Spin-up and nucleation of vortices in superfluid ^{4}He

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It is shown that the Ginzburg-Landau model corrected for the normal component describes adequately the spin-up problem for the superfluid liquid helium. An analysis of the Eckhaus instability in an inhomogeneous rotationally invariant system is presented. It has been found that the number of vortices which can be nucleated at the threshold of instability scales with the radius of the container as $R^{3/4}$. The effect of excitation of the vortex loops by thermal fluctuations is considered, and the barrier and the nucleation rate are evaluated. [S0163-1829(96)05941-3]

I. INTRODUCTION

One of the most severe and important experimental tests of a dynamical model of a superflow in helium is the well-known spin-up problem in which a container with a superfluid is spun until a steady-state rotation is reached.[1] Here the transient behavior of the cell, while transferring an angular momentum to the superfluid not only reflects interaction of the superfluid with the walls but also gives insight into the nature of the superfluidity itself. Specifically, one is interested in the mechanism of vortex nucleation at the walls due to rotation and consequently vortex-wall interaction and their role in the transient and steady-state rotation behavior of the superfluid.

A conventional approach to the spin-up problem is to describe it by a two-fluid hydrodynamic model corrected by an equation for vortex line dynamics.[1–3] The vortex lines, containing the vorticity of the superfluid component and produced topologically multiconnecting space, interact with the normal component that leads to mutual friction.[4] It is evident that this hydrodynamical description does not catch the key point of the spin-up problem, namely, the vortex nucleation which actually causes the spin-up of the superfluid component. Thus the vortex nucleation at the container walls and its dynamics are the primary superfluid relaxation mechanism toward a steady rotation.

In this paper we suggest a different approach to the spin-up problem. Let us consider a cell containing a superfluid helium rather close to the superfluid transition temperature T_λ. When the cell is rotated with an angular velocity Ω, the normal component is involved in a solid body rotation with $\vec{V}_n=\vec{\Omega}\times\vec{r}$ on rather a short time scale of the order of r^2/ν_n, where ν_n is the kinematic viscosity of the normal component. The superfluid component cannot participate in the uniform rotation up to the point where the potential flow condition is satisfied, i.e., $\vec{\nabla}\times\vec{V}_s=0$. It follows also from the order-parameter description of the Ginzburg-Landau (GL) type model, introduced by Ginzburg and Pitaevskii for a superfluid[5] (which will be discussed in detail in Sec. II). Indeed, then

$$\vec{V}_s=\frac{\hbar}{m}\nabla\chi \quad \text{and} \quad \rho_s=m|\Psi|^2, \tag{1}$$

where $\Psi=|\Psi|\exp(i\chi)$ is the complex order parameter of the superfluid condensate.[5] Pitaevskii then generalized this description by also taking into account hydrodynamics of the normal component.[6] In this framework the resolution of the spin-up of the superfluid component is the nucleation of the quantized vortices which are the topological solutions of the GL equation.[5] Outside the vortex core which is normal, one gets from Eq. (1) the superfluid velocity circulation around a single quantum vortex

$$\kappa=2\pi\hbar/m. \tag{2}$$

Moreover the GL equation also provides the mechanism of vortex nucleation due to phase instability.[7] Since we suggest starting from the GL equation for the order parameter which naturally includes the vortex nucleation, we cannot avoid discussing two main objections to the approach.

The first and most severe one is the applicability of the GL-type mean-field theory which, although well known, does not properly describe even equilibrium properties of the superfluid transition.[8] Since we are interested just in the vortex nucleation mechanism which results from the translational symmetry breaking of the condensate,[7] we believe that this mechanism will remain valid also in a more elaborate approach that takes fluctuations into account. The first step in this direction was taken recently.[9] However as has been done in the equilibrium case one can correct in an artificial manner the wrong critical exponents appearing in the mean-field approximation.[10]

The second objection is related to the comparison with already existing theories of vortex nucleation which have been used to analyze and discuss the experimental data. We will not discuss pinning models since they have already been compared with nucleation models and criticized by Varoquaux *et al.*[11] The generally accepted consideration which justifies the existence of quantum vortices in superfluid helium, goes back to Feynman's ground-breaking idea.[12] According to Feynman[12] it is energetically favorable to create a vortex ring of the radius R in a superfluid if the flow velocity exceeds the self-induced velocity of the ring

$$V_{sc} \geqslant V_R = (\hbar/2mR)\ln(8R/a-1), \tag{3}$$

where a is the core radius of the vortex related to the superfluid correlation length with the same temperature dependence near T_λ. Then the potential flow condition, introduced by Landau for the superfluid component,[13] $\vec{\nabla}\times\vec{V_s}=0$ breaks down locally at the vortex core. The predicted critical velocity is temperature independent but size dependent with a value which is equal to about 10 cm/s in a channel of about 10^{-4} cm. Firstly, this velocity is not an actual velocity of the vortex nucleation due to the existence of a large energy barrier between the states of higher and lower energies. Secondly, experimentally observed strong temperature dependence of V_{sc} and independence on the channel size contradict the theory discussed above.[12] And finally even in early experiments the observed values of V_{sc} were much larger than predicted by Eq. (3). Two mutually excluded theories of vortex nucleation were suggested later on: a classical one based on a thermodynamical approach of thermal excitation over a barrier separating two states (with and without a vortex) and applicable at higher temperatures; and a quantum one, based on an idea of quantum tunneling under the barrier which is applicable at very low temperatures. The theory of homogeneous thermal nucleation of a quantized vortex ring in a superflow due to Iordanskii,[14] Langer, and Fisher[15] (ILF) suggested the mechanism of the vortex nucleation. However, a pure thermodynamical approach to a nonequilibrium state has been applied. As an energy barrier, Eq. (3) for the energy of the critical vortex ring was taken. Then the nucleation rate per unit volume over the free energy barrier is

$$\Gamma = \Gamma_0 \exp[-E_a/k_B T], \tag{4}$$

where $E_a = \rho_s(\hbar^2\pi^2/m^2R)(\ln 8R/a-3)$ is the energy of a vortex ring of radius R. It is clear that the prefactor Γ_0 critically controls the final result on the critical velocity and the critical ring radius, and cannot be calculated in the framework of the theory.[14,15] Although the theory predicts temperature dependence of the critical velocity $V_{sc}\sim\rho_s/T$, the value of the critical velocity predicted is larger up to an order of magnitude of that found in early and recent experiments on channel flow in several microns and submicrons size channels.[11,16] And finally, very recent experiments[17] on the energy barrier for the vortex creation in the channel flow reveals a scaling with V_s which is different from that predicted by the ILF theory. At this point we would like to emphasize that a dynamic instability approach to the superflow was suggested by Kramer[18] and Mikeska[19] more than 25 years ago, however, without direct relation to the vortex nucleation mechanism. Later on this idea was reiterated in various applications, particularly in the description of the critical behavior of superfluid helium in the vicinity of T_λ under a heat current.[9] Just recently this idea was applied to the problem of vortex nucleation in a channel flow with an attempt to explain quantitatively the experimental data.[7] However, in spite of its long history, the GL model was never used to describe the spin-up problem in the superfluid helium.

Thus from this short review it follows that the existing theories are not able to explain even qualitatively the experimental data on vortex nucleation. A more elaborate theory which will explain the observed velocity and temperature dependences of the energy barrier and give reasonable value for the prefactor Γ_0, is, therefore, definitely needed. The approach suggested provides different scaling of the energy barrier with the superfluid velocity and the expression for the prefactor Γ_0. In this we will correct an erroneous statement made in the literature on the superflow, i.e., that the GL model cannot incorporate naturally the thermal excitation over the barrier.[16] On the other hand, low critical velocities observed in the experiment, in our opinion, are possibly a result of remnant vorticity, and we hope that next generation of the experiments will overcome this problem, e.g., in the way suggested about 20 years ago.[21] And finally we would like to point out that a superfluid ^{3}He is a more appropriate system to test theories since the pinning of vortices and trapped vorticity there are much less severe problems.[22] That is the reason why the steady-state rotation without vortices has been observed experimentally only in ^{3}He and not in ^{4}He.

The paper is divided into two parts. In the first part we will discuss just the mechanism of the vortex nucleation due to the intrinsic instability of the condensate, and the resulting critical velocity scaling. In the second part we will discuss the vortex nucleation due to thermal excitation over a barrier which is a relevant problem at temperatures close to T_λ.

