Critical state of periodically arranged superconducting-strip lines in perpendicular fields

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Analytical investigations of the critical state are carried out for strip-array systems, in which thin superconducting strip lines parallel to the xy plane are arranged periodically. Two types of strip-array systems are considered: a stack of strip lines piled along the z axis and an array of strip lines aligned in the xy plane. The results show that the relations between the magnetic field H and the current density J of strip-array systems in an applied magnetic field perpendicular to the strips can be transformed to the H-J relation for an isolated strip line. Therefore, H and J of strip-array systems in the Meissner state and in the Bean's critical state can be easily derived by transforming the H and J for an isolated strip line. Magnetization properties and interactions in multiple strip lines are discussed. [S0163-1829(96)06842-7]

I. INTRODUCTION

The concept of a critical state was introduced to describe the magnetic hysteresis of type-II superconductors and was first applied to slab and cylindrical superconductors in magnetic fields applied parallel to the superconductors.[1] The critical state in superconducting strips in perpendicular magnetic fields was considered by Swan[2] and Norris.[3] Recently, the electromagnetic properties (e.g., magnetic field, current density, magnetization, and ac loss) of flat superconductors in perpendicular fields, in which the demagnetization factor is close to 1, have been extensively investigated.

The magnetic field and current density in the critical state of strip lines have been analytically investigated as follows.[4–7] Consider an isolated strip line of width $2w$ and thickness d (the strip line fills the area $|x|\leqslant w$, $|z|\leqslant d/2$, and $|y|<\infty$). The width of the strip line, $2w$, is assumed to be much larger than the thickness d and the effective penetration depth in thin films[8] $2\lambda^2/d$, where λ is the London penetration depth. This assumption allows us to describe the electromagnetic property of the strip array by the z component of the magnetic field at $z=0$, $H(x)\equiv H_z(x,z=0)$, and by the y component of the mean current density, $J(x)\equiv(1/d)\int_{-d/2}^{+d/2}J_y(x,z)dz$. The basic equation of the Biot-Savart law for $H(x)$ and $J(x)$ for an isolated strip line is given by

$$H(x)=H_a-\frac{d}{2\pi}\int_{-w}^{+w}\frac{J(u)}{x-u}du, \tag{1}$$

where the magnetic field H_a is applied parallel to the z axis. In the critical state of a strip line, $H(x)$ and $J(x)$ are determined so as to satisfy the integral equations $H(|x|<a)=0$ and $|J(a<|x|<w)|=J_c$ with Eq. (1), where a is the position of the flux front and J_c is the critical current density. The integral equations for $H(x)$ and $J(x)$ can be solved analytically assuming the Bean's critical state model,[4–6] in which J_c is assumed to be field independent; on the contrary, a numerical calculation is necessary for field-dependent J_c.[7] Bean's critical state for disk superconductors in perpendicular fields has also been investigated analytically.[9–11] These results show that the $H(x)$ and $J(x)$ profiles of a strip and a disk in perpendicular fields are very different from those of a slab and a cylinder in parallel fields.

The magnetic field distribution in strip lines predicted by those previous analyses have been experimentally confirmed by using magneto-optical Faraday effects[12,13] and by using an electron spin resonance probe.[14] Magnetization properties predicted by the theoretical works have been confirmed in studies of the magnetization curves for $YBa_2Cu_3O_y$ disks.[15]

In these previous works, isolated strip lines or disks were investigated. When multiple strip lines are subject to a magnetic field, on the other hand, $H(x)$ and $J(x)$ profiles are affected by the interaction of the strip lines. For example, a stack of strip lines may behave like a slab superconductor, if the spacing between strip lines in the stack is much narrower than the width. Multiple strip lines or wires are utilized for device and power application of superconductors, and basic physical investigations of the electromagnetic response of such a strip-array system is useful for the application of superconductors.

In the present paper, analytical expressions of $H(x)$ and $J(x)$ are determined for systems of multiple strip lines in perpendicular magnetic fields on the basis of the Bean's critical state model. The results show that the basic H-J relation of the Biot-Savart law for strip-array systems can be transformed to the H-J relation for an isolated strip line, Eq. (1). Conversely, the $H(x)$ and $J(x)$ for multiple strip lines can then be obtained by a simple transformation of $H(x)$ and $J(x)$ for an isolated strip line. I consider two types of strip-array systems: a stack of strip lines along the z axis (I call this stack a Z stack) is analyzed in Sec. II and an array of strip lines in the xy plane (I call this array an X array) in Sec. III.

