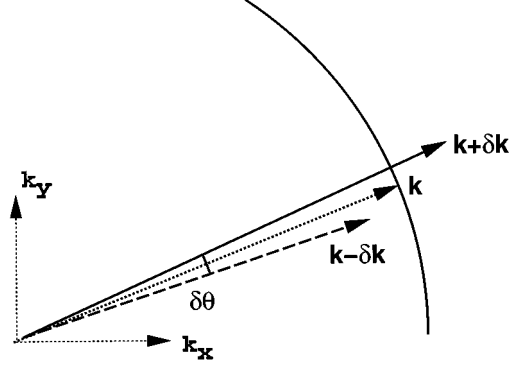


FIG. 3. The electron (solid arrow) and hole (dashed arrow) momenta do not lie exactly at \mathbf{k} on the Fermi surface; consequently their momenta differ in direction by a small angle $\delta\theta$.

with some spread of k_y . This packet will drift in the $+y$ direction at the group velocity

$$v_{\text{drift}} = \frac{\partial \epsilon(k)}{\partial k_y} = -\frac{\tan \theta \delta k_x}{m} + v_s. \quad (4.8)$$

For trajectories that are at close to normal incidence on the supercurrents ($\theta = 0$ or $k_y \approx 0$) this velocity is essentially v_s .

This drift can be accounted for by noting that the Andreev scattering is not perfectly retroreflective. Consider reflection from the right-hand supercurrent in Fig. 2. Although both the incident electron and the reflected hole have nominal momentum \mathbf{k} , they in fact have momenta $\mathbf{k} \pm \delta \mathbf{k}$ (see Fig. 3).

This means that each reflection causes a small change in the angle of incidence

$$\begin{aligned} \delta\theta &= 2(-\delta k_x \sin \theta + \delta k_y \cos \theta)/k_f \\ &= -\frac{\epsilon}{E_f} \tan \theta + \frac{2v_s}{v_f} \frac{1}{\cos \theta}. \end{aligned} \quad (4.9)$$

Because both particle and hole move with speed v_f , it is easy to see (see Fig. 4) that such a change in angle leads to the trajectory migrating up the junction with velocity

$$v_{\text{drift}} = \frac{v_f}{2 \cos \theta} \delta\theta. \quad (4.10)$$

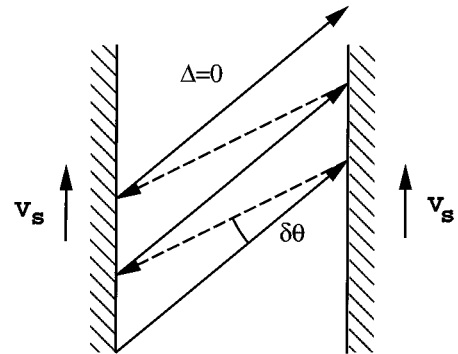


FIG. 4. The slight difference between the angle of incidence of the electron (solid arrow) and the angle of reflection of the hole (dashed arrow) causes the bound state to migrate up the SNS junction.

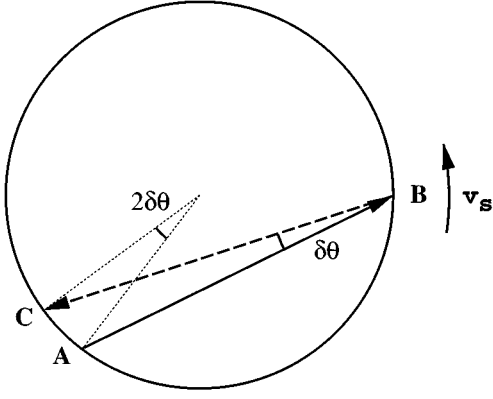


FIG. 5. Andreev reflection from the supercurrent causes the chord defining the bound state to precess in the opposite sense to the superflow.

Inserting (4.9) into (4.10) precisely reproduces (4.8).

The same result holds true for the bound states in the circular core. From Sec. III we know that the bound state energies are

$$\epsilon_n(l) = \frac{v_f}{4R \sin \chi} [2\chi + 2\pi(n + \frac{1}{2})], \quad (4.11)$$

where $l = k_f R \cos \chi$. The group angular velocity is therefore

$$\omega_g = \frac{\partial \epsilon}{\partial l} = \frac{\cos \chi}{\sin^2 \chi} \frac{\epsilon}{k_f R} - \frac{1}{2mR^2 \sin^2 \chi}. \quad (4.12)$$

For states with small ϵ (i.e., $n = -1$, $\chi \approx \pi/2$) the latter term dominates.