II. THE VORTEX NUCLEATION PROBLEM AND CRITICAL VELOCITY

According to Pitaevskii[6] dynamics and vortex nucleation of a superfluid helium near the superfluid transition temperature T_λ can be described by a set of equations, which are a generalization of a two-fluid hydrodynamical model of Landau[13,23]

$$i\hbar\,\partial_t\Psi = -\frac{\hbar^2}{2m^2}\Delta\Psi + (\mu+\mu_s)m\Psi - i\Lambda\left[\frac{1}{2}\left(\frac{i\hbar}{m}\nabla+\mathbf{V_n}\right)^2+\mu_s\right]m\Psi, \tag{5}$$

$$\begin{gathered}\partial_t\rho + \operatorname{div}\mathbf{j}=0,\\ \mathbf{j}=\rho_s\mathbf{V_s}+\rho_n\mathbf{V_n},\quad \rho_n=\rho-m|\Psi|^2,\\ \rho_s\mathbf{V_s}=-\frac{i\hbar}{2}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*),\\ \rho_s=m|\Psi|^2,\end{gathered} \tag{6}$$

$$\partial_t\mathbf{j}+\nabla\Pi=0, \tag{7}$$

$$\partial_t S + \operatorname{div}\left(S\mathbf{V_n}-\frac{\kappa}{T}\nabla T\right)=\frac{R}{T}, \tag{8}$$

where $\rho_n,\rho_s,\mu_n,\mu_s$, and V_n,V_s are the densities, chemical potentials and velocities of normal and superfluid components correspondingly, Π is the stress tensor (including viscous terms and pressure), S is an entropy, and T is a temperature, Λ is the parameter characterizing relaxation rate, and R is a function of dissipation (see, for details, Refs. 6, 23, and 10). For simplicity we restricted the analysis by small values of v_n, v_s and neglected in Eq. (7) nonlinear in v_n terms.

Near T_λ the chemical potentials of normal and superfluid components can be written in the form

$$\mu_s = \epsilon\Psi - b|\Psi|^2\Psi + \text{higher-order terms}, \tag{9}$$

$$\mu = \mu_0(S,\rho) - V_n^2, \tag{10}$$

where μ_0 is the chemical potential of motionless liquid, $\epsilon = (T_\lambda - T)/T_\lambda$ is the reduced temperature, and b is a positive constant in a mean-field approximation. In order to correct the mean-field scaling and to describe the properties of superfluid helium near T_λ one can introduce by hand the temperature dependence of ϵ and b.[10,20] Then one gets $\epsilon \sim (T-T_\lambda)^{4/3} \ll 1$ and $b \sim (T-T_\lambda)^{2/3}$.[10,20] The term $\sim V_n^2$ in the expression for μ arises from the renormalization of chemical potential of the fluid due to macroscopic motion of the normal component (see, e.g., Ref. 23).

To describe the spin-up problem, the set of equations (5)–(8) can be significantly simplified in the first order of the small parameter ϵ. We assume that there are no external temperature fluxes and the temperature is kept constant.

It is convenient to perform the following scaling of the variables (see for details Ref. 10):

$$x \to \frac{x}{\xi(\epsilon)},$$
$$\xi(\epsilon) = \sqrt{\hbar^2 \rho_{s0}/2m^2 \Delta C_\mu T_\lambda} = 1.63 \times 10^{-8} \epsilon^{-2/3} \text{ cm}, \tag{11}$$

$$\Psi \to \frac{\Psi}{\Psi_e(\epsilon)}, \quad \Psi_e(\epsilon) = \Psi_{e0}\epsilon^{1/3} = 0.23 \times 10^{12} \epsilon^{1/3} \text{ cm}^{-3/2}, \tag{12}$$

$$V_n \to \frac{V_n}{V_e(\epsilon)}, \quad V_e(\epsilon) = \frac{\hbar}{m\xi(\epsilon)} = 9.74 \times 10^3 \epsilon^{2/3} \frac{\text{cm}}{\text{s}}, \tag{13}$$

$$\rho_s \to \frac{\rho_s}{\rho_e(\epsilon)}, \tag{14}$$
$$\rho_e(\epsilon) = m|\Psi|^2 = \rho_{e0}\epsilon^{2/3} = 1.43 \rho_\lambda T_\lambda^{2/3} \epsilon^{2/3} = 0.35 \epsilon^{2/3} \frac{\text{g}}{\text{cm}^3},$$

$$t \to t/\tau(\epsilon), \quad \tau(\epsilon) = \frac{\xi(\epsilon)}{2V_e(\epsilon)} \approx 0.83 \times 10^{-12} \epsilon^{-4/3} \text{ s}, \tag{15}$$

where $\xi(\epsilon)$ is the correlation length below the λ point, ΔC_μ is the specific-heat jump, Ψ_{e0}, ρ_{e0} are amplitudes of the temperature dependence of the equilibrium values of Ψ and ρ_s in bulk helium, $\rho_\lambda = 0.146$ g/cm^3, m is the mass of the helium atom, $V_e(\epsilon)$ is the temperature dependent unit of velocity, and $\tau(\epsilon)$ is the unit of time. Under this scaling in the leading order of ϵ we are left with the set of equations:

$$\partial_t \Psi = -\frac{i}{2}(\Delta\Psi + \Psi - |\Psi|^2\Psi - V_n^2\Psi) + \frac{\Lambda}{2}[(\nabla - i\mathbf{V_n})^2\Psi + \Psi - |\Psi|^2\Psi] + O(\epsilon), \tag{16}$$

$$\partial_t \rho_n + \text{div}\rho_n \mathbf{V_n} + O(\epsilon) = 0, \tag{17}$$

$$\partial_t \rho_n \mathbf{V_n} = \eta \Delta \mathbf{V_n} + O(\epsilon), \tag{18}$$

where $\eta/\rho_n = \nu_n$. If no external temperature gradient is imposed and the temperature is kept constant, then the equation for the entropy splits off and consequently the residual part of the chemical potential $\mu_0 = \text{const}$ is gauged away. Here we neglect the coupling between the order parameter Ψ and the temperature T via dependence $\mu_0(T)$. In Ref. 24 it is claimed that this coupling might be relevant for the dynamics of superfluid. We agree that if an external heat flux is supplied, the temperature variations cause macroscopic motion of the superfluid due to counterflow convection effects. Even in an isolated system mutual friction between normal and superfluid components produces some heat.

However, for the small velocity limit that we consider, these temperature variations should not affect in the main order of ϵ the motion of the fluid. Also, the coupling coefficient from[24] is a phenomenological parameter, and there is no reason for this parameter to be large enough to provide an effective coupling.

For velocities much smaller than the speed of the first sound, we can treat the normal fluid as incompressible. The dynamics of the normal part is described by a simple linear equation

$$\partial_t \mathbf{V_n} = \nu_n \Delta \mathbf{V_n}. \tag{19}$$

Equation (19) admits a rigidly rotating stationary solution

$$V_n = \Omega r. \tag{20}$$

Equation (16) is reminiscent of that of the Ginzburg-Landau equation for type-II superconductors with a complex order parameter.[25] The role of an external magnetic field is played by the angular velocity, and that of the corresponding vector potential by the velocity of the normal component. Then by analogy one expects that at $\Omega \leq \Omega_{c1}$, there exists a motionless superfluid component with no vortices. At $\Omega > \Omega_{c1}$ vortices will be nucleated and penetrate into the fluid producing a vortex lattice in the interior of a helium container. As follows from the experiments on the superfluid ^{4}He Ω_{c1} is too low to be detected.[4] On the other hand, Ω_{c2} (which is analogous to H_{c2} in superconductors and at which superfluidity will be completely destroyed in the sample) is too high to be reached experimentally. However, as already mentioned, one should keep in mind that theoretical estimates of Ω_{c1} were based on erroneous energy considerations.[4]

We will treat the superfluid part independently from the normal one: we consider the spin-up of the superfluid part in a rigidly rotating flow of the normal component. Of course, this assumption does not describe the first stage of the spin-up because it takes some time until rigid rotation of normal fluid sets in. However, this stage is relatively short of the order of r^2/ν_n, and does not affect the vortex nucleation mechanism, which is the main subject of the studies.