II. STACK OF STRIP LINES ALONG THE z AXIS (Z STACK)

In this section we consider a Z stack in which an infinite number of strip lines are stacked along the z axis. Each strip line has width $2w$ and thickness d, and is stacked at the same interval D as shown in Fig. 1. The nth superconducting strip occupies an area where $|x|\leqslant w$, $|y|<\infty$, and $|z-nD|\leqslant d/2$

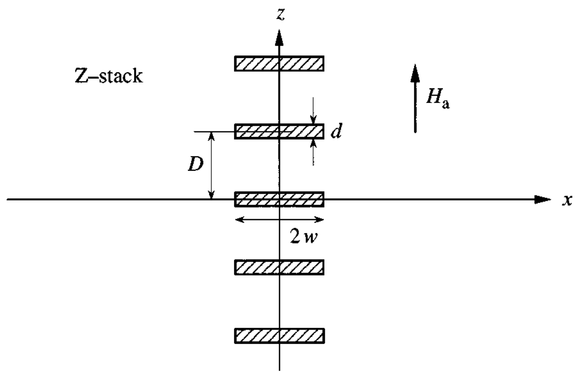

FIG. 1. Arrangement of strip lines in a Z stack, in which an infinite number of strip lines are stacked along the z axis at an interval D. Each strip line has width $2w$, thickness d, and is infinitely long along the y axis.

$(n=0,\pm 1,\pm 2,\cdots,\pm\infty)$. We assume $2w\gg d$, $2w\gg 2\lambda^2/d$, and $D\gg d$.

A. Basic equations

When H_a is applied parallel to the z axis (perpendicular to the strips), the Biot-Savart law for the z component of the magnetic field $H(x)$ and the mean current density $J(x)$ in a Z stack is given by

$$H(x)=H_a-\frac{d}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-w}^{+w}J(u)\frac{(x-u)du}{(x-u)^2+(nD)^2}$$
$$=H_a-\frac{d}{2D}\int_{-w}^{+w}J(u)\coth\left(\frac{\pi(x-u)}{D}\right)du. \tag{2}$$

By introducing the following transformation of the variables,

$$\tilde{x}=\frac{D}{\pi}\tanh\left(\frac{\pi x}{D}\right),\quad \tilde{w}=\frac{D}{\pi}\tanh\left(\frac{\pi w}{D}\right), \tag{3}$$

Eq. (2) is reduced to

$$\tilde{H}(\tilde{x})=\tilde{H}_a-\frac{d}{2\pi}\int_{-\tilde{w}}^{+\tilde{w}}\frac{\tilde{J}(\tilde{u})}{\tilde{x}-\tilde{u}}d\tilde{u}, \tag{4}$$

where $\tilde{H}(\tilde{x})$ is the transformed magnetic field, $\tilde{J}(\tilde{x})$ is the transformed current density, and $\tilde{H}_a$ is the transformed applied field given by

$$\tilde{H}_a=H_a+\frac{d}{2\pi}\int_{-\tilde{w}}^{+\tilde{w}}\tilde{J}(\tilde{u})\frac{\tilde{u}d\tilde{u}}{(D/\pi)^2-(\tilde{u})^2}$$
$$=H_a+\frac{d}{2D}\int_{-w}^{+w}J(u)\tanh\left(\frac{\pi u}{D}\right)du. \tag{5}$$

Note that Eq. (4) has the same form as Eq. (1); namely, Eq. (4) shows the $\tilde{H}(\tilde{x})$-$\tilde{J}(\tilde{x})$ relation for an isolated strip line of width $\tilde{w}$ with a magnetic field $\tilde{H}_a$ in $\tilde{x}$ space. Therefore, $H(x)$ and $J(x)$ of a Z stack can be derived from those of an isolated strip line using the transformation given by Eqs. (3) and (5). The magnetization M is then calculated from $J(x)$ as

$$M=\frac{1}{2w}\int_{-w}^{+w}xJ(x)dx. \tag{6}$$

If a Z stack carries no transport current, $I_t=0$, then $J(x)$ is asymmetric, $J(-x)=-J(x)$, and we have

$$\tilde{H}(\tilde{x}=D/\pi)=H_a. \tag{7}$$

This equation can be used to determine $\tilde{H}_a$, because $\tilde{H}_a$ is implicit on the left-hand side of the equation. Since the extension to the case of $I_t\neq 0$ is straightforward,[16] in this paper we derive $H(x)$ and $J(x)$ for $H_a\neq 0$ and $I_t=0$.

