Nonlinear dynamics of vortices in ultraclean type-II superconductors: Integrable wave equations in cylindrical geometry

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Due to their short coherence lengths and relatively large energy gaps, the high-transition temperature superconductors are very likely candidates as ultraclean materials at low temperature. This class of materials features significantly modified vortex dynamics, with very little dissipation at low temperature. The motion is then dominated by wave propagation, being in general nonlinear. Here two-dimensional vortex motion is investigated in the ultraclean regime for a superconductor described in cylindrical geometry. The small-amplitude limit is assumed, and the focus is on the long-wavelength limit. Results for both zero and nonzero Hall force are presented, with the effects of nonlocal vortex interaction and vortex inertia being included within London theory. Linear and nonlinear problems are studied, with a predisposition toward the more analytically tractable situations. For a nonlinear problem in 2+1 dimensions, the cylindrical Kadomtsev-Petviashvili equation is derived. Hall angle measurements on high-T_c superconductors indicate the need to investigate the properties of such a completely integrable wave equation. [S0163-1829(96)06526-5]

INTRODUCTION

Recently it has been shown theoretically how soliton propagation[1–3] could result from a type-II superconductor in the mixed state.[4–7] Wave propagation in an Abrikosov vortex lattice is subject to nonlinearity and dispersion. In a suitable regime, these effects counteract in such a way that a soliton is generated.

This paper treats both linear and nonlinear wave propagation problems for a type-II superconductor with cylindrical symmetry. It is demonstrated that soliton propagation is possible in two space dimensions in such geometry. As described below, the effects of nonlocal vortex interaction, vortex inertia, and the Hall force can be included.

In a superconductor dispersion comes from the Meissner screening of fields, which can be modeled either macroscopically by the London equation or microscopically with the use of an electrodynamic kernel function. If the electrodynamic fields are not too strong, then the nonlinearity is weak. A necessary condition on the magnetic-field magnitude is $H \ll H_{c2}$, where H_{c2} is the upper critical field, so that vortex cores do not interact. For sustained soliton propagation, energy dissipation needs to be sufficiently low. A remarkable property of high-T_c superconductors in this regard has recently been examined experimentally, that these materials may belong to the ultraclean class at low enough temperature.[8,9]

When the energy spacing $\hbar\omega_c$ is much larger than the energy width $\hbar/\tau$, a type-II superconductor is in the ultraclean regime. Here $\hbar\omega_c$ is the low-level spacing for bound vortex core states and τ is the lifetime of quasiparticles in the core. A signal of this property is a flux-flow Hall angle θ_H approaching $\pm\pi/2$. A consequence of this varying angle is a change of the vortex velocity from alignment with the direction of the Lorentz force $\mathbf{J}\times\hat{z}$, for vortices along $\hat{z}$, to the direction of the current density $\mathbf{J}$. Since the Hall angle may have either sign in the ultraclean regime, the direction of the vortex velocity may approach the direction of $\pm$ $\mathbf{J}$.

The high-T_c superconductors are particularly viable candidates for ultracleanliness owing to their relatively large energy gaps Δ and small Fermi energy ϵ_F. When the ratio of quasiparticle mean free path l to the coherence length ξ_c greatly exceeds $\epsilon_F/|\Delta|$, the ultraclean regime results. Corresponding to the relatively small ϵ_F, the high-T_c materials are recognized to have small coherence lengths.

A remarkable upturn of the Hall angle θ_H with decreasing temperature has recently been observed in samples of 60 K $YBa_2Cu_3O_{6+y}$.[8] The upturn starts around 40 K and continues to the lowest data point near 15 K. These measurements provide direct evidence for the approach to the ultraclean regime.[8] The parameter $\omega_c\tau$ has been estimated as roughly 14 in 90 K Y-Ba-Cu-O, putting this material in the ultraclean class.[9]

The suggestion of ultraclean high-T_c superconductors motivates a detailed study of the modified dynamics of vortices. In materials with very long quasiparticle mean free path, the dissipation may become negligible at low temperature. This fact allows new features to dominate, including vortex inertia and the Hall force. Emphasizing this point, the vortex motion is no longer overdamped, allowing for a wide range of phenomena. This paper takes up such a study for superconductors described in cylindrical coordinates. Analytically tractable situations for both zero and nonzero Hall coefficient α_H are treated. The recent characterization of the Hall angle of Y-Ba-Cu-O (Refs. 8 and 9) makes such a study especially relevant.

In an ultraclean type-II superconductor in the absence of pinning and the Hall force, the nonlinear Korteweg-de Vries (KdV) equation governs the evolution of the first-order electrodynamic field corrections for one-dimensional vortex motion.[4] For two-dimensional (2D) motion under these conditions, the Kadomtsev-Petviashvili (KP) equation[10] governs the dynamics when described in Cartesian coordinates,[6] whereas radial vortex motion is described by the cylindrical KdV (CKdV) equation.[5] With the inclusion of the Hall term, the nonlinear Schrödinger (NLS) equation appears for 1D motion.[7]

In the following section the governing equations are presented in cylindrical coordinates. (Microscopic nonlocality and vortex pinning are ignored.) Various special cases are analyzed in subsequent sections. Some facts are recalled from the derivation of the CKdV equation[5] to show how these results can be developed through the second order of the perturbation expansion. It is shown that the second-order correction can be found by solving a linear, albeit complicated variable coefficient equation.

