Subharmonic structures in Josephson tunneling

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We studied the ac Josephson effect. The phase difference is expanded in a Fourier series. The Fourier components can be evaluated by solving the equations of motion. Truncation was applied to deal with the infinite products and series of Bessel functions. It is very efficient if the nonlinearity is not too large. We give some numerical results and derive the dc current response. The latter has a different form from previous works. The Shapiro steps were also studied with the same method and a more accurate step size estimation is given. [S0163-1829(96)04838-2]

I. INTRODUCTION

The ac Josephson effect has been an active field of study since it was discovered by Shapiro.[1] Recently this effect in high-T_c superconducting tunnel junctions has attracted much attention.[2] Josephson tunnel junctions irradiated by microwave show rich phenomena[3] such as those due to imputities, types and sizes of junctions, the magnetic field effect, and effects of flux. In this work we are concerned with the Shapiro steps and the energy gap-related current steps at the subharmonics and higher harmonics of the microwave frequency. One of the reasons that make the Josephson tunneling so interesting and important is that it is a nonlinear system. The Shapiro steps can be simulated quite well using the resistively shunted junction (RSJ) model.[4] It has an analog in mechanics, namely, the damped driven pendulum. It exhibits extensive nonlinear behaviors including chaos. Thus it is tempting to pursue similar phenomena in the Josephson tunneling. Most of the works[5] on the RSJ model and its mechanical analogs used simulations and obtained satisfying results. However, analytical works are still desirable. They in many cases provide a deeper view of various physical phenomena. This work is a step in that dirction. It was also motivated by experimental[6] and theoretical[7] works on the subharmonic structures related to the superconducting energy gap. In Ref. 7 a model similar to RSJ model was studied and a self-coupling solution was proposed. There are certainly other theories, such as the multiparticle tunneling model. But they are not without shortcomings. For example, a multiparticle tunneling model cannot produce some fine structures of I-V characteristics.[7] On the other hand, the self-coupling mechanism indeed gives Shapiro steps which is a similar phenomenon and produces correct structure shapes. Therefore, we believe it is a major cause of the gap-related structures. Here, we tried to provide a rigorous solution and generalization. The voltage bias setup, which, we think, is more suitable to the high-frequency region, is studied. In Sec. II we derive the formalism. The numerical results are presented in Sec. III. We then compute the dc current in Sec. IV. The comparison with other theories is discussed in Sec. V. The Shapiro steps are studied in detail in Sec. VI. The conclusion is given in Sec. VII.

II. DERIVATION OF PHASE

We begin with the equations of motion of the phase,[2]

$$\dot{\phi}=2eV(t) \tag{1}$$

and

$$\ddot{\phi}+\frac{1}{RC}\dot{\phi}+\omega_J^2\sin\phi=2eV_{\rm ac}\omega\sin\omega t, \tag{2}$$

where Q is the charge and C is the capacitance of the junction, $V(t)$ includes the dc and ac bias voltage $V_{\rm dc}$ and $-V_{\rm ac}\cos\omega t$, ω_J is the Josephson plasma frequency, and $\hbar$ was set to 1. A term of the normal resistance of the junction is added to Eq. (2). As in the case of simple harmonic oscillators the damping force produces a phase difference (not ϕ) between the displacement and the external force and modifies the amplitude; the resistance term here has a similar effect. However, it will not change the characteristics of the solution. Therefore, we present the derivation without the resistive term to reduce the notation, but will give the results with the junction resistance taken into account.

Hasselberg *et al.*,[7] with physical insight, proposed the following solution:

$$\phi=\omega_0 t-\frac{2eV_{\rm ac}}{\omega}\sin\omega t+\frac{\omega_J^2}{\omega_0^2}\sin\left(\omega_0 t-\frac{2eV_{\rm ac}}{\omega}\sin\omega t\right), \tag{3}$$

where

$$\omega_0=2eV_{\rm dc}. \tag{4}$$

It was based on the result of a small ω_J/ω_0 perturbation. Now we found a general solution which is valid in all cases and agrees with the solution of Hasselberg *et al.* in the small ω_J/ω_0 limit. It has the form

$$\phi=\omega_0 t+\sum_{k,l}a_{kl}\sin(k\omega+l\omega_0)t+\sum_{m,n}b_{mn}\cos(m\omega+n\omega_0)t, \tag{5}$$

where k and m are set to be positive to prevent redundancy. This is motivated by the nonlinearity of the equations of motion. a's and b's are to be determined by Eq. (2). The second summation will disappear if we drop the resistive term and this is what we shall do in the following process. Using the relation[8]

$$\exp(iz\sin\theta)=\sum_n J_n(z)e^{in\theta}, \tag{6}$$

we get

$$e^{i\phi}=e^{i\omega_0 t}\prod_{k,l}\sum_\alpha J_\alpha(a_{kl})e^{i\alpha(k\omega+l\omega_0)t}$$

$$=e^{i\omega_0 t}\sum_{\alpha_{kl}}\prod_{kl} J_{\alpha_{kl}}(a_{kl})e^{i\alpha_{kl}(k\omega+l\omega_0)t}, \tag{7}$$

where the summation of α_{kl} means taking into account all the possible integer values. This convention will be used in the whole paper. Substituting into Eq. (2), we have

$$\sum_{k,l} a_{kl}(k\omega+l\omega_0)^2\sin(k\omega+l\omega_0)t+2eV_{\rm ac}\omega\sin\omega t$$

$$=\omega_J^2\sum_{\alpha_{kl}}\left[\prod_m J_{\alpha_{kl}}(a_{kl})\right]\sin\left[\omega_0 t+\sum_{k,l}\alpha_{kl}(k\omega+l\omega_0)t\right]. \tag{8}$$