B. Meissner state

In the Meissner state $\tilde{H}(\tilde{x})$ and $\tilde{J}(\tilde{x})$ for a Z stack, which are the same expressions as $H(x)$ and $J(x)$ for an isolated strip line,[4–6] are given by

$$\tilde{H}(|\tilde{x}|>\tilde{w})=\tilde{H}_a\frac{|\tilde{x}|}{\sqrt{\tilde{x}^2-\tilde{w}^2}}, \tag{8}$$

$$\tilde{J}(|\tilde{x}|<\tilde{w})=-\frac{2\tilde{H}_a}{d}\frac{\tilde{x}}{\sqrt{\tilde{w}^2-\tilde{x}^2}}, \tag{9}$$

and $\tilde{H}(|\tilde{x}|<\tilde{w})=0$. Substituting $\tilde{x}=D/\pi$ in Eq. (8) and using Eq. (7), we have

$$\tilde{H}_a=\frac{H_a}{\cosh(\pi w/D)}. \tag{10}$$

The $H(x)$ and $J(x)$ for a Z stack are obtained from Eqs. (3), (8), (9), and (10) as

$$H(|x|>w)=\frac{H_a\sinh(\pi|x|/D)}{\sqrt{\sinh^2(\pi x/D)-\sinh^2(\pi w/D)}}, \tag{11}$$

$$J(|x|<w)=-\frac{(2H_a/d)\sinh(\pi x/D)}{\sqrt{\sinh^2(\pi w/D)-\sinh^2(\pi x/D)}}, \tag{12}$$

and $H(|x|<w)=0$. At the edge of the strip line ($|x|\sim w$), $H(x)$ and $J(x)$ diverge as follows:

$$H(|x|\to w+0)\sim H_a\sqrt{\frac{\tanh(\pi w/D)}{2\pi(|x|-w)/D}}, \tag{13}$$

$$J(|x|\to w-0)\sim-\frac{H_a}{d}\,\mathrm{sgn}(x)\sqrt{\frac{\tanh(\pi w/D)}{\pi(w-|x|)/2D}}. \tag{14}$$

The magnetization M is calculated as

$$M=-H_a\frac{D^2}{\pi w d}\ln\left[\cosh\left(\frac{\pi w}{D}\right)\right]. \tag{15}$$

C. Critical state

In the critical state, when a Z stack is exposed to a magnetic field increased after zero-field cooling, $H(|x|<a)=0$ in the shielded region, and $H(a<|x|<w)>0$ in the flux-filled region, where $a>0$ is the position of the flux front. The current density J saturates as $J(a<|x|<w)=-J_c\,\mathrm{sgn}(x)$, where we assume that the critical current density J_c is field independent[1] in this paper.

In the critical state, the $\tilde{H}(\tilde{x})$ and $\tilde{J}(\tilde{x})$ are given by[4–6]

$$\tilde{H}(\tilde{x})=\begin{cases}0, & |\tilde{x}|<\tilde{a},\\ H_0\,\mathrm{arctanh}(1/|\tilde{\varphi}(\tilde{x})|), & \tilde{a}<|\tilde{x}|<\tilde{w},\\ H_0\,\mathrm{arctanh}|\tilde{\varphi}(\tilde{x})|, & |\tilde{x}|>\tilde{w},\end{cases} \tag{16}$$

$$\tilde{J}(\tilde{x})=\begin{cases}-(2/\pi)J_c\arctan\tilde{\varphi}(\tilde{x}), & |\tilde{x}|<\tilde{a},\\ -J_c\,\mathrm{sgn}(\tilde{x}), & \tilde{a}<|\tilde{x}|<\tilde{w},\end{cases} \tag{17}$$

where H_0 is the characteristic field given by

$$H_0=J_c d/\pi \tag{18}$$

and $\tilde{\varphi}(\tilde{x})$ is

$$\tilde{\varphi}(\tilde{x})=\frac{\tilde{x}}{\tilde{w}}\sqrt{\frac{\tilde{w}^2-\tilde{a}^2}{|\tilde{a}^2-\tilde{x}^2|}}. \tag{19}$$

The transformed flux front $\tilde{a}=(D/\pi)\tanh(\pi a/D)$ satisfies

$$\cosh\left(\frac{\tilde{H}_a}{H_0}\right)=\frac{\tilde{w}}{\tilde{a}}=\frac{\tanh(\pi w/D)}{\tanh(\pi a/D)}. \tag{20}$$

The relation between H_a and $\tilde{H}_a$ is then obtained from Eq. (7) as

$$\sinh\left(\frac{\tilde{H}_a}{H_0}\right)=\frac{\sinh(H_a/H_0)}{\cosh(\pi w/D)}. \tag{21}$$

Using Eqs. (20) and (21), we have the following relation between H_a and a:

$$\sinh\left(\frac{\pi a}{D}\right)=\frac{\sinh(\pi w/D)}{\cosh(H_a/H_0)}. \tag{22}$$

For a low applied field [$H_a \ll H_0$ and $w-a\ll \min(w,D)$], Eq. (22) is then approximated as

$$w-a\simeq\frac{D}{2\pi}\left(\frac{H_a}{H_0}\right)^2\tanh\left(\frac{\pi w}{D}\right). \tag{23}$$

For a high field ($H_a\gtrsim H_0$ and $a\ll D$), on the other hand, we have

$$a\simeq\frac{2D}{\pi}\exp\left(-\frac{H_a}{H_0}\right)\sinh\left(\frac{\pi w}{D}\right). \tag{24}$$

As shown in Eqs. (23) and (24), $w-a$ grows with field as $\sim H_a^2$ and saturates exponentially. These field penetration properties are similar to those for an isolated strip line.