Linear wave propagation in cylindrical coordinates including the Hall force is then examined, showing an exact analytical solution is possible for no angular variation. Next the situation of zero Hall force is readdressed for motion described by both cylindrical coordinates ρ and θ. Perhaps the main result of this paper is the derivation of a variable coefficient nonlinear wave equation in 2+1 dimensions for an ultraclean superconductor. The equation is a form of the cylindrical KP or nearly concentric KdV equation and is completely integrable.[11] This equation has been known to arise for shallow-water waves subject to a small transverse disturbance.[11] A discussion of the results and summary follows.

GOVERNING EQUATIONS

The system considered is an isothermal type-II superconductor at or near absolute zero. In this instance a normal current density contribution is ignored; the total current density **J** is the supercurrent density. (Frequencies well below the superconducting gap frequency are assumed so that the displacement current is not present.) The superconductor is assumed isotropic in the plane perpendicular to the direction of vortices, which is taken to be $\hat{z}$. It is considered to be of very large Ginzburg-Landau parameter, $\kappa \gg 1$, a condition well satisfied by high-T_c materials. Within mesoscopic London theory a continuum description is employed using a vortex areal density $n(\mathbf{x},t)$.[12,13] Nonlocal vortex interaction is a key in providing the dispersion necessary for a soliton equation.

With these assumptions, the principal equations[4] are vortex continuity (with scalar n), a vortex equation of motion (with velocity **v**), and the London equation (with total magnetic induction **B**). The equation of flux-line conservation is

$$\frac{\partial \mathbf{B}_v}{\partial t} = -\nabla \times (\mathbf{B}_v \times \mathbf{v}), \tag{1}$$

where $\mathbf{B}_v = n\phi_0 \hat{z}$ is the local vortex-generated magnetic induction and $\phi_0 = h/2e \simeq 2\times 10^{-15}$ Wb is the flux quantum.

Also written in vector form, an equation of motion for an ultraclean superconductor is

$$\mu \frac{d\mathbf{v}}{dt} + \alpha_H \mathbf{v}\times\hat{z} = \phi_0 \mathbf{J}\times\hat{z}, \tag{2}$$

where α_H is the Hall coefficient and μ is the mass per unit length of vortex.[14] In the limit as $\theta_H \to \pm\pi/2, \alpha_H \to n_s e \phi_0$ where n_s is the concentration of superholes. For high-T_c materials the vortex mass due to certain mechanisms is larger than for conventional superconductors.[14] In addition, a very recent microscopic analysis of the Hall anomaly has found a large vortex mass coming from the core, specifically in the ultraclean limit.[15]

The third, London equation with vortex term, may be written in the form

$$\mathbf{B}_v = \mathbf{B} - \lambda_L^2 \nabla^2 \mathbf{B}, \tag{3}$$

where λ_L is the London penetration depth, taken to be independent of magnetic field. Due to the neglect of a normal fluid contribution, a term with first-order time derivative does not appear. In the Meissner state the density n is absent and the left-hand side of Eq. (3) is zero. The forms of Eqs. (1) and (3) are briefly reviewed in Appendix A.

Consider an ultraclean superconductor with a static external magnetic field along $\hat{z}$, with magnitude $H > 2H_{c1}$ where H_{c1} is the lower critical field. Use the above assumptions, apply Ampere's law, and write the vortex velocity as $\mathbf{v} = v_\rho \hat{\rho} + v_\theta \hat{\theta}$. Then the governing system of 2D equations is

$$\frac{\partial n}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho n v_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \theta}(n v_\theta) = 0, \tag{4a}$$

$$\mu \frac{dv_\rho}{dt} + \alpha_H v_\theta = -\frac{\phi_0}{\mu_0}\frac{\partial B}{\partial \rho}, \tag{4b}$$

$$\mu \frac{dv_\theta}{dt} - \alpha_H v_\rho = -\frac{\phi_0}{\mu_0}\frac{1}{\rho}\frac{\partial B}{\partial \theta}, \tag{4c}$$

$$n\phi_0 = B - \frac{\lambda_L^2}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial B}{\partial \rho}\right) - \frac{\lambda_L^2}{\rho^2}\frac{\partial^2 B}{\partial \theta^2}. \tag{4d}$$

The equations are scaled as $t' = \omega_0 t$, $\rho' = \rho/\lambda_L$, $B' = B/B_0$, $n' = n/n_0$, $v'_\rho = v_\rho/\omega_0\lambda_L$, and $v'_\theta = v_\theta/\omega_0\lambda_L$, where $\omega_0\lambda_L = \sqrt{\phi_0 B_0/\mu_0\mu}$ and $n_0 = B_0/\phi_0$. For the sake of notational ease, the primes are dropped and α_H is written for $\alpha_H/\mu\omega_0$. The governing system becomes

$$\frac{\partial n}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho n v_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \theta}(n v_\theta) = 0, \tag{5a}$$

$$\frac{dv_\rho}{dt} + \alpha_H v_\theta = -\frac{\partial B}{\partial \rho}, \tag{5b}$$

$$\frac{dv_\theta}{dt} - \alpha_H v_\rho = -\frac{1}{\rho}\frac{\partial B}{\partial \theta}, \tag{5c}$$

$$n = B - \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial B}{\partial \rho}\right) - \frac{1}{\rho^2}\frac{\partial^2 B}{\partial \theta^2}, \tag{5d}$$

where $d/dt = \partial/\partial t + v_\rho \partial/\partial \rho + (v_\theta/\rho)\partial/\partial \theta$ is the convective derivative.