So we get

$$(m\omega+n\omega_0)^2 a_{mn}+2eV_{\rm ac}\omega\delta_{m1}\delta_{n0}=\omega_J^2\sum_{\alpha_{kl}}\prod_{k,l}J_{\alpha_{kl}}(a_{kl}), \tag{9}$$

with

$$\sum_{k,l}\alpha_{kl}k=m \tag{10}$$

and

$$\sum_{k,l}\alpha_{kl}l+1=n. \tag{11}$$

We consider the energy gap-related structures first. The Shapiro steps will be studied in Sec. V. There are infinite products of $J_n(x)$ in Eq. (9). We can simplify them with the following mathematical knowledge: In any Fourier series, the larger k or $|l|$, the less a_{kl} in general, and

$$J_n(x)\propto x^n \quad \text{for} \quad n>0 \quad \text{and} \quad x\to 0. \tag{12}$$

Thus, for any term containing the product $\Pi_{kl}J_{\alpha_{kl}}(a_{kl})$ to be important, α_{kl} must approach 0 for large k or $|l|$. So we can truncate the product by setting

$$\alpha_{kl}=0 \quad \text{for} \quad k \quad \text{or} \quad |l-1|>m_0, \tag{13}$$

where m_0 can be chosen for desired accuracy and thus the approximation

$$J_{\alpha_{kl}}(a_{kl})\simeq 1 \quad \text{for} \quad k \quad \text{or} \quad |l-1|>m_0 \tag{14}$$

can be applied. Below we give an example of $m_0=1$:

$$\omega^2 a_{10}+2eV_{\rm ac}\omega\simeq\omega_J^2\sum_{\alpha_{10},\alpha_{01},\alpha_{11},\alpha_{02}}J_{\alpha_{10}}(a_{10})J_{\alpha_{01}}(a_{01})\times J_{\alpha_{11}}(a_{11})J_{\alpha_{02}}(a_{02}), \tag{15}$$

with

$$\alpha_{10}+\alpha_{11}=1 \tag{16}$$

and

$$\alpha_{01}+\alpha_{11}+2\alpha_{02}=-1, \tag{17}$$

$$\omega_0^2 a_{01}\simeq\omega_J^2\sum_{\alpha_{10},\alpha_{01},\alpha_{11},\alpha_{02}}J_{\alpha_{10}}(a_{10})J_{\alpha_{01}}(a_{01})\times J_{\alpha_{11}}(a_{11})J_{\alpha_{02}}(a_{02}), \tag{18}$$

with

$$\alpha_{10}+\alpha_{11}=0 \tag{19}$$

and

$$\alpha_{01}+\alpha_{11}+2\alpha_{02}=0, \tag{20}$$

$$(\omega+\omega_0)^2 a_{11}\simeq\omega_J^2\sum_{\alpha_{10},\alpha_{01},\alpha_{11},\alpha_{02}}J_{\alpha_{10}}(a_{10})J_{\alpha_{01}}(a_{01})\times J_{\alpha_{11}}(a_{11})J_{\alpha_{02}}(a_{02}), \tag{21}$$

with

$$\alpha_{10}+\alpha_{11}=1 \tag{22}$$

and

$$\alpha_{01}+\alpha_{11}+2\alpha_{02}=0, \tag{23}$$

and

$$4\omega_0^2 a_{02}\simeq\omega_J^2\sum_{\alpha_{10},\alpha_{01},\alpha_{11},\alpha_{02}}J_{\alpha_{10}}(a_{10})J_{\alpha_{01}}(a_{01})\times J_{\alpha_{11}}(a_{11})J_{\alpha_{02}}(a_{02}), \tag{24}$$

with

$$\alpha_{10}+\alpha_{11}=1 \tag{25}$$

and

$$\alpha_{01}+\alpha_{11}+2\alpha_{02}=1. \tag{26}$$

In the above equations all other α's are set to zero. There are still infinite series involved in Eqs. (15), (18), (21), and (24). Since a_{kl} is small in magnitude, Bessel functions of lower orders usually give greater contribution. We can truncate them by retaining only lower-order terms. Now it is straightforward to solve these coupled equations. Details and some numerical results will be presented in the next section.

It can be shown that the result of the above derivation coincides with those of Hasselberg *et al.*[7] for small ω_J/ω_0. The solution in Eq. (3) gives

$$e^{i\phi}=\sum_{k,l}J_k\left(\frac{2(l+1)eV_{\rm ac}}{\omega}\right)J_l\left(\frac{\omega_J^2}{\omega_0^2}\right)e^{i[(l+1)\omega_0-k\omega]t}. \tag{27}$$

On the other hand, in view of Eqs. (15) and (18), in the limit of small ω_J/ω_0, we found

$$a_{10}\simeq\frac{2eV_{\rm ac}}{\omega}, \tag{28}$$

$$a_{01}\simeq\frac{\omega_J^2}{\omega_0^2}, \tag{29}$$

and

$$a_{kl}\simeq 0 \quad \text{for others.} \tag{30}$$

Substituting Eqs. (28) and (29) into Eq. (5), we get the leading terms of Eq. (3).