As a result of these assumptions we are left with one equation for the order parameter of the superfluid Eq. (16) in the given velocity field of the normal component. This equation is asymptotically correct in the vicinity of the λ point. Moreover, one can speculate that the equation also describes some aspects of spin-up for ^{4}He near zero temperature with $\Lambda \to 0$. Of course, the assumption is that $\rho_n \gg \rho_s$ is no longer valid, and opposite inequality takes place. However, we can imagine the situation when a small part of superfluid ^{4}He is in still normal ^{3}He, which plays the role of normal component. Here we expect that the vortex nucleation is not affected by this circumstance. Therefore, we will consider Eq. (16) approximately valid for all temperatures with the parameter $\Lambda \to 0$ for $T \to 0$ and diverges as $\Lambda \sim \epsilon^{-1/3}$ for $T \to T_\lambda$.

III. CRITICAL VELOCITY

Critical velocity for the bulk instability of the superflow in a channel was found in Refs. 18 and 19, which in our scaled variables reads as $V_n = 1/\sqrt{3}$. It appears that the critical velocity is not Λ dependent. This value coincides with the well-known critical wave number of the Eckhaus instability in pattern-forming systems.[26]

For the cylindrical geometry the method employed in Refs. 18 and 19 cannot be implemented because of the dependence of the normal velocity and as a result of the superfluid density on a radial coordinate in the presence of rotation. In order to define the critical angular velocity we use a method similar to that of Ref. 27 implemented for thin superconducting films in a magnetic field. It turns out that the analysis of the critical velocity for the circular container is much more complicated than that for the channel. We show that the critical velocity (or angular frequency) is always higher than the one for the parallel flow. Moreover, at the threshold of the instability only a certain angular mode with the azimuthal number $n \sim R^{3/4}$ grows. This selection mechanism has no analog for the parallel flow. In this section we consider the stability of stationary solution, corresponding to nonrotating superfluid. Our aim is to find the critical frequency of the rotation as the function of the container radius R, and the critical angular number of the most unstable perturbation. We carry out our calculation in the physically relevant limit of very large R (for real vessels of the diameter 1 cm the radius R after the scaling is of order 10^6 depending on the closeness to the λ point). We employ the method of matched asymptotic expansions in the bulk of the container and near the container wall. Analytical results appear to be in very good agreement with the results of numerical simulation even for moderate values of $R = 100-200$.

A. Stability of the nonrotating solution

Equation (16) possesses a stationary solution corresponding to nonrotating superfluid in the presence of the solid body rotation of the normal component. For this solution the $\arg\Psi = 0$, whereas the superfluid density $|\Psi|^2$ dependence on the radius is given by the following equation:

$$\partial_r^2 F + \frac{\partial_r F}{r} + F - F^3 - \Omega^2 r^2 F = 0, \tag{21}$$

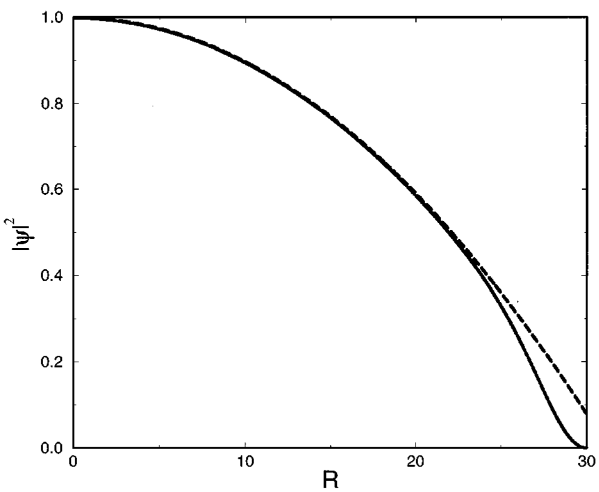

FIG. 1. The order-parameter amplitude $|\Psi|$ as a function of radius for $R = 30$, $\Omega = 0.032$, and $\gamma = \infty$. The solid line shows numerical solution, the dashed line is the adiabatic approximation Eq. (A5).

where $F = |\Psi|$. Equation (21) has to be complimented by the conditions at the $r = 0$ and the boundary condition at the wall of the container $r = R$, where R is the radius of the container. As a boundary condition at the wall we will take a condition of finite suppression of the superfluid density by the wall, i.e.,

$$\partial_r \Psi + \gamma \Psi = 0 \quad \text{for } r = R, \tag{22}$$

where γ characterizes the suppression rate of the order parameter. For $\gamma \to 0$ we have a no-flux boundary condition ($\partial_r \Psi = 0$) and for the $\gamma \to \infty$ zero condition, $\Psi = 0$ correspondingly.

The solution of Eq. (21) for general Ω, R is accessible only numerically. For some particular choice of parameters Ω, R, and for $\gamma \to \infty$ the solution is depicted in Fig. 1.

The stationary solution is stable for $\Omega < \Omega_c$ and loses stability above the critical angular velocity Ω_c. In order to find Ω_c we need to perform a linear stability analysis near the stationary solution of Eq. (21). Instability of the stationary solution leads to nucleation of vortices and the corresponding spin-up of superfluid. It is similar to that of penetration of a magnetic field into type-II superconductors and creating of the vortex state.[25]

We substitute a perturbative solution of the form

$$\Psi = F(r) + W(r, \theta, t), \tag{23}$$

where W is a small generic perturbation.

It is convenient to transform to the frame rotating with frequency Ω. In the corotating frame we obtain in the linear order in W

$$\partial_t W = \frac{\Lambda + i}{2}[\Delta W + (1 - 2F^2 - \Omega^2 r^2) W - F^2 W^* - 2i\Omega \partial_\theta W]. \tag{24}$$

At the threshold of the instability (which appears to be stationary bifurcation) the growth rate of the perturbations vanishes and we can set $\partial_t W = 0$. Representing $W = A + iB$, and looking for solution in the form

$$\begin{pmatrix} A \\ B \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \exp[in\theta], \tag{25}$$

we arrive at the following equations for A_n, B_n (where n is the rotation or azimuthal number of the perturbation):

$$\partial_r^2 A_n + \frac{\partial_r A_n}{r} - \frac{n^2 A_n}{r^2} + (1 - 3F^2 - \omega^2 r^2) A_n + 2\Omega n i B_n = 0, \tag{26}$$

$$\partial_r^2 B_n + \frac{\partial_r B_n}{r} - \frac{n^2 B_n}{r^2} + (1 - F^2 - \omega^2 r^2) B_n - 2\Omega n i A_n = 0. \tag{27}$$

The boundary conditions are $A_n(R) + \gamma A_n(R) = B_n(R) + \gamma B_n(R) = 0$ and $A_n(r), B_n(r)$ remain bounded for $r \to 0$.

The critical angular velocity Ω_c is defined by the condition when Eqs. (26) and (27) have a nontrivial eigenfunction satisfying boundary conditions as we increase Ω from zero (see, for details, Ref. 27). For $R \gg 1$ it is a tough numerical problem. However, the solution of the problem can be achieved in the limit of very large R. Finding the most unstable azimuthal number comprises three steps. Firstly, we determine the solution in the bulk of container using the advantage of so-called adiabatic approximation. This approximation is similar to the adiabatic approximation used in derivation of the phase diffusion equation in the pattern-forming systems. Physically it means that the amplitude (or real part in the present case) adiabatically slaves the phase dynamics due to a difference in time scales. The ''adiabatic'' solution loses its validity near the wall of the container (because the boundary condition is violated) and another approximation has to be applied. Secondly, we apply another valid approximation upon the wall of the container (inner approximation). This approximation breaks down at some distance from the wall. Finally, we match both solutions in the region where both approximations are valid. Matching of the adiabatic solution with the solution at the wall of the container fixes critical frequency of the rotation and selects the most unstable mode with the rotation number n. The details of the procedure are described in the Appendixes A–C; here we present the answer (the calculations are rather cumbersome, and were carried out with the aid of a program for analytical computations, MAPLE).