REVIEW OF CKDV CASE AND DERIVATION OF SECOND-ORDER FIELD CORRECTION

In the absence of the Hall force, Eqs. (5) can be combined to yield fourth-order equations for the velocity components,

$$\frac{d^2 v_\rho}{\partial t\ dt} = \frac{\partial^2}{\partial t \partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{d v_\rho}{dt}\right) + \frac{\partial^2}{\partial t \partial \rho} \frac{1}{\rho} \frac{\partial^2 v_\theta}{\partial \theta\ dt} + \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho n v_\rho) + \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \theta}(n v_\theta), \tag{6a}$$

$$\rho^2 \frac{d^2 v_\theta}{\partial t\ dt} = \frac{\partial^2}{\partial t\ \partial \rho}\left(\rho \frac{\partial}{\partial \rho} \rho \frac{d v_\theta}{dt}\right) + \frac{\partial^4 v_\theta}{\partial t\ dt\ \partial \theta^2} + \frac{\partial^2(\rho n v_\rho)}{\partial \theta\ \partial \rho} + \frac{\partial^2 (n v_\theta)}{\partial \theta^2}, \tag{6b}$$

where B has been eliminated. These two equations are useful alternatives in determining the dispersion relation of the linear wave propagation problem.

If further there is no angular dependence of the electrodynamic fields, v_θ drops out of the system (5). Then Eq. (6b) becomes void and two terms drop out of the right-hand side of Eq. (6a). This special case has been shown to give the cylindrical Korteweg-de Vries (CKdV) equation for the first-order field correction.[5] Here the analysis is extended by finding the linear equation satisfied by the second-order field correction. This equation is third-order, variable coefficient, making its exact solution even for the single soliton case much more complicated than for its KdV counterpart.[16]

In the present geometry, the supercurrent density is strictly azimuthally directed, so that the Lorentz force is only in the radial direction. In terms of the stretched variables $\xi = \omega(\rho - t)$, $\eta = \omega^3 \rho$, the $O(\omega^6)$ equations need to be determined from the system (10) of Ref. 5 in order to find the second-order correction equation. The vortex density, magnetic induction, and (radial) velocity are expanded as

$$n(\xi,\eta) = 1 + \sum_{j=1}^{\infty} \omega^{2j} n^{(j)}(\xi,\eta), \tag{7a}$$

$$B(\xi,\eta) = 1 + \sum_{j=1}^{\infty} \omega^{2j} B^{(j)}(\xi,\eta), \tag{7b}$$

and

$$v(\xi,\eta) = \sum_{j=1}^{\infty} \omega^{2j} v^{(j)}(\xi,\eta). \tag{7c}$$

Then the equations

$$-\frac{\partial n^{(3)}}{\partial \xi} + \frac{1}{\eta} \frac{\partial}{\partial \eta}[\eta n^{(1)} v^{(1)} + \eta v^{(2)}] + \frac{\partial v^{(3)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n^{(1)} v^{(2)}) + \frac{\partial}{\partial \xi}(n^{(2)} v^{(1)}) = 0, \tag{8a}$$

$$-\frac{\partial v^{(3)}}{\partial \xi} + v^{(1)} \frac{\partial v^{(2)}}{\partial \xi} + v^{(2)} \frac{\partial v^{(1)}}{\partial \xi} + v^{(1)} \frac{\partial v^{(1)}}{\partial \eta} + \frac{\partial B^{(3)}}{\partial \xi} + \frac{\partial B^{(2)}}{\partial \eta} = 0, \tag{8b}$$

$$B^{(3)} - n^{(3)} = \frac{\partial^2 B^{(2)}}{\partial \xi^2} + \frac{\partial^2 B^{(1)}}{\partial \xi\ \partial \eta} + \frac{1}{\eta} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial B^{(1)}}{\partial \xi}\right), \tag{8c}$$

follow as a special case of Eqs. (5).

By combining Eqs. (8) it is possible to eliminate the third-order correction fields. The second-order density correction is known in terms of $B^{(2)}$ by the London equation: $n^{(2)} = B^{(2)} - \partial^2 B^{(1)}/\partial \xi^2$. In order to obtain an equation for $B^{(2)}$ alone it remains to express the second-order velocity correction $v^{(2)}$ in terms of $B^{(2)}$ and the first-order correction fields $v^{(1)} = n^{(1)} + f(\eta)$, $B^{(1)} = n^{(1)}$. By integrating the vortex equation of motion at $O(\omega^4)$ (Ref. 5) with respect to ξ it is possible to write

$$v^{(2)} = B^{(2)} + \frac{1}{2}[v^{(1)}]^2 + \frac{\partial}{\partial \eta} \int B^{(1)}(\xi,\eta) d\xi. \tag{9}$$