We note that the truncation scheme is built upon the interplay of two frequencies ω and ω_0. Without ac bias, it is no longer applicable or in need of major modification. However, there is no self-contradiction. For example, if $V_{\rm ac}=0$, then $a_{10}=0$. In fact all the Fourier coefficients $a_{kl}=b_{mn}=0$ for $k\neq 0$ and $m\neq 0$. In this case the form of ϕ in Eq. (5) goes back to that proposed in the pioneer works in this field.[9]

III. NUMERICAL RESULTS

As stated in the last section, we have to truncate the infinite products and infinite series in order to perform the computation. The truncation is based on Eqs. (12)–(14). The value of m_0 depends on two dimensionless parameters

$$x=\frac{\omega_J}{\omega} \tag{31}$$

and

$$x_d=\frac{\omega_J}{\omega_0}. \tag{32}$$

They are measures of nonlinearity. The greater these parameters, the larger m_0 should be. Once m_0 is chosen the numbers of terms in Eqs. (15), (18), (21), and (24) that should be kept are more or less fixed. For example, if $m_0=2$, then $|\alpha_{01}|\leqslant 2$ and $|\alpha_{11}|\leqslant 1$. Taking into account more terms will not enhance accuracy. In our experience, when $x<1$ and $x_d<1$, taking $m_0=1$ is already a very good approximation. The results agree with those of taking $m_0=2$ within a tenth of a percent. Below we give an example of considering only a few of the leading terms. All the results can be derived from the equations in the previous section with the resistive term taken into account:

$$\omega^2 a_{10}+\frac{\omega}{RC}b_{10}+2eV_{\rm ac}\omega\simeq\omega_J^2[J_0(a_{10})J_0(a_{01})J_1(a_{1-1}) - J_1(a_{10})J_1(a_{01})J_0(a_{1-1})], \tag{33}$$

$$\omega_0^2 a_{01}+\frac{\omega_0}{RC}b_{01}\simeq\omega_J^2 J_0(a_{10})J_0(a_{01}), \tag{34}$$

$$4\omega_0^2 a_{02}+\frac{2\omega_0}{RC}b_{02}\simeq\omega_J^2 J_0(a_{10})J_1(a_{01}), \tag{35}$$

$$(\omega+\omega_0)^2 a_{11}+\frac{\omega+\omega_0}{RC}b_{11}\simeq\omega_J^2 J_1(a_{10})J_0(a_{01}), \tag{36}$$

$$(\omega-\omega_0)^2 a_{1-1}+\frac{\omega-\omega_0}{RC}b_{1-1}\simeq\omega_J^2 J_1(a_{10})J_2(a_{01}), \tag{37}$$

$$\omega^2 b_{10}-\frac{\omega}{RC}a_{10}\simeq\omega_J^2 J_1(a_{10})J_1(b_{01}), \tag{38}$$

$$\omega_0^2 b_{01}-\frac{\omega_0}{RC}a_{01}\simeq 0, \tag{39}$$

$$4\omega_0^2 b_{02}-\frac{2\omega_0}{RC}a_{02}\simeq\omega_J^2 J_0(a_{10})J_1(b_{01}), \tag{40}$$

$$(\omega+\omega_0)^2 b_{11}-\frac{\omega+\omega_0}{RC}a_{11}\simeq\omega_J^2 J_0(a_{10})J_1(b_{10}), \tag{41}$$

and

$$(\omega-\omega_0)^2 b_{1-1}-\frac{\omega-\omega_0}{RC}a_{1-1}\simeq\omega_J^2 J_1(a_{10})J_2(a_{01}). \tag{42}$$

The above results were derived with the assumption that the junction resistance is very large so that $|b_{ij}|\ll|a_{ij}|$. Actually it depends on the experimental situation. However, we anticipate no problem if the resistance is small. The procedure can be carried out in almost the same way except that more b's have to be considered. We have also used the fact that $|a_{ij}|$ becomes smaller if either $|i|$ or $|j|$ get larger and have kept only the leading terms. This approximation is valid if the nonlinearity is not too large $(x,x_d\geqslant 1)$. Though the list is not complete, we can see that a_{10} is approximately proportional to $2eV_{\rm ac}/\omega$. Thus we consider it to be the zeroth-order term. The next term large in magnitude is a_{01}. a_{02} and a_{11} are smaller in magnitude and a_{1-1} is the smallest coefficient among the a's in the list.

According to Eqs. (33)–(42), we plotted a few Fourier components against $1/x$ in Figs. 1(a) and 1(b) under the conditions $1/RC\omega=0.1$, $1/x_d=1.1$, and $1/x_a=1.2$, respectively. In Figs. 2(a) and 2(b) these coefficients are plotted against $1/x_d$ with $1/RC\omega=0.1$, $1/x=1.1$, and $1/x_a=1.2$. One can clearly see that the Fourier components converge very fast.

We derive a general rule for truncation below. According to Eq. (9), if $x/k^2<1$ and $x_d/l^2<1$, then $a_{kl}<1$. One also found that $a_{k+1,l}<a_{kl}$ and $a_{k,l+1}<a_{kl}$. The ratios $a_{k+1,l}/a_{kl}$ and $a_{k,l+1}/a_{kl}$ do not vary much with k and l in our experience. Thus we can define loosely two small parameters

$$\epsilon_a\sim\frac{a_{k+1,l}}{a_{kl}} \tag{43}$$

and

$$\epsilon_b\sim\frac{a_{k,l+1}}{a_{kl}}. \tag{44}$$

The above argument is valid for $l>0$. For $l<0$, $l-1$ should be used instead of $l+1$. In view of Eq. (14), we have