Substituting Eq. (3) into Eqs. (16), (17), and (19), we obtain $H(x)$ and $J(x)$ in the critical state as

$$H(x)=\begin{cases}0, & |x|<a,\\ H_0\,\mathrm{arctanh}(1/|\varphi(x)|), & a<|x|<w,\\ H_0\,\mathrm{arctanh}|\varphi(x)|, & |x|>w,\end{cases} \tag{25}$$

$$J(x)=\begin{cases}-(2/\pi)J_c\arctan\varphi(x), & |x|<a,\\ -J_c\,\mathrm{sgn}(x), & a<|x|<w,\end{cases} \tag{26}$$

where $\varphi(x)$ is

$$\varphi(x)=\frac{\tanh(\pi x/D)}{\tanh(\pi w/D)}\sqrt{\frac{\tanh^2(\pi w/D)-\tanh^2(\pi a/D)}{|\tanh^2(\pi a/D)-\tanh^2(\pi x/D)|}}. \tag{27}$$

At the edge of the strip line ($|x|\sim w$), $H(x)$ diverges logarithmically as

$$H(|x|\to w)\sim\frac{H_0}{2}\ln\left[\frac{\tanh(\pi w/D)}{\pi||x|-w|/2D}\sinh^2\left(\frac{H_a}{H_0}\right)\right]. \tag{28}$$

At the flux front ($|x|\sim a$), $J(x)$ and $H(x)$ have vertical slopes,

$$J_c-|J(|x|\to a-0)|\sim\frac{2}{d}H(|x|\to a+0)$$
$$\sim\frac{2}{\pi}J_c\coth\left(\frac{H_a}{H_0}\right)\sqrt{\frac{2\pi(|x|-a)/D}{\tanh(\pi a/D)}}. \tag{29}$$

Logarithmic divergence of $H(x)$ at $|x|\sim w$ and vertical slopes of $H(x)$ and $J(x)$ at $|x|\sim a$ appear in a Z stack as well as in an isolated strip line. For a high field ($H_a>\tilde{H}_a\gtrsim H_0$), $\tilde{H}_a$ given by Eq. (21) is reduced to

$$\tilde{H}_a=H_a-H_0\ln\left[\cosh\left(\frac{\pi w}{D}\right)\right], \tag{30}$$

and $H(x)$ is given by

$$H(x)=H_a-\frac{H_0}{2}\ln\left|\frac{\sinh^2(\pi w/D)}{\sinh^2(\pi x/D)}-1\right|. \tag{31}$$

Figure 2 shows $H(x)$ and $J(x)$ profiles for a Z stack (solid lines) and for an isolated strip line (dashed lines). This figure clearly shows that the field shielding effect is larger and the width of the flux-filled region $w-a$ is smaller for a Z stack than for an isolated strip line.

In the critical state, M for a Z stack is calculated by substituting Eqs. (26) and (27) into Eq. (6); thus we have

$$M=-\frac{D^2}{2\pi w d}\int_0^{H_a}\ln\left[1+\frac{\sinh^2(\pi w/D)}{\cosh^2(H'/H_0)}\right]dH'$$
$$=-\frac{M_0}{\nu_z^2}\int_0^{\nu_z}\ln\left[\frac{\cosh(s+\eta)}{\cosh(s-\eta)}\right]ds, \tag{32}$$

where M_0, ν_z, and η are defined as

$$M_0=\frac{1}{2}wJ_c,\quad \nu_z=\frac{\pi w}{D},\quad \eta=\frac{H_a}{H_0}=\frac{\pi H_a}{J_c d}. \tag{33}$$

For $\eta<\nu_z$ (i.e., $H_a<\pi H_0 w/D=J_c w d/D$), Eq. (32) can be rewritten as the following series:

$$-\frac{M}{M_0}=\frac{2\eta}{\nu_z}-\left(\frac{\eta}{\nu_z}\right)^2-\frac{1}{\nu_z^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$
$$\times[1-e^{-2k\eta}-e^{-2k\nu_z}\sinh(2k\eta)]. \tag{34}$$