The geometric optics picture works here also. We can use the plane interface formula for $\delta\theta$ after noting that the angle of incidence, called θ in (4.9) is now $\theta = \pi/2 - \chi$. Notice, though, that a positive $\delta\theta$ means that the point of impact of the return trajectory moves *backwards* through an angle of $2\delta\theta$ (see Fig. 5). This is the origin of the minus sign in $\epsilon(l) \approx -\omega_0 l$. The other information we need is the time between impacts at the points A and C. This is $\delta t = 4R \sin \chi / v_f$. Putting all the ingredients together gives the precession rate as

$$\omega_g = -2 \frac{\delta\theta}{\delta t} = \frac{\cos \chi}{\sin^2 \chi} \frac{\epsilon}{k_f R} - \frac{1}{2mR^2 \sin^2 \chi} \quad (4.13)$$

as before.

The l dependence of our core-state spectrum is therefore entirely determined by the geometry of Andreev reflection. The only role of the wave character of the states is in the condition that l be an integer, and in the n dependence of the spectrum. Since only the zero-crossing branch, $n = -1$, is of interest in the spectral flow problem, the n dependence is irrelevant here.

The lesson we learn from this section is that the quasiclassical motion of the chord is governed by two processes whose effects add. One, proportional to the bound-state energy is independent of the superflow, and the other, proportional to the superflow, is independent of the energy. The

additivity of the effects also applies when the superflow is not parallel to the SN interface. We will need this fact in the next section.

V. FRUSTRATED FLOW AND MUTUAL FRICTION

If we insert our vortex in a transport supercurrent by replacing the $\Delta e^{i\theta}$ gap function by $\Delta e^{i\theta} e^{2imv_s x}$ then either considerations of the group velocity or of the geometry of Andreev reflection will show that the wave packets will try to migrate across the core at speed v_s , just as in the parallel sided junction. In the absence of precession, this motion across the core will cause a monotonic change of $(\theta_R - \theta_L)$ from 0 (where $\epsilon = -\Delta$) to 2π (where $\epsilon = +\Delta$) leading to a steady stream of states crossing the gap and becoming unbound quasiparticles, just as in Sec. II.

This migration is, however, in competition with the $-\omega_0$ precession. The competition can be described quantitatively by writing an effective Hamiltonian governing the evolution of the wave packets

$$H = -\omega_0 l + \mathbf{v}_s \cdot \mathbf{k}, \quad (5.1)$$

where $\mathbf{k} = (k_f \cos \theta, k_f \sin \theta)$. The observables l and θ are canonically conjugate and have commutation relations $[\theta, l] = i$.

For a supercurrent in the $+x$ direction, the orientation and impact parameter of a wave packet of bound states evolve according to the quasiclassical equations

$$\dot{\theta} = \frac{\partial H}{\partial l} = -\omega_0,$$

$$\dot{l} = -\frac{\partial H}{\partial \theta} = k_f v_s \sin \theta. \quad (5.2)$$

The orbits are

$$l = \frac{k_f v_s}{\omega_0} \cos \omega_0 t + \text{const.} \quad (5.3)$$

We see that a packet with an initially positive value of $\sin \theta$ tries to cross the core from $-|l_{\max}|$ to $+|l_{\max}|$ but, because of the evolution of θ , it is reflected back. This is not surprising since the Hamiltonian (5.1) is identical to the Wannier-Stark Hamiltonian for a tight-binding model. In this interpretation l would label the atomic site, ω_0 would be the external electric field, and θ the crystal momentum. The 2π periodicity of the latter is a direct consequence of the discreteness of the values of l —just as in the vortex the discreteness of l is a direct consequence of the 2π periodicity of θ . The to-and-fro motions of $l(t)$ are therefore Bloch oscillations caused by the discreteness of the bound state spectrum.

The only way a state can make the journey from $-|l_{\max}|$ to $+|l_{\max}|$ without being reflected would be for $v_s k_f / \omega_0$ to be larger than l_{\max} . But this is equivalent to the condition that v_s exceed the pair-breaking velocity, $v_{\text{pair}} k_f = \Delta$.