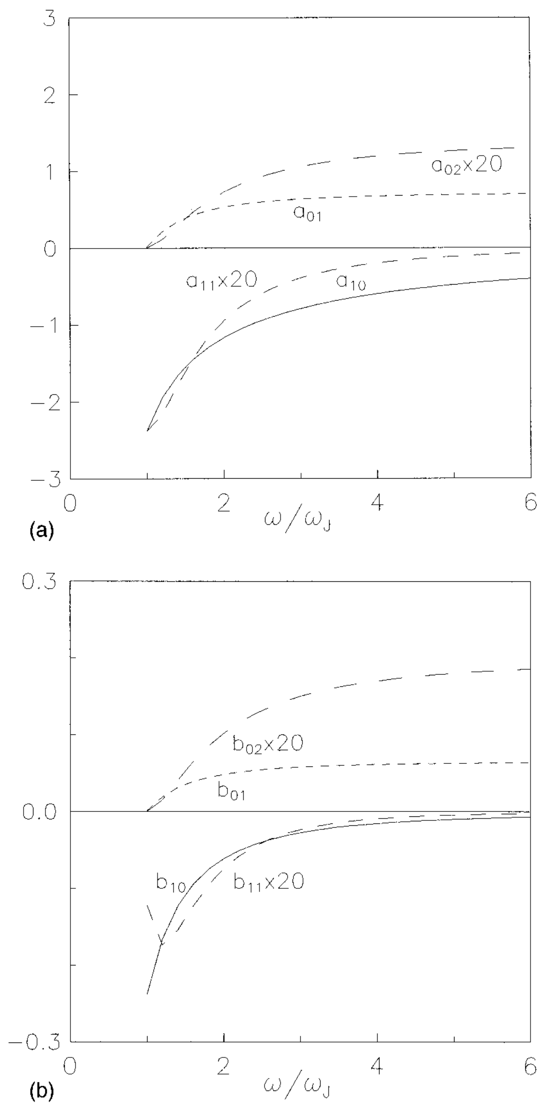

FIG. 1. (a) a_{10}, a_{01}, a_{11}, and a_{02} were plotted against ω/ω_p with $\omega_0=1.1\omega_p$ and $2eV_{\mathrm{ac}}=1.2\omega_p$. Note that both a_{11} and a_{02} were multiplied by a factor of 20. (b) b_{10}, b_{01}, b_{11}, and b_{02} were plotted against ω/ω_p with $\omega_0=1.1\omega_p$ and $2eV_{\mathrm{ac}}=1.2\omega_p$. Note that both b_{11} and b_{02} were multiplied by a factor of 20.

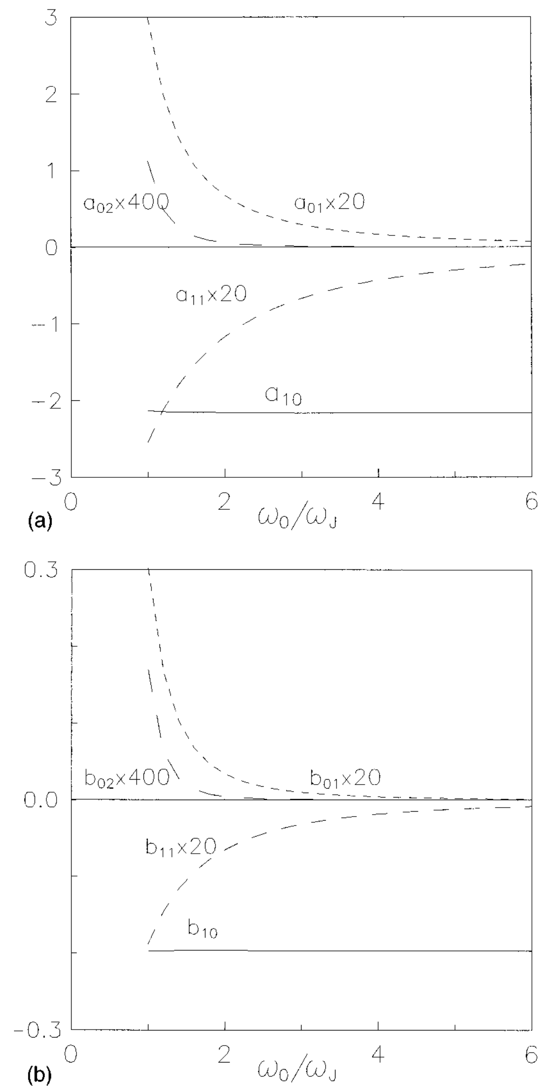

FIG. 2. (a) a_{10}, a_{01}, a_{11}, and a_{02} were plotted against ω_0/ω_p with $\omega=1.1\omega_p$ and $2eV_{\mathrm{ac}}=1.2\omega_p$. Note that both a_{01} and a_{11} were multiplied by a factor of 20 and a_{02} was multiplied by a factor of 400. (b) b_{10}, b_{01}, b_{11}, and b_{02} were plotted against ω_0/ω_p with $\omega=1.1\omega_p$ and $2eV_{\mathrm{ac}}=1.2\omega_p$. Note that both b_{01} and b_{11} were multiplied by a factor of 20 and b_{02} was multiplied by a factor of 400.

$$J_n(a_{kl}/2)\sim\epsilon_a^{n(k-\delta_{1k})}\epsilon_b^{n|l|}, \tag{45}$$

except for a_{10}. The order of magnitude of each Bessel function can be judged by the power of ϵ_a or ϵ_b. This way one can decide the contribution of any term and m_0 can be chosen for a desired accuracy. The infinite series can also be truncated accordingly. We conclude this section by stating that the truncation method is very efficient for $x,x_d<2$.