The analysis shows that the most unstable eigenmodes are localized in the narrow layer of the width r_b near the container wall. We obtained $r_b \sim \sqrt{R} \ll R$ for large R. The value of most unstable azimuthal number n and the critical frequency Ω_c for the container radius R is given by the expressions:

$$n = Q(\gamma) R^{3/4}, \tag{28}$$

$$\Omega_c = \frac{1}{R} \sqrt{\frac{1}{3} + \frac{\Delta(\gamma)}{R^{1/2}}}. \tag{29}$$

The dimensionless parameters $Q(\gamma), \Delta(\gamma)$ are the functions of the suppression rate γ and are obtained by the matching of outer and inner expansions. The values of Δ and corresponding Q as a function of the suppression rate γ are shown in Fig. 2. Note that for the zero boundary condition (which corresponds to $\gamma \to \infty$) the parameter Δ vanishes.

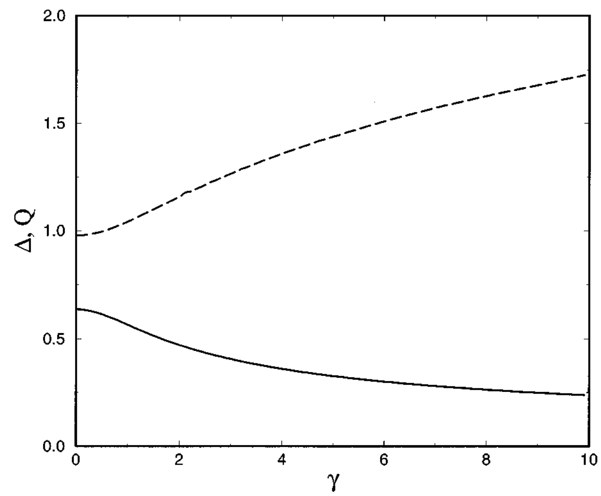

FIG. 2. Dependence of Δ (solid line) and Q (dashed line) as functions of γ.

B. Comparison of critical velocities

The quantity $\Omega_c R$ in Eq. (29) has a dimension of velocity. Corresponding critical velocity for parallel flow is smaller ($V_c = 1/\sqrt{3} = 0.5773$). Therefore, inhomogeneous suppression of the order parameter due to rotation results in increase of the critical velocity.

According to Refs. 18, 19, and 10 the critical velocity of parallel flow in the physical units is

$$V_c = \Omega_c R = \frac{1}{\sqrt{3}} V_e \approx 3.3 \times 10^3 \epsilon^{2/3} \frac{\text{cm}}{\text{s}}, \tag{30}$$

where V_e is the velocity above which the density of superfluid vanishes [see Eq. (13)]. The power 2/3 is introduced to assure correct scaling of the superfluid density near the λ point. For $\tilde{R} = 1$ cm and $\epsilon = 10^{-4}$ from this formula one gets $\Omega_c \approx 11\ \text{s}^{-1}$. Correction to the frequency due to finite R is irrelevant in this case, since scaled R is of the order 10^5 ($\tilde{R}$ is in the dimensional unit).

In mean-field theory one has $\Omega_c R = (\hbar/m)(\epsilon^{1/2}/\sqrt{3}\xi_0)$. For $\tilde{R} = 1$ cm and $\epsilon = 10^{-4}$ the difference in the angular velocity is rather significant, e.g., $\Omega_c = 50\ \text{s}^{-1}$. From the conventional Feynman equation one obtains $\Omega_c = 1.7 \times 10^{-2}\ \text{s}^{-1}$, i.e., the difference is more than 3 orders of magnitude. At $\epsilon = 10^{-6}$ just about an order of magnitude difference still remains and is equal $\Omega_c = 0.5\ \text{s}^{-1}$. At $\tilde{R} = 0.1$ cm and $\epsilon = 10^{-6}$ both values for Ω_c will be of the same order of magnitude of several Hz. What is crucial here is the different scaling of Ω_c with R and the strong dependence on ϵ in our case. For the case $\epsilon = 10^{-6}$ and $\tilde{R} = 0.1$ cm scaled R is of the order 10^3, and the correction to the frequency can be of the order of several percent.

C. Phase equation

The nonlinear stage of the dynamics can be partially recovered by a simplified phase diffusion equation. Indeed, the modulus of Ψ follows the phase according to the relation $|\Psi|^2 \approx 1-(\nabla\phi-\mathbf{V_n})^2$. Substituting this expression for the equation for the phase we obtain a single equation (for $\Lambda\to\infty$, close to T_λ)

$$\partial_t \phi = \frac{\Lambda}{2}\left(\Delta\phi-(\nabla\phi-\mathbf{V_n})\frac{\nabla(\nabla\phi-\mathbf{V_n})^2}{1-(\nabla\phi-\mathbf{V_n})^2}\right) + \text{higher-order terms}. \quad (31)$$

This equation reminds us of that from the theory of a nonlinear stage of Eckhaus instability (see, e.g., Ref. 28). However, the difference is that velocity of a normal component cannot be gauged away by substitution $\nabla\phi_{\rm new} = \nabla\phi-\mathbf{V_n}$ because field V_n has nonzero vorticity. This circumstance results in a completely different asymptotic state of the system when the instability saturates: if ''regular'' Eckhaus instability just drives the system to a new value of the velocity in the stable region, whereas in our case the instability results in the creation of an array of vortices. This difference can be explained by the following considerations: the vortex which is nucleated at the boundary climbs normally to $\vec{V}_n$ towards the center of the container. The resulting superflow around the vortex reduces the velocity difference V_R-V_s at the boundary. Vortices will be generated at the boundary till the moment when $V_R-V_s\leqslant V_{cs}$ is satisfied. This new state with n vortices will be stable at a given value of Ω. The mechanism of nonlinear relaxation of the unstable solution and a particular selection of a new stable solution following the instability (that is analogous to the wavenumber selection problem in pattern-forming systems) are issues which can be studied by analysis of nonlinear evolution, e.g., using the nonlinear phase diffusion equation (31). As shown recently by numerical simulations of the Swift-Hohenberg equation,[32] thermal fluctuations smear the Eckhaus boundary and change the scaling of the finite wave number (or superfluid velocity in our case) as a function of the initial one. The temporal evolution of the unstable state changes as well.[32] However in any case the result differs from conventionally accepted opinion that the steady state of rotating superfluid helium can be described by $\Omega_i = N\pi\hbar/m$ where N is the number of vortices per unit area. Thus if $\Omega_i \geqslant \Omega_c$ the number of vortices will be defined by the relation $(\Omega_i-\Omega_f)=n\pi\hbar/m$ where $(\Omega_i-\Omega_f)\sim\Omega_i-\Omega_c$ and $(\Omega_i-\Omega_f)\leqslant\Omega_c$. Here $\Omega_f\leqslant\Omega_c$ is the final velocity of the superflow. Therefore the difference between N and n can be considerable.

The difference with pattern-forming systems can be followed even further. In principle one can imagine that inside the Eckhaus stable region one can bring the system locally into an unstable state by a local perturbation of a superfluid velocity, and due to weak conservation law $|\Psi|^2(V_s-V_n)=\text{const}$ it results in a local superfluid density dip. This solution is unstable to small perturbations of the superfluid density which causes further deepening of its amplitude to zero, and finalizes in a phase slip. This process only involves a homoclinic unstable solution. The resulting vortex climbs into the container center, stays there, and reduces the velocity difference at the boundary. Thus one creates different states with $V_s\leqslant V_{cs}$ and a different number of vortices at the center of the container.

IV. NUMERICAL RESULTS

To go beyond stability analysis we performed simulations of Eq. (16). We used a quasispectral code based on fast-Fourier transform (FFT) for the rectangular domain. Boundary conditions modeled the $\Psi(R)\to 0$ condition. This was achieved by adding the attenuation factor $d\Psi$ to the linear term in Ψ term in Eq. (16) beyond the radius R according the following formula: $d=0$ for $r\leqslant R$ and $d=-\sinh[0.5\sqrt{r^2-R^2}]$ for $r>R$. Although it is difficult to evaluate the effective suppression rate γ corresponding to this boundary condition, we note that the function ψ vanishes at the distance of the order of few dimensionless units away from the wall, which roughly can be modeled by $\gamma\approx 2$–3. We used typically 128×128 harmonics of FFT and the time step $0.05/|\Lambda+i|$ and $R=65$. We also performed simulations with 256×256 harmonics in $R=140$.