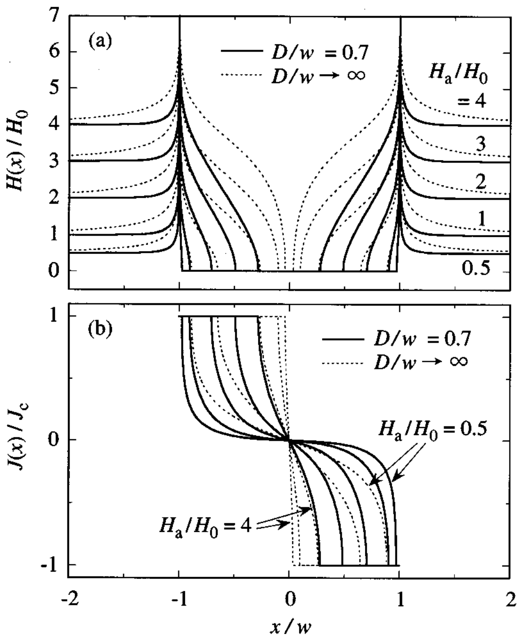

FIG. 2. Profiles of (a) magnetic field $H(x)$ and (b) current density $J(x)$ in a Z stack where $D/w=0.7$ (solid lines) and in an isolated strip line where $D/w\to\infty$ (dashed lines) at $H_a/H_0=0.5,1,2,3,$ and 4.

On the other hand, for $\eta>\nu_z$ (i.e., $H_a>\pi H_0 w/D$), we have

$$-\frac{M}{M_0}=1-\frac{2}{\nu_z^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}e^{-2k\eta}\sinh^2(k\nu_z). \quad (35)$$

In a low applied field ($\eta\ll 1$), Eq. (32) is reduced to

$$-\frac{M}{M_0}\simeq\frac{1}{\nu_z^2}\left[2\eta\ln(\cosh\nu_z)-\frac{\eta^3}{3}\tanh^2\nu_z+\cdots\right], \quad (36)$$

and in a high field ($\eta\gg 1$), we have

$$-\frac{M}{M_0}\simeq 1-2e^{-2\eta}\frac{\sinh^2\nu_z}{\nu_z^2}+\cdots. \quad (37)$$

When the spacing in a Z stack is large ($\nu_z\ll 1$), we have

$$-\frac{M}{M_0}\simeq\tanh\eta-\frac{\nu_z^2}{6}\frac{\tanh\eta}{\cosh^2\eta}+\cdots. \quad (38)$$

The first term of the right-hand side of Eq. (38) corresponds to the magnetization for an isolated strip line.

We denote the magnetization M in an increasing field ($H_a>0$) after zero-field cooling as $M_{\rm zfc}(H_a)$, which is given by Eq. (32). Then, the magnetization $M_\downarrow(H_a)$ when H_a is decreasing from a maximum field $H_m>0$ ($|H_a|<H_m$) and the magnetization $M_\uparrow(H_a)$ when H_a is increasing from $-H_m$ are given by[4,15,17]

$$M_\downarrow(H_a)=M_{\rm zfc}(H_m)-2M_{\rm zfc}\left(\frac{H_m-H_a}{2}\right), \quad (39)$$

$$M_\uparrow(H_a)=-M_{\rm zfc}(H_m)+2M_{\rm zfc}\left(\frac{H_m+H_a}{2}\right)=-M_\downarrow(-H_a). \quad (40)$$

Hysteretic ac loss per unit volume for one field cycle, $P(H_m)$, is calculated as[17]

$$\begin{aligned}P(H_m)&=-\mu_0\oint M(H_a)dH_a\\&=2\mu_0\int_{-H_m}^{+H_m}M_\downarrow(H_a)dH_a\\&=4\mu_0H_mM_{\rm zfc}(H_m)-8\mu_0\int_0^{H_m}M_{\rm zfc}(H_a)dH_a.\end{aligned} \quad (41)$$

Substituting Eq. (32) for $M_{\rm zfc}(H_a)$, Eq. (41) becomes

$$P(H_m)=\frac{2\mu_0D^2}{\pi wd}\int_0^{H_m}dH_a(H_m-2H_a)\times\ln\left[1+\frac{\sinh^2(\pi w/D)}{\cosh^2(H_a/H_0)}\right], \quad (42)$$

D. Densely piled Z stack

All results presented in Secs. IIA–IIC with $D\to\infty$ correspond to the results for an isolated strip line, which is considered in Refs. 4–6. For a densely piled Z stack in which spacing between strip lines is small ($D\lesssim w$), on the other hand, the $H(x)$ and $J(x)$ profiles are significantly affected by the interactions between multiple strip lines. Here we consider such a Z stack where $D\lesssim w$ ($e^{-2\pi w/D}\ll 1$).

For $|x|-w\gtrsim D$, $H(x)$ in the Meissner state given by Eq. (11) is approximated as

$$H(|x|>w)\simeq H_a[1+\tfrac{1}{2}e^{-2\pi(|x|-w)/D}]. \quad (43)$$

For $|x|\gtrsim D$ and $w-|x|\gtrsim D$, $J(x)$ given by Eq. (12) is approximated as

$$J(|x|<w)\simeq-\frac{2H_a}{d}\,{\rm sgn}(x)e^{-\pi(w-|x|)/D}. \quad (44)$$

These two equations show that $H(x)$ is almost same as H_a except near the edge of the strips, and that the current flows only near the edge.