IV. CURRENT RESPONSE

A direct consequence of the bias voltage is the current response. It has the form[10]

$$j(t)=\mathrm{Im}\left\{e^{-i\phi(t)/2}\left[\int_{-\infty}^{\infty}dt'\,e^{i\phi(t+t')/2}j_1(t') + \int_{-\infty}^{\infty}dt'\,e^{-i\phi(t-t')/2+\alpha}j_2(t')\right]\right\}, \tag{46}$$

where $j_1(t)$ is the quasiparticle current, $j_2(t)$ is the Cooper-pair current, and α is a constant phase. With the Fourier transform

$$j_{1(2)}(t)=\int\frac{d\omega}{2\pi}e^{-i\omega t}j'_{1(2)}(\omega) \tag{47}$$

and substitution of Eq. (7) into Eq. (46), we obtain the quasiparticle part of the dc current:

$$j_{\mathrm{dc}}^{q}=\sum_{\alpha_{kl},\gamma_{mn}}\left[\prod_{k,l}J_{\alpha_{kl}}\left(\frac{a_{kl}}{2}\right)\right]\left[\prod_{m,n}J_{\gamma_{mn}}\left(\frac{a_{mn}}{2}\right)\right]\times\mathrm{Im}j_1'\left(\sum_{m,n}m\gamma_{mn}\omega+\frac{\omega_0}{2}+\sum_{m,n}n\gamma_{mn}\omega_0\right),\quad(48)$$

with the requirement

$$\sum_{k,l}k\alpha_{kl}=\sum_{m,n}m\gamma_{mn}\quad(49)$$

and

$$\sum_{k,l}l\alpha_{kl}=\sum_{m,n}n\gamma_{mn}.\quad(50)$$

The Cooper-pair part has the form

$$j_{\mathrm{dc}}^{p}=\sum_{\alpha_{kl},\gamma_{mn}}\left[\prod_{k,l}J_{\alpha_{kl}}\left(\frac{a_{kl}}{2}\right)\right]\left[\prod_{m,n}J_{\gamma_{mn}}\left(\frac{a_{mn}}{2}\right)\right]\times\mathrm{Im}\left[e^{i\alpha}j_2'\left(\sum_{m,n}m\gamma_{kl}\omega+\frac{\omega_0}{2}+\sum_{m,n}n\gamma_{mn}\omega_0\right)\right],\quad(51)$$

with the requirement

$$\sum_{k,l}k\alpha_{kl}+\sum_{m,n}m\gamma_{mn}=0\quad(52)$$

and

$$\sum_{l}l\alpha_{kl}+\sum_{n}n\gamma_{mn}+1=0.\quad(53)$$

It is obvious that the structures of both currents are very rich and complicated. Consider the quasiparticle current first. For the purpose of analysis, we truncate the product by setting $m_0=1$ as in Sec. III. Now Eqs. (49), (50), (52), and (53) become, respectively,

$$\alpha_{10}+\alpha_{11}=\gamma_{10}+\gamma_{11},\quad(54)$$

$$\alpha_{01}+\alpha_{11}+2\alpha_{02}=\gamma_{01}+\gamma_{11}+2\gamma_{02},\quad(55)$$

$$\alpha_{10}+\alpha_{11}+\gamma_{10}+\gamma_{11}=0,\quad(56)$$

and

$$\alpha_{01}+\alpha_{11}+2\alpha_{02}+\gamma_{01}+\gamma_{11}+2\gamma_{02}=-1.\quad(57)$$

We can have

$$\alpha_{kl}=\gamma_{kl}\quad(58)$$

for all the k and l and many other combinations. The most important ones are

$$j_{\mathrm{dc}}^{q}\simeq\sum_{k,l,m,n}\left\{J_k^2\left(\frac{a_{10}}{2}\right)J_l^2\left(\frac{a_{01}}{2}\right)J_m^2\left(\frac{a_{11}}{2}\right)J_n^2\left(\frac{a_{02}}{2}\right)+\sum_j\left[J_k^2\left(\frac{a_{10}}{2}\right)J_m^2\left(\frac{a_{11}}{2}\right)J_l\left(\frac{a_{01}}{2}\right)J_{l+2j}\left(\frac{a_{01}}{2}\right)J_n\left(\frac{a_{02}}{2}\right)J_{n-j}\left(\frac{a_{02}}{2}\right)\right.\right.$$
$$\left.\left.+J_k\left(\frac{a_{10}}{2}\right)J_{k+j}\left(\frac{a_{10}}{2}\right)J_l\left(\frac{a_{01}}{2}\right)J_{l+j}\left(\frac{a_{01}}{2}\right)J_m\left(\frac{a_{11}}{2}\right)J_{m-j}\left(\frac{a_{11}}{2}\right)J_n^2\left(\frac{a_{02}}{2}\right)\right]\right\}$$
$$\times\mathrm{Im}j_2'\left(\sum_{k,m}(k+m)\omega+\frac{\omega_0}{2}+\sum_{l,m,n}(l+m+2n)\omega_0\right).\quad(59)$$