For $R=65$ we have observed instability at frequency $\Omega=0.009\,45$. For this radius and a rough estimate for $\gamma\approx 3$ our analytic expression (29) gives $\Omega=0.009\,52$, which corresponds to the numerical result within 1%. The reason for the discrepancy is obviously uncertainty in the choice of γ. Note that the result without the $1/\sqrt{R}$ correction is $\Omega=1/(\sqrt{3}R)\approx 0.0088$, and is much below the numerical value.

We observed nucleation and consequent tearing off of the vortices for $\Omega\geqslant\Omega_c$ irrespective of Λ. However, the character of the nucleation and asymptotic states depends strongly on Λ.

Let us discuss first the case $\Lambda\to\infty$. Slightly above the critical value Ω_c we observed nucleation of several vortices. Nucleation occurs at nonlinear stage of the instability when a set of single zeros (four zeros for $R=65$) is torn off at the radius R. These zeros are the seeds for the vortex cores. The vortices propagate into the interior of the container and finally form a lattice, reminiscent of that of the Abrikosov lattice[25] (see Fig. 3). Further increase of Ω results in formation of additional vortices. Also, the number of vortices does not depend linearly on Ω, however it approaches a linear law with increase of Ω (see Fig. 4).

It turns out that the character of nucleation is drastically different for $\Lambda\to 0$. Typically one has torn off whole clusters of vortices (see Fig. 4). These clusters can be considered as a perturbed multicharged vortex. The multicharged vortex is definitely unstable and breaks down into single-charged vortices after some time. However, the lifetime of the vortex happens to be proportional to Λ and diverges for small Λ. Explanation of this phenomenon is reported in Ref. 29. The reason for multivortex nucleation is that for $\Lambda\to 0$ simultaneously many azimuthal modes are unstable and grow independently. As a result, these modes have close zeros at the same place. Finally, a perfect vortex lattice is formed, which can be considered as an analog of the Abrikosov lattice.

V. THERMAL EXCITATION OF THE VORTEX OVER THE BARRIER AT $\Omega<\Omega_C$

Contrary to hydrodynamical pattern-forming systems, in a superfluid helium thermal noise plays a crucial role in vor-

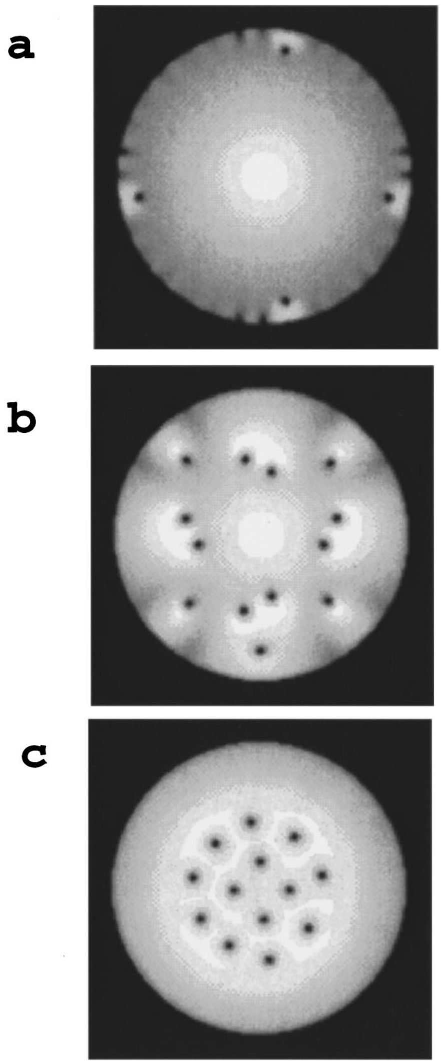

FIG. 3. The sequence of grey-coded images of $|\Psi|$ demonstrating nucleation of vortices and creation of the vortex lattice. The dark shade corresponds to zero of $|\Psi|$; the white one to its maximum value. Vortices are seen as black dots. The parameters are $\Omega = 0.01$, $R = 65$, $\Lambda \gg 1$. The initial condition is $\Psi = 1$ plus small-amplitude broadband noise. (a) $t = 100\Lambda$, (b) $t = 200\Lambda$, (c) $t = 1100\Lambda$.

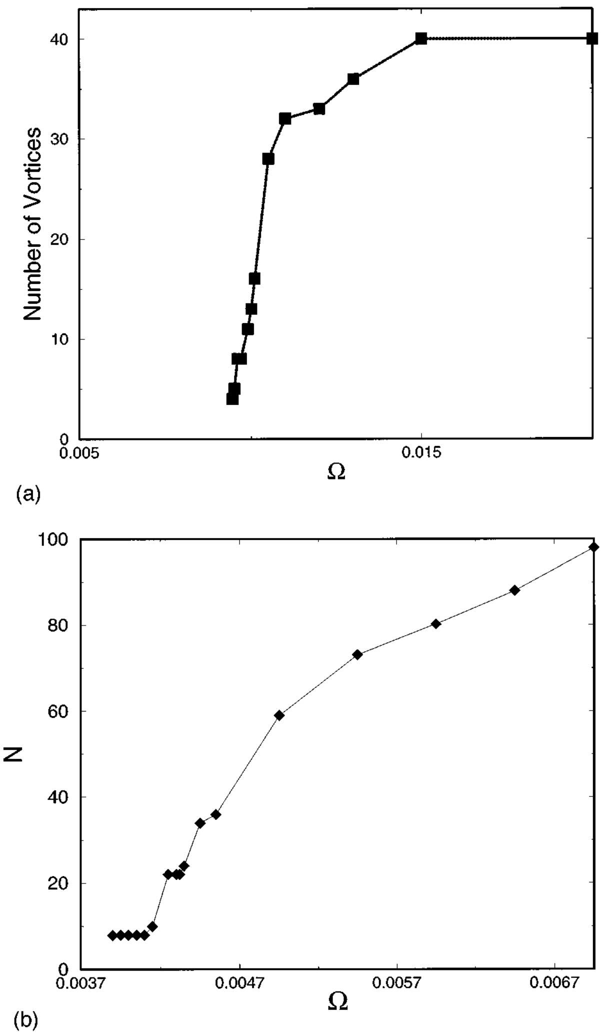

FIG. 4. Number of vortices as a function of Ω. (a) $R = 65$; (b) $R = 140$.

tex nucleation over the barrier particularly at high enough temperatures (above 1 K). The reason for this difference is in the macroscopic nature of hydrodynamic flow compared with the microscopic nature of the quantum vortex. According to the currently accepted ILF theory the nucleation rate over the barrier may be described by the Kramers' equation

$$\Gamma = \Gamma_0 \exp[-E_0/k_B T]. \tag{32}$$

There have been several attempts to estimate Γ_0 from general considerations or from experimental data. The difference scans many orders of magnitude from early estimates of Langer and Reppy[30] which gave $\Gamma_0 = 10^{34}$ Hz, up to the recent experiment for the microscopic channels, by Steinhauer *et al.*,[17] who obtained $\Gamma_0 = 10^7$ Hz. It is clear that this parameter fixes the fluctuation time scale, and it remains a challenge to find it from theory.

McCumber and Halperin[31] evaluated the thermoactivation rate for thin superconducting wires from the Ginzburg-Landau equation. This problem is similar to that considered by us, and we will use the analogy in the following estimates. They have obtained the following expression:

$$\Gamma_0 = \frac{N(T)}{\tau(T)}, \tag{33}$$

where $N(T)$ is in effect the number of statistically independent subsystems along the wire and $\tau(T)$ is the microscopic diffusion time inversely proportional to ΔT. The effective number $N(T)$ is approximately equal to the length of the wire measured in units of the coherence length $\xi(T)$

$$N(T) \approx \frac{L}{\xi(T)},$$

where L is the wire's length.

For the energetic dimensionless barrier of single phase-slip nucleation McCumber and Halperin[31] have obtained the following expression:

$$E_0=\left(\frac{4\sqrt{2}}{3}\sqrt{1-3k^2}-4k(1-k^2)\arctan\sqrt{\frac{1-3k^2}{2k^2}}\right), \tag{34}$$

where the wave number k is defined by the total supercurrent by the condition $j=k(1-k^2)$.