In the critical state, Eq. (22) is approximated as

$$\frac{\pi}{D}(w-a)\simeq\frac{H_a}{H_0}-\ln 2, \quad (45)$$

for $a\gtrsim D$ and $w-a\gtrsim D$. Here, the flux-filled region $w-a$ grows linearly with increasing H_a, similar to growth seen in slab superconductors with parallel fields. Note that $w-a$ grows as Eq. (23) in the vicinity of zero field ($w-a\ll D$) even in the densely piled Z stack. For $w-a\gtrsim D$, $w>a>|x|\gtrsim D$, and $a-|x|\gtrsim D$, $J(x)$ is given by

$$J(|x|<a)\simeq-\frac{2}{\pi}J_c\,{\rm sgn}(x)e^{-\pi(a-|x|)/D}, \quad (46)$$

namely, $|J(|x|<a)|\ll J_c$. For $w-a\gtrsim D$, $H(x)$ outside the strip and not so close to the edge ($|x|-w\gtrsim D$) is given by

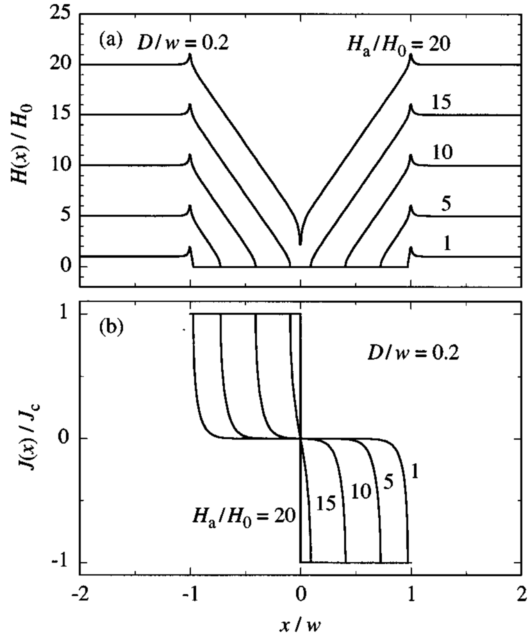

FIG. 3. Profiles of (a) magnetic field $H(x)$ and (b) current density $J(x)$ in a densely piled Z stack where $D/w=0.2$ at $H_a/H_0=1,5,10,15$, and 20.

$$H(|x|>w)\simeq H_a+\frac{H_0}{2}e^{-2\pi(|x|-w)/D}. \tag{47}$$

In the flux-filled region not so close to either the edge $(w-|x|\gtrsim D)$ or the flux front $(|x|-a\gtrsim D)$, we have

$$H(a<|x|<w)\simeq H_a-\pi H_0\left[\frac{w-|x|}{D}-\frac{1}{2\pi}e^{-2\pi(w-|x|)/D}\right]. \tag{48}$$

As shown in Eqs. (47) and (48), except near the flux front and near the edge of the strip lines, $H(x)$ in a densely piled Z stack is approximately given by $H(|x|>w)\simeq H_a$ and $H(a<|x|<w)\simeq H_a-(d/D)J_c(w-|x|)$. Figure 3 shows that the $H(x)$ and $J(x)$ profiles in a Z stack with narrow spacing $(D/w=0.2)$ are similar to those in slab superconductors. The anomaly of $H(x)$ at the edge $(|x|\sim w)$, and at the flux front $(|x|\sim a)$, becomes small as D/w decreases. Furthermore, at the narrow spacing limit $(\nu_z\gg 1)$, M given by Eq. (34) is approximated as $M/M_0\sim-(2\eta/\nu_z)+(\eta/\nu_z)^2$ for low fields except in the vicinity of zero field $(\eta<\nu_z$ and $e^{-2\eta}\ll 1)$, and becomes saturated as $M/M_0\sim-1$ in high fields $(\eta>\nu_z)$. These results show that the electromagnetic properties of a densely piled Z stack with a narrow spacing of $D\ll w$ in the critical state are similar to those of a slab in which critical current density is J_cd/D.

III. PARALLEL STRIP LINES IN THE *xy* PLANE (*X* ARRAY)

In this section we consider an X array in which an infinite number of strip lines are aligned in the xy plane. Each superconducting strip has width $2w$, thickness d $(\ll w)$, and is placed at the same interval L $(>2w)$ as shown in Fig. 4. The nth superconducting strip occupies an area where $|x-nL|\leqslant w$ $(n=0,\pm1,\pm2,\ldots,\pm\infty)$, $|y|<\infty$, and $|z|\leqslant d/2$.