When the arbitrary function $f = 0$, for simplicity, and setting

$$I_v^{(1)} \equiv \frac{1}{2}[B^{(1)}]^2 + \frac{\partial}{\partial \eta} \int B^{(1)}(\xi,\eta) d\xi,$$

the equation for the second-order correction to the magnetic induction can be written as

$$\frac{\partial^3 B^{(2)}}{\partial \xi^3} + 2\frac{\partial B^{(2)}}{\partial \eta} + 3\frac{\partial}{\partial \xi}[B^{(1)} B^{(2)}] + \frac{B^{(2)}}{\eta} = R(B^{(1)}), \tag{10}$$

where

$$-R(B^{(1)}) = 2\frac{\partial^3 B^{(1)}}{\partial \xi^2 \partial \eta} + \frac{1}{\eta} \frac{\partial^2 B^{(1)}}{\partial \xi^2} + \frac{\partial}{\partial \xi}[B^{(1)} I_v^{(1)}] - \frac{\partial}{\partial \xi} B^{(1)} \frac{\partial^2 B^{(1)}}{\partial \xi^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta}[\eta (B^{(1)})^2] + \frac{1}{\eta} \frac{\partial}{\partial \eta}[\eta I_v^{(1)}] + I_v^{(1)} \frac{\partial B^{(1)}}{\partial \xi} + 2B^{(1)} \frac{\partial B^{(1)}}{\partial \eta} + [B^{(1)}]^2 \frac{\partial B^{(1)}}{\partial \xi}. \tag{11}$$

Although the inhomogeneous Eq. (10) is linear, there are two terms with variable coefficients. The form of this combination increases the analytical complexity of solutions, even those of traveling wave shape.

Recalling that the exact single-soliton solution of the CKdV equation for $B^{(1)}$ requires an integral representation with the Airy function Bi,[5] even in this case it appears that the exact solution of Eq. (10) requires substantial effort. On the other hand, by exploiting an approximate similarity solution of the CKdV equation, e.g., as constructed in Ref. 17, it may be possible to develop an approximation for the second-order field correction. (Such an approximation for a soliton solution of CKdV is partly based upon consideration of the energy.)

ROTATIONALLY SYMMETRIC DYNAMICS

In this section the Hall force is retained but the assumption of rotational symmetry is invoked. In this case $\partial_\theta = 0$ and the equations in the system (5) simplify. It is demonstrated that the solution of the linear problem can be explicitly written and the dispersion relation is found. Considerations which may lead to a form(s) of a cylindrical nonlinear Schrödinger equations (NLS) for the nonlinear problem are not pursued.

For the linearized problem, let

$$n(\rho,t)=1+\delta n J_0(k\rho)e^{-i\omega t}, \tag{12a}$$

$$B(\rho,t)=1+\delta B J_0(k\rho)e^{-i\omega t}, \tag{12b}$$

$$v_\rho(\rho,t)=\delta v_\rho J_1(k\rho)e^{-i\omega t}, \tag{12c}$$

and

$$v_\theta(\rho,t)=\delta v_\theta J_1(k\rho)e^{-i\omega t}, \tag{12d}$$

where J_ν is the Bessel function of order ν. By using the ordinary differential equation satisfied by J_0 and the relation $(d/dz)zJ_1(z)=zJ_0(z)$, the coefficients are found from system (5) to be related by

$$\delta B=-\delta n/(1+k^2), \quad \delta v_\rho=(i\omega/k)\delta n, \quad \delta v_\theta=-(\alpha_H/k)\delta n. \tag{13}$$

Then the dispersion relation is given by

$$\omega^2(k)=\alpha_H^2+\frac{k^2}{1+k^2}. \tag{14}$$

Therefore, in the long-wavelength limit, the angular frequency $\omega(k)$ does not vanish. This suggests, as in the Cartesian case,[7] that a form of the NLS equation may appear for the correction fields for the nonlinear problem.

When $\alpha_H \to 0$, the above results reduce to those expected for the cylindrical KdV case.

The necessary multiscale analysis for the nonlinear problem will not be taken up here, however a very useful scalar equation for the linearized problem is recorded. Letting n temporarily stand for the perturbation δn for notational ease, it can be shown that the following single wave equation of sixth order for the vortex density results:

$$\partial_{tt}\left[\frac{1}{\rho}\partial_\rho(\rho\ \partial_\rho n_{tt})+(1+\alpha_H^2)\frac{1}{\rho}\partial_\rho(\rho\ \partial_\rho n)-n_{tt}-\alpha_H^2 n\right]=0. \tag{15}$$

With the time dependence $\exp(-i\omega t)$, it can be checked that Eq. (15) leads to the zero-order Bessel differential equation, with $k^2(\omega)=(\omega^2-\alpha_H^2)/(1+\alpha_H^2-\omega^2)$, giving the dispersion relation (14).

Similarly, coupled equations can be written in the case that $\partial_\rho=0$, and combined in the linearized regime, but being of less physical interest, are omitted here.