Direct implementation of the McCumber and Halperin method is technically impossible for our case. Our situation is essentially three-dimensional, and instead of one-dimensional phase slips we expect nucleation of the vortex loops. Even saddle-point calculations are technically impossible because of the explicit dependence of the equation on the radial coordinate. However, strong inhomogeneity of the problem is also a simplifying factor. Indeed, the superflow velocity is maximal near the edge of the container. Therefore, the activation barrier is smallest at the edge, whereas the bulk contribution is negligibly small (activation in the bulk has a much higher energetic barrier).

It seems plausible to consider an effective quasi-one-dimensional model for thermoactivation. We assume that only the boundary layer plays a role in the thermoactivation process. It follows from that consideration that the small vortex rings nucleated in the bulk shrink, whereas they will be metastable near the wall in the layer of the order of ξ. The vortex rings are stretched by the velocity field and cause the phase-slip events on the scale of the effective boundary layer. The width of the effective boundary layer r_b at the critical frequency Ω_c is of the order $\sqrt{R}\ll R$ since our analysis shows that the eigenfunctions for instability are localized in this layer.

We consider the thermoactivation of vortex loops in the cylindrical container of the radius R and the height H. The number of statistically independent degrees of freedom $N(T)$ in our context is simply $\tilde{R}/\xi(\epsilon)\times H/\xi(\epsilon)$, since we assume that the nucleating vortex loop is the size ξ. We neglect the contribution from the activation of the vortex rings in the bulk and take into account only boundary nucleation. The characteristic time in our case is $\tau_c=\tau/\Lambda$. In the context of the superfluid we obtain for the prefactor Γ_0, the following expression:

$$\Gamma_0=\frac{\tilde{R}H}{\xi(\epsilon)^2\tau_c(\epsilon)}\approx 10^{27}\epsilon^{7/3}\ \text{Hz}. \tag{35}$$

For $\epsilon\sim 0.001$ and for $\tilde{R}=H=0.1$ cm we obtain $\Gamma_0\sim 10^{17}$, which is reasonable for the experimental data.[11]

For the energetic barrier of the vortex nucleation in the superfluid helium we have in the dimensional units[10]

$$\Delta E=T_\lambda\Delta C_\mu\epsilon^2\sigma\xi(T)\left(\frac{4\sqrt{2}}{3}\sqrt{1-3k^2}-4k(1-k^2)\arctan\sqrt{\frac{1-3k^2}{2k^2}}\right)\ \text{erg}, \tag{36}$$

where $\Delta C_\mu=0.76\times 10^7$ erg cm^{-3} K^{-1} is the specific-heat jump at the λ point, and σ is the effective cross section corresponding to the excitation of a single phase-slip event.

The effective cross section (for the vortex loops of the radius ξ) in our case is the product of the correlation length $\xi(\epsilon)$ by the effective boundary layer width r_b. Our analysis shows (see Sec. III and the Appendixes) that the unstable modes leading to the vortex penetration are indeed localized in this layer near the container's wall. We can evaluate the value of r_b from the condition

$$r_b\sim\xi(\epsilon)\times\sqrt{\tilde{R}/\xi(\epsilon)}=\sqrt{\xi(\epsilon)\tilde{R}}.$$

For the container of the radius $\tilde{R}=0.1$ cm the width r_b is of the order 1 μm. Finally, we obtain the following expression for the dimensionless thermoactivation exponent $E^*=\Delta F/(k_BT_\lambda)$ for $k=\Omega R\rightarrow 1/\sqrt{3}$

$$E^*\approx\frac{\Delta C_\mu\epsilon^2\xi(\epsilon)^2\sqrt{\xi(\epsilon)\tilde{R}}}{k_B}(1-3\Omega^2R^2)^{3/2}\approx 10^3\epsilon^{1/3}(1-3\Omega^2R^2)^{3/2}. \tag{37}$$

For $\epsilon=0.001$ we estimate that the transition broadening at the critical rotation velocity $\Omega_cR=1/\sqrt{3}$ due to thermal fluctuations is about 1–2% of the total value of Ω.

We can use the same considerations to estimate the nucleation rate of a vortex loop in a narrow channel flow. Typical channel volume is of the order of 1 μm.[3,11,17] Then one gets $\Gamma_0\approx 10^{22}\epsilon^3$ that gives at $\epsilon=0.001$ $\Gamma_0=10^{15}$ Hz. The relative thermoactivation barrier is $E^*\approx 10r_c^3(1-3V^2)^{3/2}$ where $r_c\approx 3-5$ correlation lengths is the minimum radius of the vortex loop which can grow.[11] This leads to the broadening of the transition of the order to 1–2%. Thus both the nucleation rate and the broadening of the transition are in reasonable agreement with experiment.[11,17]

VI. CONCLUSIONS

The main results of our studies are the following:

(1) We showed that one can describe adequately the spin-up problem in superfluid helium starting from vortex-free superflow using the GL model corrected for the normal component velocity field. The superfluid is brought into rotation by the vortex nucleation process at the container wall where an instability of the condensate takes place. This instability is related to the translational symmetry breaking of the complex order parameter describing the superfluid state.

(2) We found that the number of vortices that can be nucleated at once scales with the radius of a container as $R^{3/4}$, since it is a maximum number of zeros in the wave function. On the other hand, it scales with $\Omega_i-\Omega_c$ linearly. It means that a derivative $dn/d\Omega_i$ is a finite constant contrary to analogous quantity in type-II superconductors dn/dH where it diverges at $H\rightarrow H_c$. These differences mean that in superfluid helium one can observe, in principle, nucleation of vortices one by one contrary to superconductors where already at the threshold a great number of vortices are generated.[25]

(3) We present a complete analysis of the Eckhaus instability of the initial nonuniform state and have derived the corresponding nonlinear phase diffusion equation. It differs from the well-known result for the uniform initial state by the appearance of additional gauge field which cannot be renormalized out due to time translation symmetry breaking by rotation. As a result, instead of the unsaturated Eckhaus instability one finds in the spin-up problem a new situation where the steady state includes vortices located at the center

of the container and effectively renormalizes the velocity of rotation.

(4) Scaling of the critical rotation velocity as a function of closeness to T_λ and of the radius of a container are given.

(5)By analogy with a pattern-forming system we consider also the finite-size effect on the instability. As a result, one can find a size of a channel below which the instability will not occur if in the mean-field approximation $\Omega_c=(1/R\sqrt{3})(1+2\pi^2\xi_0^2/\epsilon R^2)^{1/2}\geqslant 1$, i.e., at $R\leqslant\pi\xi/\epsilon^{1/2}$ a vortex cannot be created in a channel of radius R. Of course, the scaling should be corrected for deviation from the mean-field approach, namely the scaling $\epsilon^{2/3}$ instead of $\epsilon^{1/2}$.

(6) The same considerations lead to a simple formula for T_λ suppression due to rotation, namely $\delta\epsilon=(\Omega_c R m\xi_0\sqrt{3}/\hbar)^{3/2}$.

(7) Thermal nucleation over the barrier give scaling for the barrier and for the nucleation rate which are different from those given by previous theories.

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APPENDIX A: OUTER EXPANSION

In this section we focus on the solution, valid in the interior of the container, which we will call the outer solution. This solution can be obtained analytically in the limit of very large R with the framework of adiabatic approximation. In order to apply the adiabatic approximation, we scale the coordinates in Eqs. (21), (26), and (27) as follows:

$$\eta=r/R. \tag{A1}$$

This scaling brings the equations into the form

$$\frac{1}{R^2}\left(\partial_\eta^2 F+\frac{\partial_\eta F}{\eta}\right)+F-F^3-(\Omega R)^2\eta^2 F=0, \tag{A2}$$

$$\frac{1}{R^2}\left(\partial_\eta^2 A_n+\frac{\partial_\eta A_n}{\eta}\right)-\frac{(n/R)^2 A_n}{\eta^2}+(1-3F^2-(\Omega R)^2\eta^2)A_n +2\Omega n i B_n=0, \tag{A3}$$

$$\frac{1}{R^2}\left(\partial_\eta^2 B_n+\frac{\partial_\eta B_n}{\eta}\right)-\frac{(n/R)^2 B_n}{\eta^2}+(1-F^2-(\Omega R)^2\eta^2)B_n -2\Omega n i A_n=0. \tag{A4}$$