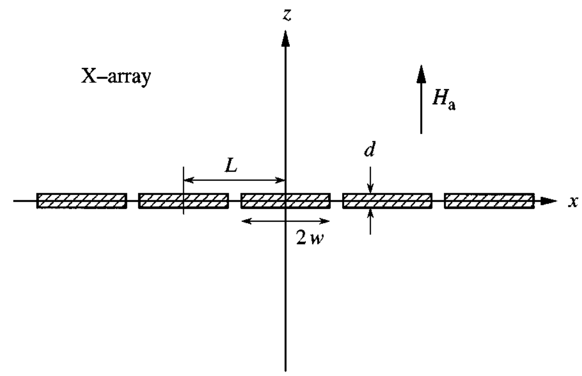

FIG. 4. Arrangement of strip lines in an X array, in which an infinite number of strip lines are aligned along the x axis at a period L. Each strip line has width $2w$ and thickness d, and is infinitely long along the y axis.

A. Basic equations

When H_a is applied parallel to the z axis, the relation between $H(x)$ and $J(x)$ in an X array is given by

$$H(x)=H_a-\frac{d}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-w}^{+w}\frac{J(u)du}{x-(u+nL)}$$
$$=H_a-\frac{d}{2L}\int_{-w}^{+w}J(u)\cot\left(\frac{\pi(x-u)}{L}\right)du. \tag{49}$$

Since $J(x)$ and $H(x)$ are periodic with period L as $J(x+nL)=J(x)$ and $H(x+nL)=H(x)$, it is sufficient to consider only the electromagnetic field for $|x|<L/2$. By introducing the following transformation of the variables,

$$\tilde{x}=\frac{L}{\pi}\tan\left(\frac{\pi x}{L}\right),\quad \tilde{w}=\frac{L}{\pi}\tan\left(\frac{\pi w}{L}\right), \tag{50}$$

Eq. (49) is reduced to

$$\tilde{H}(\tilde{x})=\tilde{H}_a-\frac{d}{2\pi}\int_{-\tilde{w}}^{+\tilde{w}}\frac{\tilde{J}(\tilde{u})}{\tilde{x}-\tilde{u}}d\tilde{u}, \tag{51}$$

where $\tilde{H}(\tilde{x})$ is the transformed magnetic field, $\tilde{J}(\tilde{x})$ is the transformed current, and $\tilde{H}_a$ is the transformed applied field given by

$$\tilde{H}_a=H_a-\frac{d}{2\pi}\int_{-\tilde{w}}^{+\tilde{w}}\tilde{J}(\tilde{u})\frac{\tilde{u}d\tilde{u}}{(L/\pi)^2+(\tilde{u})^2}$$
$$=H_a-\frac{d}{2L}\int_{-w}^{+w}J(u)\tan\left(\frac{\pi u}{L}\right)du=H(x=L/2). \tag{52}$$

Note that Eq. (51) has the same form as Eq. (1). Therefore, $H(x)$ and $J(x)$ of an X array can be derived from those of an isolated strip line using the transformation given by Eqs. (50) and (52).

Furthermore, the transformation of $D \rightarrow iL$ (where $i=\sqrt{-1}$) in Eqs. (2), (3), and (5) lead to Eqs. (49), (50), and (52), respectively. The expressions for H, J, M, and P for an X array are then easily obtained by the simple transformation $D \rightarrow iL$ of the expressions for a Z stack given in Secs. II B and II C.

B. Meissner state

In the Meissner state, $H(x)$ and $J(x)$ for an X array are

$$H(w<|x|<L/2)=\frac{H_a \sin(\pi|x|/L)}{\sqrt{\sin^2(\pi x/L)-\sin^2(\pi w/L)}}, \quad (53)$$

$$J(|x|<w)=\frac{-(2H_a/d)\sin(\pi x/L)}{\sqrt{\sin^2(\pi w/L)-\sin^2(\pi x/L)}}, \quad (54)$$

and $H(|x|<w)=0$.

For $L-2w \ll L$, Eq. (54) is approximated as

$$J(|x|<w) \simeq -\frac{2H_a}{d}\tan\left(\frac{\pi x}{L}\right), \quad (55)$$

except near the edge $(w-|x| \ll w \sim L/2)$. The $H(x)$ at the center of the spacing of strip lines for $L-2w \ll L$ is much larger than H_a,

$$H(|x|=L/2)=\frac{H_a}{\cos(\pi w/L)} \simeq \frac{4w}{\pi(L-2w)}H_a. \quad (56)$$

In the Meissner state, M for an X array is

$$M=H_a\frac{L^2}{\pi w d}\ln\left[\cos\left(\frac{\pi w}{L}\right)\right]. \quad (57)$$