DERIVATION OF CYLINDRICAL KP EQUATION

For zero Hall coefficient, the dispersion relation (14) suggests the possibility to find a variable coefficient nonlinear wave equation for the first-order field corrections in two spatial dimensions. This expectation is borne out, with the derivation here of a cylindrical Kadomtsev-Petviashvili equation.[11] This form of the cylindrical KP equation is completely integrable. It is possible to develop explicit solutions for it based upon Painlevé analysis, Bäcklund transformations, or symmetry reduction.[11] This is an appearance of a variable coefficient, completely integrable nonlinear wave equation in the 2D vortex dynamics of a type-II superconductor. The key is to find an appropriate ordering(s) within reductive perturbation theory for the angle θ and the transverse velocity component v_θ.

A solitary disturbance is assumed to propagate mainly in the ρ direction with a constant speed V, and accordingly the stretched variables

$$\xi=\omega(\rho-Vt), \quad \eta=\omega^3\rho, \quad \tau=\omega^{-1}\theta, \tag{16}$$

are introduced. There is a strong coordinate dependence upon the angle θ. The uniqueness of the ordering of some of the variables of Eq. (16) is briefly discussed below. An alternative choice of stretched variables is presented in Appendix B, which also leads to the cylindrical KP equation.

Written in terms of the new independent variables, the scaled system (5) reads

$$-V\frac{\partial n}{\partial \xi}+\frac{1}{\eta}\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)(\eta n v_\rho)+\frac{\omega}{\eta}\frac{\partial}{\partial \tau}(n v_\theta)=0, \tag{17a}$$

$$-V\frac{\partial v_\rho}{\partial \xi}+v_\rho\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)v_\rho+v_\theta\frac{\omega}{\eta}\frac{\partial v_\rho}{\partial \tau} =-\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)B, \tag{17b}$$

$$-V\frac{\partial v_\theta}{\partial \xi}+v_\rho\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)v_\theta+v_\theta\frac{\omega}{\eta}\frac{\partial v_\theta}{\partial \tau}=-\frac{\omega}{\eta}\frac{\partial B}{\partial \tau}, \tag{17c}$$

$$n-B=-\frac{\omega^2}{\eta}\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)\left[\eta\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)\right]B -\frac{\omega^4}{\eta^2}\frac{\partial^2 B}{\partial \tau^2}. \tag{17d}$$

The associated perturbation expansions for the vortex density, velocity components, and magnetic induction are, respectively,

$$n(\xi,\eta,\tau)=\sum_{j=0}^{\infty}\omega^{2j}n^{(j)}(\xi,\eta,\tau), \tag{18a}$$

$$v_\rho(\xi,\eta,\tau)=\sum_{j=1}^{\infty}\omega^{2j}v_\rho^{(j)}(\xi,\eta,\tau), \tag{18b}$$

$$v_\theta(\xi,\eta,\tau)=\sum_{j=1}^{\infty}\omega^{2j+1}v_\theta^{(j)}(\xi,\eta,\tau), \tag{18c}$$

and

$$B(\xi,\eta,\tau)=\sum_{j=0}^{\infty}\omega^{2j}B^{(j)}(\xi,\eta,\tau), \tag{18d}$$

with $n^{(0)}=B^{(0)}\equiv 1$. All of these expansions are consistent with the small amplitude limit.

Substituting the expansions (18) into Eqs. (17) and equating coefficients of like powers of ω gives recursion relations for the higher-order field corrections. The lowest-order equations, at $O(\omega^2)$, can be integrated to yield

$$n^{(1)}=B^{(1)},\quad v_\rho^{(1)}=Vn^{(1)}+f_1(\eta,\tau),\quad B^{(1)}=Vv_\rho^{(1)}+f_2(\eta,\tau), \tag{19}$$

where $f_1=-f_2/V$ and $V^2=1$. The $O(\omega^3)$ relation from Eq. (17c) reads

$$V\frac{\partial v_\theta^{(1)}}{\partial\xi}=\frac{1}{\eta}\frac{\partial B^{(1)}}{\partial\tau}. \tag{20}$$

The next order contributions are at $O(\omega^4)$ and $O(\omega^5)$:

$$-V\frac{\partial n^{(2)}}{\partial\xi}+\frac{1}{\eta}\frac{\partial}{\partial\eta}[\eta v_\rho^{(1)}]+\frac{\partial v_\rho^{(2)}}{\partial\xi}+\frac{\partial}{\partial\xi}(n^{(1)}v_\rho^{(1)})+\frac{1}{\eta}\frac{\partial v_\theta^{(1)}}{\partial\tau}=0, \tag{21a}$$

$$-V\frac{\partial v_\rho^{(2)}}{\partial\xi}+v_\rho^{(1)}\frac{\partial v_\rho^{(1)}}{\partial\xi}=-\frac{\partial B^{(2)}}{\partial\xi}-\frac{\partial B^{(1)}}{\partial\eta}, \tag{21b}$$

$$-V\frac{\partial v_\theta^{(2)}}{\partial\xi}+v_\rho^{(1)}\frac{\partial v_\theta^{(1)}}{\partial\xi}=-\frac{1}{\eta}\frac{\partial B^{(2)}}{\partial\tau}, \tag{21c}$$