We assume that critical frequency Ω is of the order $1/R$, i.e. $\Omega R\sim 1$. Also the critical number of the azimuthal mode is $n\gg 1$. Let us first focus on Eq. (39). For $R\gg 1$ the term $1/R^2(\partial_\eta^2 F+\partial_\eta F/\eta)$ is small and the solution in a zero order in R^{-2} is given simply by

$$F^2=1-\Omega^2R^2\eta^2+O(R^{-2}). \tag{A5}$$

This solution (which we call ''adiabatic'') is valid everywhere except the edge of container. From Eq. (A3) in the limit $R\to\infty$ we obtain the relation between A_n and B_n

$$A_n=\frac{\Omega i n B_n}{1-\Omega^2R^2\eta^2+n^2/(2R^2\eta^2)}. \tag{A6}$$

Substituting Eqs. (A2) and (A3) in Eq. (A4) we arrive at a single equation for B_n:

$$\partial_\eta^2 B_n+\frac{\partial_\eta B_n}{\eta}=\frac{n^2}{\eta^2}\left(\frac{1-3\Omega^2R^2\eta^2+n^2/(2R^2\eta^2)}{1-\Omega^2R^2\eta^2+n^2/(2R^2\eta^2)}\right)B_n=0. \tag{A7}$$

We assume that for $R\gg 1$ also $n\gg 1$. Therefore, the right-hand side (rhs) of Eq. (A7) is in general very large. In order to obey the boundary condition at $\eta=1$ the rhs has to become small as $\eta\to 1$. This can be achieved by setting $n/R\to 0$, $(\Omega R)^2=1/3$. Thus, we obtain the familiar expression for the critical frequency Ω_c,

$$\Omega_c\to\frac{1}{\sqrt{3}R}\quad\text{for } R\to\infty. \tag{A8}$$

In order to obtain the correction to the critical frequency for finite R and the relation between n and R we have to focus on the behavior of the solution in the vicinity of $\eta=1$, i.e., near the wall. We assume now that for $R\gg 1$ $(\Omega R)^2=1/3+\delta$, where $\delta\ll 1$ is the correction to the critical frequency due to finite R. Changing the variables $\eta=1-\xi$ (not to be confused with the correlation length in this context) and keeping only the terms linear in ξ, we arrive at the following equation:

$$\partial_\xi^2 B_n+(1+\xi)\partial_\xi B_n=\frac{3n^2}{2}\left(-3\delta+2\xi+\frac{n^2}{2R^2}\right)B_n \tag{A9}$$

with the boundary condition $B_n(\xi\to\infty)\to 0$. We cannot formally apply the boundary condition for $\xi=0$ because outer expansion loses its validity near the wall at the distance of order 1. At this point we have to match outer solution with inner expansion, which has to be obtained in a different approximation.

Equation (A9) is reduced to the Airy equation in the limit of large n. To see that, we apply the following scaling of the variables, $\tilde{\xi}=\xi(3n^2)^{1/3}$, and arrive at the equation

$$\partial_{\tilde{\xi}}^2 B_n=(\tilde{\xi}+\alpha)B_n+(2n^2)^{-1/3}[1+(2n^2)^{-1/3}\tilde{\xi}]\partial_{\tilde{\xi}}B_n, \tag{A10}$$

where

$$\alpha=3/4(-6\delta n^2+n^4/R^2)/(3n^2)^{2/3} \tag{A11}$$

is the constant of order one. For $n\gg 1$ the last two terms in Eq. (A10) can be dropped and we obtain the Airy equation $\partial_{\tilde{\xi}}^2 B_n=(\tilde{\xi}+\alpha)B_n$. The solution, obeying the boundary condition at $\xi\to\infty$ is

$$B_n=CAi(\tilde{\xi}+\alpha)=CAi\left((3n^2)^{1/3}\frac{\overline{r}}{R}+\alpha\right), \tag{A12}$$

where C is the arbitrary constant (due to linearity of the equation) and $\overline{r}=\xi R=R-r$. The constant α involves as yet unknown values n,δ which have to be fixed by matching with the solution which is valid near the boundary of the container (inner solution).

APPENDIX B: DERIVATION OF INNER SOLUTION

Let us now consider the solution of Eqs. (26) and (27) valid also at the wall, $r\sim R$. The key simplification in the inner region is that we ignore the explicit dependence of Eqs. (26) and (27) on the radial coordinate r and replace it by its value at the boundary R. As above, we introduce new coordinate $r=R-\overline{r}$. The inner region of the solution is defined by the condition $0\leqslant\overline{r}\ll R$. In this region in the leading order in $1/R$, Eq. (21) for stationary solution F and Eqs. (26) and (27) for the perturbations assume the following form:

$$\partial_{\overline{r}}^2 F_0+F_0-F_0^3-\Omega^2R^2F_0=0, \tag{B1}$$

$$\partial_{\overline{r}}^2 A_n-\frac{n^2A_n}{R^2}+(1-3F_0^2-\Omega^2R^2)A_n+2\Omega niB_n=0, \tag{B2}$$

$$\partial_{\overline{r}}^2 B_n-\frac{n^2B_n}{R^2}+(1-F_0^2-\Omega^2R^2)B_n-2\Omega niA_n=0. \tag{B3}$$

These equations are subjected to the boundary condition Eq. (22) at $\overline{r}=0$. The boundary condition at $\overline{r}\to\infty$ is defined by matching with the outer solution.

The solution of Eq. (B1) is of the form

$$F_0=\sqrt{1-\nu^2}\tanh\left[\sqrt{1-\nu^2}\frac{\overline{r}}{\sqrt{2}}+u\right], \tag{B4}$$

where $\nu=\Omega R$ and the constant u is given by the boundary condition Eq. (22):

$$\sinh 2u=\frac{\sqrt{2(1-\nu^2)}}{\gamma}. \tag{B5}$$

For the solution to the set of linear equations (B2) and (B3) we search in the form of series

$$\begin{pmatrix}A_n\\B_n\end{pmatrix}=\begin{pmatrix}A_0\\B_0\end{pmatrix}+\frac{n}{R}\begin{pmatrix}A_1\\B_1\end{pmatrix}+\frac{n^2}{R^2}\begin{pmatrix}A_2\\B_2\end{pmatrix}+\cdots . \tag{B6}$$

In zero order we find immediately

$$A_0=0\,,\quad B_0=F_0=\sqrt{1-\nu^2}\tanh\left[\sqrt{1-\nu^2}\frac{\overline{r}}{\sqrt{2}}+u\right] \tag{B7}$$

(taking into account that the critical frequency obeys the relation $\Omega\sim 1/R$.) The first order is also trivial, and we readily obtain

$$A_1=-i\partial_\nu F_0\,,\quad B_1=0. \tag{B8}$$

This expression is obtained taking the derivative with respect to ν from Eq. (B4). However, the solution Eq. (B8) in general does not satisfy the boundary condition at $\overline{r}=0$. This problem can be solved by modifying the solution Eq. (B8) by a linear combination with fundamental solution of the homogeneous Eq. (B2), bounded at $\overline{r}\to\infty$. This solution is simply $A_h=\partial_{\overline{r}}F_0$. The corrected solution is of the form

$$A_1=-i(\partial_\nu F_0+\tilde{\alpha}\partial_{\overline{r}}F_0),\quad B_1=0, \tag{B9}$$

where constant $\tilde{\alpha}$, defined from the condition $\partial_{\overline{r}}A_1-\gamma A_1=0$, is of the form (for $\nu=1/\sqrt{3}$):

$$\tilde{\alpha}=\frac{3\sqrt{3}\gamma\cosh^2 u}{4+6\gamma^2\cosh^2 u}. \tag{B10}$$

Nontrivial behavior arises in the second order in n/R for Eq. (B3). Using Eqs. (B7) and (B9) we obtain the following:

$$\partial_{\overline{r}}^2 B_2+(1-F_0^2-\Omega^2R^2)B_2 =\left(F_0+\frac{2\Omega^2R^2}{\nu}(\partial_\nu F_0+\tilde{\alpha}\partial_{\overline{r}}F_0)\right). \tag{B11}$$