C. Critical state

The flux front a for an X array in the critical state is given by

$$\sin\left(\frac{\pi a}{L}\right)=\frac{\sin(\pi w/L)}{\cosh(H_a/H_0)}, \quad (58)$$

and $H(x)$ and $J(x)$ are given by Eqs. (25) and (26), where $\varphi(x)$ for an X array is

$$\varphi(x)=\frac{\tan(\pi x/L)}{\tan(\pi w/L)}\sqrt{\frac{\tan^2(\pi w/L)-\tan^2(\pi a/L)}{|\tan^2(\pi a/L)-\tan^2(\pi x/L)|}}. \quad (59)$$

For a high field $(H_a \gtrsim H_0)$, $H(x)$ is given by

$$H(|x|>a) \simeq H_a-\frac{H_0}{2}\ln\left|\frac{\sin^2(\pi w/L)}{\sin^2(\pi x/L)}-1\right|, \quad (60)$$

which is approximated for $L \simeq 2w$ as

$$H(|x|>a) \simeq H_a+H_0\ln\left[\tan\left(\frac{\pi|x|}{L}\right)\right]. \quad (61)$$

Figure 5 shows $H(x)$ and $J(x)$ profiles for an X array with $L/w=3$ (solid lines) and for an isolated strip line (dashed lines). Note that the edges of the neighboring strips are at $x/w=\pm 2$. Both profiles for $|x|<w$ are similar to those for an isolated strip line. The magnetic field for $w<|x|<L/2$ is enhanced due to the demagnetization effect of neighboring strip lines.

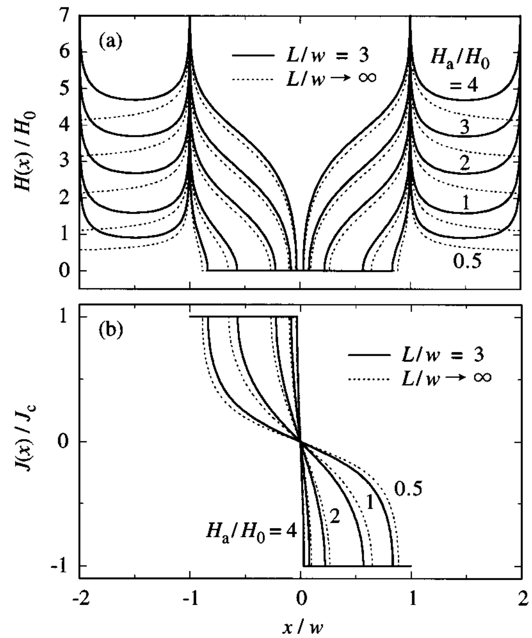

FIG. 5. Profiles of (a) magnetic field $H(x)$ and (b) current density $J(x)$ in an X array where $L/w=3$ (solid lines) and in an isolated strip line where $L/w \rightarrow \infty$ (dashed lines) at $H_a/H_0=0.5,1,2,3,$ and 4.

The magnetization M for an X array in the critical state is given by

$$M=\frac{L^2}{2\pi w d}\int_0^{H_a}\ln\left[1-\frac{\sin^2(\pi w/L)}{\cosh^2(H'/H_0)}\right]dH' \quad (62)$$

and is expanded as the following series:

$$-\frac{M}{M_0}=1-\frac{2}{\nu_x^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}e^{-2k\eta}\sin^2(k\nu_x), \quad (63)$$

where $M_0=wJ_c/2$, $\nu_x=\pi w/L$, and $\eta=H_a/H_0$. For low fields $(\eta \ll 1)$, we have

$$-\frac{M}{M_0} \simeq -\frac{1}{\nu_x^2}\left[2\eta\ln(\cos\nu_x)+\frac{\eta^3}{3}\tan^2\nu_x+\cdots\right] \quad (64)$$

and, for high fields $(\eta \gg 1)$,

$$-\frac{M}{M_0} \simeq 1-2e^{-2\eta}\frac{\sin^2\nu_x}{\nu_x^2}+\cdots. \quad (65)$$

When the spacing in an X array is large $(\nu_x \ll 1)$, Eq. (62) is reduced to

$$-\frac{M}{M_0} \simeq \tanh\eta+\frac{\nu_x^2}{6}\frac{\tanh\eta}{\cosh^2\eta}+\cdots. \quad (66)$$

Hysteretic ac loss per unit volume for one field cycle $P(H_m)$ is

$$P(H_m) = -\frac{2\mu_0 L^2}{\pi w d}\int_0^{H_m} dH_a (H_m - 2H_a) \times \ln\left[1 - \frac{\sin^2(\pi w/L)}{\cosh^2(H_a/H_0)}\right]. \quad (67)$$

IV. CONCLUSIONS

The electromagnetic properties in the Meissner state and the critical state were determined analytically for two types of superconducting strip-array systems, a Z stack and an X array, in perpendicular magnetic fields. The magnetic field H and the current density J for both systems can be derived by using simple transformations of H and J for an isolated strip line. For a Z stack in which the distance between the strips is much smaller than the width, electromagnetic properties in the critical state are similar to those for slab superconductors.

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