$$n^{(2)}-B^{(2)}=-\frac{\partial^2B^{(1)}}{\partial\xi^2}. \tag{21d}$$

Combining Eqs. (21) and using Eq. (19) gives the third-order nonlinear equation

$$V\frac{\partial^3v_\rho^{(1)}}{\partial\xi^3}+2V\frac{\partial v_\rho^{(1)}}{\partial\eta}+3v_\rho^{(1)}\frac{\partial v_\rho^{(1)}}{\partial\xi}-f_1\frac{\partial v_\rho^{(1)}}{\partial\xi}+V\frac{v_\rho^{(1)}}{\eta}+\frac{V}{\eta}\frac{\partial v_\theta^{(1)}}{\partial\tau}=-\frac{\partial f_2}{\partial\eta}, \tag{22}$$

which for the 1D case ($\partial/\partial\tau=0$) is the CKdV equation. The change of coordinates

$$\xi'=\xi+\frac{V}{2}\int f_1d\eta,\qquad \eta'=\eta, \tag{23}$$

can be used to eliminate the term with the function f_1 on the left-hand side. If Eq. (22) is differentiated with respect to ξ and Eq. (20) is differentiated with respect to τ, and the two equations are combined, there results

$$(Vv_{\rho\xi\xi\xi}^{(1)}+3v_\rho^{(1)}v_{\rho\xi}^{(1)}+2Vv_{\rho\eta}^{(1)}-f_1v_{\rho\xi}^{(1)}+Vv_\rho^{(1)}/\eta)_\xi+\frac{V}{\eta^2}v_{\rho\tau\tau}^{(1)}=-f_{2\tau\tau}(\eta,\tau)/\eta^2, \tag{24}$$

where subscript notation has been used for the partial derivatives. The variable coefficient $1/\eta^2$ of the $v_{\rho\tau\tau}^{(1)}$ term shows this equation to be the cylindrical Kadomtsev-Petviashvili equation.[11] As indicated by results of Painlevé analysis, and the inverse scattering formalism of Dryuma,[11,18,20] Eq. (24) is completely integrable.

The Zakharov-Shabat representation of Eq. (24) can be found by setting $v_\rho^{(1)}=v_1/v_2$ and requiring compatibility conditions on the τ and η derivatives of v_1 and v_2.[11,18] Then Eq. (24) takes the form $[\partial/\partial\tau-L,\partial/\partial\eta-A]v=0$, where the operator L is second order in ξ, A is third order in ξ, and nonlocal.[11,18,19]

Truncated Painlevé expansion can be used to generate explicit solutions. With the scalings $u=v_\rho^{(1)}/2,\tau'=4\tau$, $\eta'=\eta/2$, Eq. (24) takes a standard form[21]

$$(u_{\xi\xi\xi}+6uu_\xi+u_{\eta'}+u/2\eta')_\xi+\frac{4}{\eta'^2}u_{\tau'\tau'}=0. \tag{25}$$

A solution with two arbitrary functions α and β and an arbitrary constant b is[21]

$$u(\xi,\eta',\tau')=-\frac{b^2}{2\eta'}\mathrm{sech}^2\left\{\frac{i}{2}[b\xi(\eta')^{-1/2}+\alpha(\eta')\tau'+\beta(\eta')]\right\}+\frac{1}{6}\left[\frac{b^2}{\eta'}-4\frac{\alpha^2(\eta')}{b^2\eta'}-\frac{(\eta')^{1/2}}{b}\times\left(-\frac{b\xi}{2(\eta')^{3/2}}+\alpha'(\eta')\tau'+\beta'(\tau')\right)\right]. \tag{26}$$

Taking b, $\alpha(\eta')$, and $\beta(\eta')$ purely imaginary gives soliton-type solutions.[21]

More generally than Eqs. (16) and (18c), one may consider order rings with

$$\tau=\omega^p\theta,\quad v_\theta=\omega^mv_\theta^{(1)}+\omega^{m+2}v_\theta^{(2)}+\cdots, \tag{27}$$

where the exponents m and p are to be determined by different types of balances in the system (5). The equations extending the system (17) for general m and p are not recorded here. Requiring a balance of lowest order terms in the equation of motion (5c) gives the condition $m=p+4$. Demanding that the last term of the continuity equation not contribute at lowest order leads to $m+p>0$. Taking m to be positive, for the weak-amplitude limit, and restricting $m\leqslant4$ and p to integral values for simplicity, allows the orderings $(m,p)=(4,0)$ or $(3,-1)$. With the first of these orderings the last term of the continuity equation [similar to Eq. (17a)] does not contribute until $O(\omega^6)$, which is too late to provide the second spatial coordinate dependence in a KP-type equation. With this ordering the angular velocity component couples very weakly to the rest of the motion.

Other possibilities exist to satisfy the lowest-order equations, leading to other possible orderings. Of particular note, one can require $m<p+4$ from Eq. (5c) so that $v_\theta^{(1)}$ must be taken as independent of the ξ coordinate. By considering the terms of the other equations with v_θ, other inequalities can be developed depending upon functional dependences and the types of lowest-order balances. Of special notice, there is the possibility to develop nonlocal generalizations of the cylindrical KdV equation, which is not pursued here. For comparison to the Cartesian case, see Ref. 6.