For the frequency, close to the critical one we have $\Omega R=\nu\approx 1\%\sqrt{3}$. As a result, Eq. (B11) is reduced to

$$\partial_{\overline{r}}^2 B_2+\frac{2}{3\cosh^2[\overline{r}/\sqrt{3}+u]}B_2 =\left(-\frac{2\overline{r}}{3\sqrt{2}\cosh^2[\overline{r}/\sqrt{3}+u]}+\frac{2}{\sqrt{3}}\tilde{\alpha}\partial_{\overline{r}}F_0\right). \tag{B12}$$

The solution to Eq. (B12) is simply linear in $\overline{r}$ function

$$B_2=-\frac{\overline{r}}{\sqrt{2}}+\tilde{\alpha}\sqrt{2/3}. \tag{B13}$$

However, for $\gamma\neq 0$ this solution does not satisfy the boundary condition for $\overline{r}=0$. The solution requires modification by some amount of second fundamental solution to homogeneous Eq. (B12) which is of the form

$$B^{(2)}=\frac{3}{\sqrt{2}}\{\tanh[\overline{r}/\sqrt{3}+u](\overline{r}/\sqrt{3}+u)-1\}. \tag{B14}$$

This fundamental solution does not obey the boundary condition, and grows linearly for large $\overline{r}$. The solution, obeying the boundary condition is of the form

$$B_2=\left(-\frac{\overline{r}}{\sqrt{2}}+\tilde{\alpha}\sqrt{2/3}+\zeta B^{(2)}\right), \tag{B15}$$

where ζ is given in terms of γ,u as follows:

$$\zeta=\frac{2\gamma\cosh^2 u}{2+3\gamma^2\cosh^2 u}. \tag{B16}$$

The constant ζ vanishes for $\gamma\to\infty$ as $\zeta=2/3\gamma$ and $\zeta=1/\sqrt{3}$ for $\gamma=0$.

APPENDIX C: MATCHING OF THE INNER AND OUTER EXPANSIONS

The outer solution Eq. (A12) must be matched with the inner solution Eq. (B15) in their overlap region. The outer solution is valid through $\overline{r}=R-r\gg 1$ and breaks at $\overline{r}=0$. The inner limit of the outer solution is defined by $(3n^2)^{1/3}\overline{r}/R\ll 1$ which gives $\overline{r}\ll R/(3n^2)^{1/3}$, which gives rise to the following bilateral condition:

$$1\ll\overline{r}\ll R/(3n^2)^{1/3}.$$

The inner solution is valid in the interval

$$0<\overline{r}\sim R^2/n^2.$$

We see that the expansions overlap at $\overline{r}\approx\sqrt{R}$ if we set $n\sim R^{3/4}$. The exact relation is fixed by matching inner limit ($\overline{r}\rightarrow 0$) of the outer solution with the outer limit ($\overline{r}\rightarrow\infty$) of the inner solution.

Expansion of the outer solution for $(3n^2)^{1/3}(\overline{r}/R)\ll 1$ gives rise to

$$B_n=C\left(Ai(\alpha)+(3n^2)^{1/3}\frac{\overline{r}}{R}Ai'(\alpha)\right). \tag{C1}$$

Outer expansion of the inner solution in the region $\overline{r}\gg 1$ but $n^2/R^2\overline{r}\ll 1$ gives [using Eq. (B14) and $\nu\approx 1/\sqrt{3}$]

$$B_n=\sqrt{\frac{2}{3}-\frac{n^2}{\sqrt{2}R^2}(1-\sqrt{3}\zeta)\overline{r}.} \tag{C2}$$

The matching of the above expansions is possible if the following relation holds:

$$Ai'(\alpha)+\frac{\sqrt{3}n^2(1-\sqrt{3}\zeta)}{2R(3n^2)^{1/3}}Ai(\alpha)=0. \tag{C3}$$

Equation (C3) can be written in terms of scaled variables

$$Q=n/R^{3/4},\quad \Delta=\delta R^{1/2}. \tag{C4}$$

In this scaling the dependence on R in Eq. (C3) disappears. Then Eq. (C3) assumes the form

$$Ai'\left(\frac{3Q^4-18\Delta Q^2}{4(3Q^2)^{2/3}}\right)+\frac{\sqrt{3}Q^{4/3}(1-\sqrt{3}\zeta)}{2(3^{1/3})}Ai\left(\frac{3Q^4-18\Delta Q^2}{4(3Q^2)^{2/3}}\right)=0. \tag{C5}$$

Equation (C5) is solved numerically. The instability threshold corresponds to the minimal value of Δ as a function of Q. The minimal value of Δ and corresponding Q as a function of the suppression rate γ are shown in Fig. 2. Note that for the zero boundary condition (which corresponds to $\gamma\rightarrow\infty$) the parameter Δ vanishes.

Using Eq. (C4) we obtain finally the value of the most unstable azimuthal number n and the shift of critical frequency due to finite radius of the container δ :

$$n=Q(\gamma)R^{3/4}, \tag{C6}$$

$$\Omega=\frac{1}{R}\sqrt{\frac{1}{3}+\frac{\Delta(\gamma)}{R^{1/2}}}. \tag{C7}$$

Since $(3n^2)^{1/3}/R\sim\sqrt{R}$ we obtain that the eigenfunctions $B_n\sim Ai[(3n^2)^{1/3}(\overline{r}/R)+\alpha]$, corresponding to the most unstable modes are localized in the layer of the width $r_b\sim\sqrt{R}$ near the container wall.

[1] A. Reisenegger, J. Low Temp. Phys. **92**, 77 (1993) and references therein.
[2] P. Adams, M. Cieplak, and W. Glaberson, Phys. Rev. B **32**, 171 (1985).
[3] Z. Peradzynski, S. Filipkovski, and W. Fiszdon, Eur. J. Mech. B **9**, 259 (1990).
[4] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991).
[5] V. L. Ginzburg and L. Pitaevskii, Sov. Phys. JETP **34**, 858 (1958).
[6] L. Pitaevskii, Sov. Phys. JETP **35**, 282 (1959).
[7] P. Soininen and N. Kopnin, Phys. Rev. B **49**, 12 087 (1994).
[8] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, London, 1976).
[9] V. Dohm and R. Hausmann, Physica B **197**, 215 (1994).
[10] V.L. Ginzburg and A.A. Sobaynin, Sov. Phys. Usp. **19**, 773 (1976); J. Low. Temp. Phys. **49**, 507 (1982).
[11] E. Varoquaux, W. Zimmermann, Jr., and O. Avenel, in *Excitations in Two-Dimensional and Three-Dimensional Quantum Fluids*, edited by A. Wyatt and H. Lauter (Plenum, New York, 1991).
[12] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1955), Vol. 1, p. 17.
[13] L. D. Landau and E. M. Lifshitz, *Hydrodynamics* (Pergamon, New York, 1987).
[14] S. V. Iordanskii, Sov. Phys. JETP **21**, 467 (1967).
[15] J. S. Langer and M. E. Fisher, Phys. Rev. Lett. **19**, 560 (1967).
[16] G. Shifflett and G. Hess, J. Low Temp. Phys. **98**, 591 (1995).
[17] J. Steinhauer *et al.*, Phys. Rev. Lett. **74**, 5056 (1995).
[18] L. Kramer, Phys. Rev. **179**, 149 (1969).
[19] H. Mikeska, Phys. Rev. **179**, 166 (1969).
[20] Y. Mamaladze, Sov. Phys. JETP **25**, 479 (1967).
[21] J. Hulin *et al.*, Phys. Rev. A **9**, 885 (1974).
[22] U. Parts *et al.*, Europhys. Lett. **31**, 449 (1995).
[23] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).
[24] A. Onuki, J. Low Temp. Phys. **51**, 601 (1983).
[25] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, Redwood City, CA, 1989).
[26] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[27] I. Aranson, M. Gitterman, and B. Shapiro, Phys. Rev. B **51**, 3092 (1995).
[28] L. Kramer and W. Zimmermann, Physica D **16**, 221 (1985).
[29] I. Aranson and V. Steinberg, Phys. Rev. B **53**, 75 (1996).
[30] J. S. Langer and J. D. Reppy, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1970), Vol. 6, Chap. 1.
[31] D. McCumber and B. I. Halperin, Phys. Rev. B **1**, 1054 (1970).
[32] E. Hernandez-Garcia *et al.*, Phys. Rev. Lett. **70**, 3576 (1993).