The Cooper-pair current has the approximate form

$$j_{\mathrm{dc}}^{p}\simeq\sum_{k,l,m,n}\left\{J_k\left(\frac{a_{10}}{2}\right)J_{-k}\left(\frac{a_{10}}{2}\right)J_l\left(\frac{a_{01}}{2}\right)J_{-l-1}\left(\frac{a_{01}}{2}\right)J_m\left(\frac{a_{11}}{2}\right)J_{-m}\left(\frac{a_{11}}{2}\right)J_n\left(\frac{a_{02}}{2}\right)J_{-n}\left(\frac{a_{02}}{2}\right)\right.$$
$$+\sum_j\left[J_k\left(\frac{a_{10}}{2}\right)J_{-k}\left(\frac{a_{10}}{2}\right)J_m\left(\frac{a_{11}}{2}\right)J_{-m}\left(\frac{a_{11}}{2}\right)J_l\left(\frac{a_{01}}{2}\right)J_{2j-l-1}\left(\frac{a_{01}}{2}\right)J_n\left(\frac{a_{02}}{2}\right)J_{-j-n}\left(\frac{a_{02}}{2}\right)\right.$$
$$\left.\left.+J_k\left(\frac{a_{10}}{2}\right)J_{j-k}\left(\frac{a_{10}}{2}\right)J_l\left(\frac{a_{01}}{2}\right)J_{j-l-1}\left(\frac{a_{01}}{2}\right)J_m\left(\frac{a_{11}}{2}\right)J_{-j-m}\left(\frac{a_{11}}{2}\right)J_n\left(\frac{a_{02}}{2}\right)J_{-n}\left(\frac{a_{02}}{2}\right)\right]\right\}$$
$$\times\mathrm{Im}\left[e^{i\alpha}j_2'\left(\sum_{k,m}(k+m)\omega+\frac{\omega_0}{2}+\sum_{l,m,n}(l+m+2n)\omega_0\right)\right].\quad(60)$$

Profound structures occur when the arguments of $j_1'(\omega)$ and $j_2'(\omega)$ are equal to 2Δ where Δ is the energy gap. For example, the Cooper-pair current shows Riedel peaks.[11] In view of Eqs. (58) and (59), the structures occur when $(N+1/2)\omega_0=\pm(2\Delta\pm M\omega)$. The leading order terms of the quasiparticle current are

$$\sum_k J_k^2\left(\frac{a_{10}}{2}\right)J_0^2\left(\frac{a_{01}}{2}\right)\mathrm{Im}j_1'\left(k\omega+\frac{\omega_0}{2}\right),$$

and those of Cooper-pair current are

$$\sum_k J_k\left(\frac{a_{10}}{2}\right)J_{-k}\left(\frac{a_{10}}{2}\right)J_0\left(\frac{a_{01}}{2}\right)J_{-1}\left(\frac{a_{01}}{2}\right)$$
$$\times\mathrm{Im}\left[e^{i\alpha}j_2'\left(k\omega+\frac{\omega_0}{2}\right)\right].$$

This prediction is not different from the leading order terms of the self-coupling theory of Ref. 7. However, Eqs. (59) and (60) give a more complicated dc current amplitude.

V. COMPARISON

In this section we compare the truncation method with other theories. The present work is built upon the self-coupling mechanism. Another model, namely, the multiparticle tunneling model, has been developed by several groups.[12,7] This model considers the possibility of the tunneling of many electrons, thus gaining enough energy from the bias voltage to overcome the energy gap. Its prediction of the occurrence of steps was indeed seen in experiments.[13] However, the sizes are not always consistent with the experimental data. For the structure at $meV_{dc}=2\Delta+n\hbar\omega$, this model gives a size proportional to $|t|^{2m}$ where t is the tunneling matrix and m is the number of electrons which tunneled through. This would make the size too small for large m. The other inconsistency is that this model predicts only current steps but current peaks were found in experiments. Therefore, the consensus is that the multiparticle tunneling model can only account for part of the picture.

Klapwijk *et al.*[14] cleverly invoked Andreev reflection[15] to explain subharmonic structures. They found that with multi-Andreev reflection, electrons can also obtain energy to overcome the energy gap. This mechanism should occur in the superconducting-normal-superconducting (SNS) junction. Similar to the multiparticle tunneling model, the amplitude of the structures (peaks of dV/dI) is proportional to r^m where r is the reflection coeffient and m is the number of reflections. r is not easy to evaluate and Ref. 14 only gave a phenomenological calculation. At this stage, a more accurate calculation is needed to give a quantitative comparison with the experimental data.

The present form sometimes predicts results quite different from those of Ref. 7. For example, consider the case $eV_{dc}=2\Delta-\omega$. In view of Eqs. (59) and (60), the leading terms of the current densities are ($k=-1$ and $l=0$)

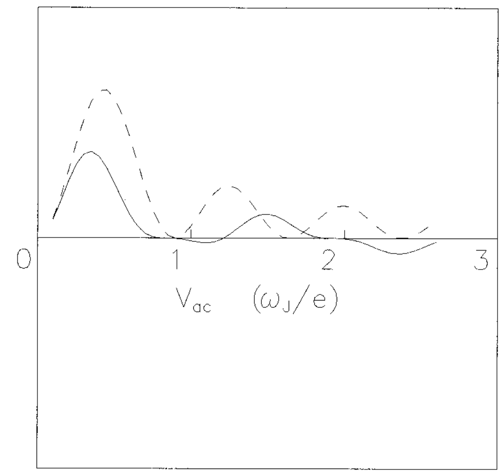

FIG. 3. The magnitudes of the Cooper-pair current vs ac bias voltage in the large resistance limit under the conditions $eV_{dc}=\omega_J$, $\omega=0.7\omega_J$ and V_{ac} is also in units of ω_J/e. The solid line is the result of the present calculation multiplied by a factor of 10 and the dashed line is from the form in Ref. 7.