Sometimes an equation of the form (25) is called the nearly concentric KdV equation,[21] which has arisen in the study of shallow-water waves. It differs from the modified KP equation derived by Giambo and Pantano in the case of an ion acoustic wave in a cold plasma,[22] in the $u_{\tau'\tau'}$ term. In their derivation, they accounted for a weak third (vertical) spatial coordinate dependence, leading to a constant coefficient of this term. Then the τ' variable is to be identified as coming from z in cylindrical coordinates. The resulting modified KP equation is also completely integrable. However, in this paper the motion is strictly two dimensional, in the plane perpendicular to the vortex axis.

SUMMARY

Due to their short coherence lengths and relatively large energy gaps, the high-transition temperature superconductors are very likely candidates as ultraclean materials. This class of materials has significantly modified vortex dynamics, featuring the propagation of linear and nonlinear waves, including solitons. At very low temperature there is negligible dissipation in ultraclean materials and the Hall force can dominate in the dynamics. In this paper results for both zero and nonzero Hall force were given. A continuum theory was used to account for nonlocal vortex interaction. Linear and nonlinear problems were studied in cylindrical coordinates. It was shown that results appropriate for the cylindrical KdV equation can be developed through the second order of the perturbation expansion. The second-order correction can be found by solving a linear, third-order variable coefficient equation.

The reductive perturbation theory applied in this paper showed that there is an ordering for which 2D soliton propagation arises in cylindrical geometry in an ultraclean type-II superconductor in the absence of pinning and the Hall force. Equation (24) is a form of the cylindrical Kadomtsev-Petviashvili or nearly concentric Korteweg-de Vries equation.[11,20,21] (The cylindrical KdV equation is a special case of this equation.) The cylindrical KP equation can be solved by the inverse scattering transformation.[18,19] It has been studied as an integrable case of a generalized variable coefficient KP equation.[20] This 2D soliton equation was derived for both weak nonlinearity and dispersion. The soliton solutions, with characteristic speed $c_s=\sqrt{\phi_0 B_0/\mu_0\mu}$, may be able to provide information on the vortex mass per unit length μ.[14]

The weak nonlinearities in the model include bilinearity in the vortex continuity equation (4a) and convective differentiation in the equation of motion (4b) and (4c). In this derivation, Eq. (4d) takes into account nonlocal vortex interaction, over the characteristic distance λ_L. Magnetic-field nonlinearity in the penetration depth was ignored, making this London equation linear. This assumption is very well justified for the high-T_c superconductors owing to their very large upper critical fields.

An analogous dynamical system may exist in fluid dynamics, e.g., in shallow-water waves described in cylindrical coordinates.[11] However, the vortex dynamics here is distinct from the ion-acoustic wave dynamics modeled for a cold plasma.[22] There a small disturbance transverse to the plane of motion was assumed, leading to a constant coefficient in place of the $1/\eta^2$ coefficient of Eq. (24). Therefore the ion dynamics of Ref. 22 is based upon 3D considerations, in contrast to the 2D motion modeled here for straight vortices.

This paper, like Ref. 5 on the cylindrical KdV equation, has considered the bulk superconductor response. For the variable coefficient equations governing the nonlinear wave propagation, satisfying boundary conditions in an analytical form is likely to be a difficult task. It appears that approximations will need to be developed in order to take into account a finite radius of the superconducting sample.

In an experimental investigation, whether using a waveguide or resonant cavity operating at microwave frequencies and very low temperature, it appears advantageous to employ a transverse electric (TE) mode. In this case the electric field and current density have only an angular component in a lowest-order approximation, more closely matching the geometry of this paper than a TM mode.

APPENDIX A: ON VORTEX CONSERVATION AND THE LONDON EQUATION

Equation (1) of the text is the same as the statement of Faraday's law for the local vortex-motion-induced electric field $\mathbf{E}_v=-\mathbf{v}\times\mathbf{B}_v$. This relation for $\mathbf{E}_v$ follows from putting the rate of change of the magnetic flux enclosed by a simple closed curve equal to the rate at which flux crosses the boundary, and is due to Josephson.[23]

The mesoscopic average of the London equation with gradient of the phase of the order parameter term can be performed. If the curl is then taken, the result can be written as $\nabla\times\mathbf{J}_s=-(\mathbf{B}-\mathbf{B}_v)/\mu_0\lambda_L^2$, where $\mathbf{J}_s$ is the supercurrent density. This equation together with Ampere's law gives Eq. (3) of the text, in the absence of a normal current density. Taking the time derivative of this equation, using $\partial\mathbf{A}/\partial t=-\mathbf{E}-\nabla\phi$, where $\mathbf{A}$ is the vector potential, $\mathbf{E}$ is the total electric field, and ϕ is a scalar potential, shows that $\mathbf{E}-\mathbf{E}_v=\mu_0\lambda_L^2\partial\mathbf{J}_s/\partial t$.

APPENDIX B: ALTERNATIVE DERIVATION OF CYLINDRICAL KP EQUATION

It is shown here that it is possible to obtain the cylindrical Kadomtsev-Petviashvili (KP) equation[11,20,21] with a choice of stretched variables and an ordering different from Eq. (16) of the text. The strong angular dependence of Eq. (16) is avoided. Let

$$\xi=k(\rho-Vt),\quad \eta=k^3t,\quad \tau=k^p\theta, \tag{B1}$$

suitable for the long-wavelength limit. In Eq. (B1) V is again a constant speed.