Although the spectral flow is thwarted by the Bloch oscillations there are still effects from the evolution of the states. We can analyze these by writing a Boltzmann equation for the distribution function, $n(l, \theta)$, in the l, θ phase space. We have

$$\frac{\partial n}{\partial t} - \omega_0 \frac{\partial n}{\partial \theta} + k_f v_s \sin \theta \frac{\partial n}{\partial l} = I, \quad (5.4)$$

where I is the collision integral. We will use a simple relaxation time approximation to (5.4) by setting

$$I = -\frac{1}{\tau} (n(l, \theta) - \bar{n}(l, \theta)), \quad (5.5)$$

where $\bar{n}(l, \theta)$ is an equilibrium distribution whose exact form is determined by the normal velocity \mathbf{v}_n . (In a superconductor the normal component relaxes to the lattice, so in this case $v_n \equiv v_{\text{lattice}}$.) For example if $\mathbf{v}_n = \mathbf{v}_s = 0$ we would expect $n \rightarrow n_0 \equiv \theta(l)$ so that all negative energy states are occupied.

It is actually more convenient to write equations for the moments of n that give the total momentum in the core

$$\begin{aligned} \langle k_x \rangle &= \frac{1}{2} \int dl \frac{d\theta}{2\pi} (n(l, \theta) - n_0(l, \theta)) k_f \cos \theta, \\ \langle k_y \rangle &= \frac{1}{2} \int dl \frac{d\theta}{2\pi} (n(l, \theta) - n_0(l, \theta)) k_f \sin \theta. \end{aligned} \quad (5.6)$$

The factor of $\frac{1}{2}$ in front of these integrals compensates for the double counting of particles and holes, and $n_0(l, \theta) = \theta(l)$ reflects the normal ordering which ensures that $\langle \mathbf{k} \rangle = 0$ when everything is at rest.

As an example suppose we fill all the negative energy states of (5.1) by setting $n(l, \theta) = \theta(\omega_0 l - v_s k_f \cos \theta)$. We find $\langle k_y \rangle = 0$ and

$$\langle k_x \rangle = -\frac{1}{4} \frac{k_f^2 v_s}{\omega_0} = -\rho \kappa \frac{v_s}{\omega_0}, \quad (5.7)$$

where $\kappa = \pi/m$ is the quantum of circulation and $\rho = m k_f^2 / 4\pi$ is the equilibrium mass density of the electron fluid. This would be the appropriate equilibrium distribution for the case $\mathbf{v}_n = 0$. The nonzero value of $\langle k_x \rangle$ reflects the reduction in current through the vortex core because of the nonzero value of ρ_n there. If $\mathbf{v}_n = \mathbf{v}_s$, on the other hand, we expect $\langle \mathbf{k} \rangle = 0$ because, even though $\rho_n \neq 0$ in the core, the total current $\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$ is unaffected by the vanishing gap.

Using these definitions and insights we find

$$\begin{aligned} \frac{d\langle k_x \rangle}{dt} &= \omega_0 \langle k_y \rangle + \kappa \rho v_{sy} - \frac{1}{\tau} (\langle k_x \rangle - \langle k_x \rangle_0), \\ \frac{d\langle k_y \rangle}{dt} &= -\omega_0 \langle k_x \rangle - \kappa \rho v_{sx} - \frac{1}{\tau} (\langle k_y \rangle - \langle k_y \rangle_0), \end{aligned} \quad (5.8)$$

where

$$\langle \mathbf{k} \rangle_0 = \frac{\kappa \rho}{\omega_0} (\mathbf{v}_n - \mathbf{v}_s). \quad (5.9)$$

In the steady state flow $\langle k_{x,y} \rangle$ will be constant, so the left-hand side of (5.8) will be zero. We can therefore solve for $\langle k_{x,y} \rangle$ and find the rate at which momentum is being transferred to the lattice

$$\left(\frac{d\langle \mathbf{k} \rangle}{dt} \right)_{\text{lattice}} = \frac{1}{\tau} (\langle \mathbf{k} \rangle - \langle \mathbf{k} \rangle_0). \quad (5.10)$$

We find

$$\begin{aligned} \left(\frac{d\langle k_x \rangle}{dt} \right)_{\text{lattice}} &= -\kappa \rho \frac{\omega_0 \tau}{1 + \omega_0^2 \tau^2} v_{nx} + \kappa \rho \frac{1}{1 + \omega_0^2 \tau^2} v_{ny}, \\ \left(\frac{d\langle k_y \rangle}{dt} \right)_{\text{lattice}} &= -\kappa \rho \frac{\omega_0 \tau}{1 + \omega_0^2 \tau^2} v_{ny} - \kappa \rho \frac{1}{1 + \omega_0^2 \tau^2} v_{nx}. \end{aligned} \quad (5.11)$$

Notice that \mathbf{v}_s does not occur in (5.11).