$$j_{dc}^q\simeq J_1^2\left(\frac{a_{10}}{2}\right)J_0^2\left(\frac{a_{01}}{2}\right)\mathrm{Im}j_2'\left(\frac{\omega_0}{2}-\omega\right) \tag{61}$$

and

$$j_{dc}^p\simeq J_1^2\left(\frac{a_{10}}{2}\right)J_0\left(\frac{a_{01}}{2}\right)J_{-1}\left(\frac{a_{01}}{2}\right)\mathrm{Im}\left[e^{i\alpha}j_2'\left(\frac{\omega_0}{2}-\omega\right)\right], \tag{62}$$

where we have used the approximation

$$J_0\left(\frac{a_{11}}{2}\right)\simeq J_0\left(\frac{a_{02}}{2}\right)\simeq 0. \tag{63}$$

Their forms are the same as those in Eq. (3.17) of Ref. 7 if the following replacements were made:

$$\alpha\rightarrow -a_{10}/2 \tag{64}$$

and

$$\alpha_1\rightarrow a_{01}/2. \tag{65}$$

Indeed we found they are approximately true in view of Eqs. (33) and (34) since, in Ref. 7,

$$\alpha=eV_{ac}/\omega \tag{66}$$

and

$$\alpha_1=\omega_J^2/2\omega_0^2 \tag{67}$$

in the high-V_{dc} limit. However, a_{01} varies with V_{ac} while α_1 in Ref. 7 is constant. This produces a difference. To present another case, we plotted in Fig. 3 the magnitudes of the Cooper-pair current at $eV_{dc}=(2\Delta-\hbar\omega)/3$ versus ac bias voltage in the large resistance limit under the conditions

$eV_{dc}=4\omega_J$ and $\omega=0.7\omega_J$, and V_{ac} is also in units of ω_J/e. The solid line is the result of the present calculation multiplied by a factor of 10 and the dashed line is from the form in Ref. 7.

VI. SHAPIRO STEPS

The dc current shows Shapiro steps provided the ratio of ω to ω_0 is a rational number. In this section we introduce a neater version of Eq. (5). In fact, both are the same. Note that if $V_{ac}=0$, one had better go back to the process of Sec. II. We define

$$\Omega \equiv \frac{\omega}{M} = \frac{\omega_0}{N}, \tag{68}$$

where M and N are integers and $\omega/\omega_0=M/N$. Instead of the form in Eq. (5) we now have

$$\phi=\omega_0 t+\sum_m c_m \sin m\Omega t+\sum_n d_n \sin n\Omega t. \tag{69}$$

Now the derivation will take the resistive term into account. However, we continue to assume the resistance to be very large so that $d_n \to 0$. Substituting into Eq. (2), we get

$$\sum_m c_m m^2\Omega^2 \sin m\Omega t+\frac{1}{RC}\sum_n d_n n\Omega \cos n\Omega t+2eV_{ac}\omega \sin\omega t=\omega_J^2\sum_{\alpha_m,\beta_n}\left[\prod_m J_{\alpha_m}(c_m)\right]\left[\prod_n J_{\beta_n}(d_n)\right] \times[e^{i(\Sigma_m m\alpha_m\Omega t+\Sigma_n n\beta_n\Omega t+\omega_0 t)} i^{\Sigma_n\beta_n} - e^{-i(\Sigma_m m\alpha_m\Omega t+\Sigma_n n\beta_n\Omega t+\omega_0 t)} i^{-\Sigma_n \beta_n}]. \tag{70}$$

The Fourier components can be found easily:

$$l^2\Omega^2 c_l+\frac{l\Omega}{RC}d_l+2eV_{ac}\omega\,\delta_{lM} = \omega_J^2\sum_{\alpha_m,\beta_n} i^{\Sigma_n \beta_n}\left[\prod_m J_{\alpha_m}(c_m)\right]\left[\prod_n J_{\beta_n}(d_n)\right], \tag{71}$$

with

$$N+\sum_m m\alpha_m+\sum_n n\beta_n=l \tag{72}$$

and $\Sigma_n\beta_n$ is an even integer, and

$$l^2\Omega^2 d_l-\frac{l\Omega}{RC}c_l=\omega_J^2\sum_{\alpha_m,\beta_n} i^{\sum_n \beta_n}\left[\prod_m J_{\alpha_m}(c_m)\right] \times\left[\prod_n J_{\beta_n}(d_n)\right], \tag{73}$$

with

$$N+\sum_m m\alpha_m+\sum_n n\beta_n=l \tag{74}$$

and $\Sigma_n\beta_n$ is an odd integer. Apparently the coefficients largest in magnitude are c_M and c_N. But as we shall see below, $c_{N\pm M}$ is not negligible. The magnitude of c_M will be determined by V_{ac}. If the nonlinearity is not too large, i.e., if x is of the order of 1 or less, the magnitudes of the Bessel functions on the right-hand side (RHS) of Eq. (71) will be small. Hence,

$$c_M \simeq -\frac{2eV_{ac}\omega}{\omega^2+(1/RC)^2}. \tag{75}$$

For c_N the requirement of Eq. (65) becomes

$$\sum_m m\alpha_m=0. \tag{76}$$

Therefore the most important term on the RHS of Eq. (71) is that with all the $\alpha_m=0$. Therefore,

$$c_N \simeq \frac{\omega_J^2}{\omega_0^2+(1/RC)^2} J_0\left(\frac{c_N}{2}\right). \tag{77}$$

It can be solved easily. Other Fourier coefficients can be computed with iterations.