When writing the scaled system (5) for $\alpha_H=0$ in the new independent variables, it is useful to express $1/k\rho=1/(\xi+Vk^{-2}\eta)$ for small values of k. The coupled equations become

$$(-V\partial_\xi+k^2\partial_\eta)n+\frac{1}{V\eta}\left(1-\frac{\xi}{V\eta}k^2+\cdots\right)\partial_\xi[(\xi k^2+V\eta)\times nv_\rho]+\frac{k^{p+2}}{V\eta}\left(1-\frac{\xi}{v\eta}k^2+\cdots\right)\partial_\tau(nv_\theta)=0, \tag{B2a}$$

$$\left[-V\partial_\xi+k^2\partial_\eta+v_\rho\partial_\xi+v_\theta\frac{k^{p+2}}{V\eta}\left(1-\frac{\xi}{V\eta}k^2+\cdots\right)\partial_\tau\right]v_\rho=-\partial_\xi B, \tag{B2b}$$

$$\left[-V\partial_\xi+k^2\partial_\eta+v_\rho\partial_\xi+v_\theta\frac{k^{p+2}}{V\eta}\left(1-\frac{\xi}{V\eta}k^2+\cdots\right)\partial_\tau\right]v_\theta$$

$$=-\frac{k^{p+2}}{V\eta}\left(1-\frac{\xi}{V\eta}k^2+\cdots\right)\partial_\tau B, \qquad \text{(B2c)}$$

$$n-B=-\frac{1}{V\eta}\left(1-\frac{\xi}{V\eta}k^2+\cdots\right)\partial_\xi[(\xi k^4+Vk^2\eta)\partial_\xi B]$$

$$-\frac{k^{6+2p}}{V^2\eta^2}\left(1-\frac{\xi}{V\eta}k^2+\cdots\right)^2\partial_{\tau\tau}B. \qquad \text{(B2d)}$$

The perturbation expansions taken for the dependent variables are

$$n=\sum_{j=0}^{\infty}k^{2j}n^{(j)}, \quad v_\rho=\sum_{j=1}^{\infty}k^{2j}v_\rho^{(j)}, \qquad \text{(B3a)}$$

$$v_\theta=\sum_{j=0}^{\infty}k^{m+2j}v_\theta^{(j+1)}, \quad B=\sum_{j=0}^{\infty}k^{2j}B^{(j)}, \qquad \text{(B3b)}$$

with $n^{(0)}=B^{(0)}\equiv 1$. With the choice $p=0$, $m=4$, the lowest-order equations, at $O(k^2)$, again yield Eq. (19) of the text, with $V^2=1$.

The $O(k^4)$ condition from the v_θ equation of motion gives

$$\frac{\partial v_\theta^{(1)}}{\partial\xi}=\frac{1}{\eta}\frac{\partial B^{(1)}}{\partial\tau}. \qquad \text{(B4)}$$

The other equations at $O(k^4)$ yield

$$-V\frac{\partial n^{(2)}}{\partial\xi}+\frac{\partial n^{(1)}}{\partial\eta}+\frac{v_\rho^{(1)}}{V\eta}+\frac{\partial v_\rho^{(2)}}{\partial\xi}+\frac{\partial}{\partial\xi}(n^{(1)}v_\rho^{(1)})$$

$$+\frac{1}{V\eta}\frac{\partial v_\theta^{(1)}}{\partial\tau}=0, \qquad \text{(B5a)}$$

$$-V\frac{\partial v_\rho^{(2)}}{\partial\xi}+\frac{\partial v_\rho^{(1)}}{\partial\eta}+v_\rho^{(1)}\frac{\partial v_\rho^{(1)}}{\partial\xi}=-\frac{\partial B^{(2)}}{\partial\xi}, \qquad \text{(B5b)}$$

$$n^{(2)}-B^{(2)}=-\frac{\partial^2 B^{(1)}}{\partial\xi^2}. \qquad \text{(B5c)}$$

When Eqs. (B5) are combined there results

$$V\frac{\partial^3 v_\rho^{(1)}}{\partial\xi^3}+2\frac{\partial v_\rho^{(1)}}{\partial\eta}+3v_\rho^{(1)}\frac{\partial v_\rho^{(1)}}{\partial\xi}-f_1\frac{\partial v_\rho^{(1)}}{\partial\xi}+\frac{v_\rho^{(1)}}{\eta}+\frac{1}{\eta}\frac{\partial v_\theta^{(1)}}{\partial\tau}$$

$$=\frac{\partial f_1}{\partial\eta}. \qquad \text{(B6)}$$

If Eq. (B6) is differentiated with respect to ξ and Eq. (B4) is used, the cylindrical KP equation follows:

$$(Vv_{\rho\xi\xi\xi}^{(1)}+3v_\rho^{(1)}v_{\rho\xi}^{(1)}+2v_{\rho\eta}^{(1)}-f_1v_{\rho\xi}^{(1)}+v_\rho^{(1)}/\eta)_\xi+\frac{V}{\eta^2}v_{\rho\tau\tau}^{(1)}$$

$$=-f_{2\tau\tau}(\eta,\tau)/\eta^2. \qquad \text{(B7)}$$

As shown by Eqs. (B1) and (B3b) the cylindrical KP equation occurs for the angle θ unchanged and weak transverse velocity component.

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