These calculations have been made with the vortex stationary. Because of the Galilean invariance of the overall system, we may replace \mathbf{v}_n by $(\mathbf{v}_n - \mathbf{v}_L)$ when the vortex is moving at velocity \mathbf{v}_L . The rate of loss of momentum from the core to the lattice is then

$$\left(\frac{d\langle \mathbf{k} \rangle}{dt} \right)_{\text{lattice}} = D(\mathbf{v}_L - \mathbf{v}_n) + D' \hat{\mathbf{z}} \times (\mathbf{v}_L - \mathbf{v}_n). \quad (5.12)$$

The quantities D, D' are the mutual friction coefficients. Their values

$$\begin{aligned} D &= \kappa \rho \frac{\omega_0 \tau}{1 + \omega_0^2 \tau^2}, \\ D' &= \kappa \rho \frac{1}{1 + \omega_0^2 \tau^2} \end{aligned} \quad (5.13)$$

are the same as found, for example, in Ref. 18. For the purposes of making comparisons with other work we must point out that in deriving these expressions we have assumed that we are at sufficiently low temperatures that there are very few quasiparticles outside the vortex core. We have therefore made no distinction between ρ and ρ_s . Similarly the linearization inherent in our use of the Andreev equations makes it difficult to distinguish between ρ and the constant C_0 which is defined as the electron mass times number of states within the Fermi sphere, and so coincides with the normal-state density. This distinction may be connected with the change of sign of the Hall effect near T_c .¹⁹

Equation (5.13) determines the Hall angle. While the task of finding a solution to the full dynamical gap equations for a vortex immersed in a superflow is rather daunting, we do know that such a solution must satisfy the law of momentum conservation. The vortex core velocity v_L must therefore satisfy the momentum balance equation

$$0 = \kappa \rho \hat{\mathbf{z}} \times (\mathbf{v}_L - \mathbf{v}_s) - D(\mathbf{v}_L - \mathbf{v}_n) - D' \hat{\mathbf{z}} \times (\mathbf{v}_L - \mathbf{v}_n) \quad (5.14)$$

as a consistency condition. The first term is the Magnus force which gives the rate at which momentum enters the core. The terms with D and D' give the rate at which momentum is being lost from the core to the lattice by mutual friction.

There are two extreme cases. In the collisionless regime, $\omega_0 \tau \gg 1$, the vortex has no choice but to move with the superflow. In the opposite, hydrodynamic, regime $\omega_0 \tau$ is small. Then the \mathbf{v}_L part of the D' term almost cancels the \mathbf{v}_L part of the Magnus force term. This allows the vortex to move at

right angles to the superflow, unwinding the order parameter phase gradient and dissipating the superflow.

It is worth pointing out that the cancellation between the two v_L terms in the hydrodynamic limit should *not* be regarded as a “cancellation” of the Magnus force. To claim that it is is akin to saying that the “ F ” cancels the “ ma ” in $F=ma$. This may seem a mere quibble, but an inappropriate choice of language has caused confusion in the literature. It is also worth stressing that while the reactive term $D'\hat{\mathbf{z}}\times(\mathbf{v}_L-\mathbf{v}_n)$ does not cause any dissipation of energy, its presence is entirely due to quantum incoherent, entropy-generating relaxation processes.

VI. CONCLUSION AND DISCUSSION

We have seen that a quasiclassical geometric optics model gives a good description of the processes in the core of a vortex immersed in a transport current. Even though there is no explicit time dependence in the Bogoliubov–de Gennes equations, spectral flow would still occur were it not suppressed by an analogue of Bloch oscillations originating in the discrete nature of the spectrum. The spectral evolution does, however, lead to a nonequilibrium occupation of momentum-carrying core states. When we account for the processes by which the occupation distribution attempts to relax, we find expressions for the mutual friction parameters which coincide with those obtained by traditional Green

function methods.^{6,7,18} It is therefore clear that the Green functions tacitly include the physics of spectral flow. The new method of explicitly following the evolution of the states does however have the advantage of being much more physical.

Our general conclusions are consistent with recent work by Kopnin *et al.* which is briefly reviewed in Ref. 20. Some other recent discussions of the Magnus force and momentum balance are by Hoffmann and Kümmel,²¹ Gaitan,²² Ao,²³ and Šimánek.²⁴ The first of these papers has much in common with the present approach. In particular these authors focus on the role Andreev scattering plays in transferring momentum from the condensate to the core states. The next two use the Berry phase approach to confirm that the Magnus force correctly gives the momentum flow into the vortex core. The last combines effects of external and core states to find an effective “Magnus” force that is a combination of the true Magnus force and the mutual friction terms.

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