The important implication of Eq. (71) is that there exists significant subharmonic structures at $(N\pm M)\Omega$. In order to satisfy Eq. (72) we can choose $\alpha_M=\pm 1$ and others vanish. Thus, we have

$$c_{N\pm M}\simeq\pm\frac{\omega_J^2}{[(N\pm M)\Omega]^2+(1/RC)^2}J_1\left(\frac{c_M}{2}\right)\times J_0\left(\frac{c_N}{2}\right)J_0\left(\frac{c_{N\pm M}}{2}\right). \tag{78}$$

$c_{N\pm M}$ is comparable with c_N in magnitude.

The dc current can be calculated in the same way as in the last section. We simply gave the result

$$j_{dc}^{p}=\sum_{\alpha_k,\beta_l,\gamma_m,\zeta_n}\left[\prod_k J_{\alpha_k}\left(\frac{c_k}{2}\right)\right]\left[\prod_l J_{\beta_l}\left(\frac{d_l}{2}\right)\right]\times\left[\prod_m J_{\gamma_m}\left(\frac{c_m}{2}\right)\right]\left[\prod_n J_{\zeta_m}\left(\frac{d_n}{2}\right)\right]\times\mathrm{Im}\left\{e^{i\alpha}j_2'\left(\sum_m m\gamma_m\Omega+\sum_n n\zeta_n\Omega+\frac{\omega_0}{2}\right)\right\}, \tag{79}$$

with the requirement

$$\sum_k k\alpha_k+\sum_l l\beta_l+\sum_m m\gamma_m+\sum_n n\zeta_n+N=0. \tag{80}$$

If $M=1$, Shapiro steps occur. The most important terms are all the

$$\alpha_k=\beta_l=\gamma_m=\zeta_n=0, \tag{81}$$

except

$$\alpha_1+\gamma_1+N=0. \tag{82}$$

This produces a series

$$\sum_n J_n\left(\frac{c_1}{2}\right)J_{-N-n}\left(\frac{c_1}{2}\right)=J_{-N}(c_1), \tag{83}$$

if

$$j_2'\left(\sum_m m\gamma_m\Omega+\sum_n n\zeta_n\Omega+\frac{\omega_0}{2}\right)\simeq j_2'=\text{const.} \tag{84}$$

The next important terms are

$$\beta_1=l\neq 0, \tag{85}$$

so that

$$\alpha_1+\gamma_1+\beta_1+N=0. \tag{86}$$

It is also possible that $\beta_1=0$ but $\zeta_1=l$. Hence

$$j_{\rm dc}^{p}\simeq J_{-N}(c_1)j_2'+2\sum_l J_{-N-l}(c_1)J_l(d_1)j_2'. \tag{87}$$

In view of Eq. (75) we see that the first term is the usual form of the size of the Shapiro steps while the second term is the modification of the step size by the resistivity.

In general ($N\neq 1$), we shall have steps at subharmonics. For example, a significant contribution comes from

$$\alpha_k=\beta_l=\gamma_m=\zeta_n=0, \tag{88}$$

except

$$\alpha_M=\pm N \tag{89}$$

and

$$\gamma_N=\mp M-1. \tag{90}$$

The resistance term can also contribute. The terms of largest magnitude are those with

$$\beta_N=-1, \tag{91}$$

$$\alpha_M+\gamma_M=0, \tag{92}$$

or

$$\beta_{N+M}=-1, \tag{93}$$

$$\alpha_M+\gamma_M=1, \tag{94}$$

or

$$\beta_{N-M}=-1, \tag{95}$$

$$\alpha_M+\gamma_M=-1. \tag{96}$$

Using Eqs. (83) and (84) we get

$$j_{\rm dc}^{p}=2j_2'\left[J_N\left(\frac{c_M}{2}\right)J_{-M-1}\left(\frac{c_N}{2}\right)+J_{-N}\left(\frac{c_M}{2}\right)J_{M-1}\left(\frac{c_N}{2}\right)-J_0(c_M)J_1\left(\frac{d_N}{2}\right)-J_1(c_M)J_1\left(\frac{d_{N+M}}{2}\right)+J_1(c_M)J_1\left(\frac{d_{N-M}}{2}\right)\right]. \tag{97}$$

Clearly Eq. (97) predicts the existence of the subharmonic steps with the largest occurring at $M=2$, i.e., the half-integer steps. The size is given mainly by the second term.

VII. CONCLUSION

We have found a general solution for the Josephson junction by expanding the phase in a Fourier series of Bessel functions. The Fourier components can be calculated with equations of motion. A truncation method was proposed to deal with the infinite products and series. If the system is not too nonlinear, i.e., if x and x_d are not greater than 2, the truncation is very efficient. We presented numerical results which showed the relationship between the Fourier components and x, x_d, and x_a. The form of dc current was derived and new predictions were made. The essential difference between our prediction and those of earlier works is that the dc current form is the product of many Bessel functions. The present theory also gives a more complex form for the Shapiro steps. Significant subharmonic structures were found at

$(N-M)\Omega$ where $\Omega=\omega/M=\omega_0/N$. Its amplitude is comparable with that of ω_0 and should be detectable in microwave radiation. The size of the Shapiro steps reduces to the conventional form in the limit of small plasma frequency. We also predict the subharmonic steps with the half-integer ones being the largest. As nonlinearity increases, the complexity grows very fast. Many additional Bessel functions have to be accounted for. However, considering the computation time, the analytical method still compared favorably with the simulation approach. It also has the advantage of being easier to analyze and being able to make predictions.

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