Screening in Josephson-junction ladders

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Both critical current and Shapiro steps in Josephson ladders are found to show effects of vortex exclusion below a critical magnetic field. Vortices are excluded even though the flux itself is *not* screened out. When current is injected perpendicular to the ladder edges, the critical current is unchanged from its $f=0$ value up to a penetration field of $f_{c1\perp}\approx0.12$ flux quanta per plaquette. Similarly, there are only integer Shapiro steps below a critical $f^*_{c1\perp}$, where vortices first penetrate. A narrower critical current plateau also appears to form near $f=\frac{1}{2}$. No such plateaus occur when current is injected in the parallel direction. We attribute the exclusion in the perpendicular geometry to screening currents which flow along the edges of the ladder.

The state of a Josephson junction array is usually specified by the phases of the order parameters on each grain.[1] If an external magnetic field is applied to such an array, the ground state is a rather complicated arrangement of these phases,[2] commonly characterized by a vortex structure. This vortex structure is specified by a set of integer "vortex numbers," defined for each plaquette in the array.[3] In the "weak screening limit," where the local magnetic field approximately equals the applied field B, it is widely believed that the total number of vortices in the array equals BS/Φ_0, where S is the array area and Φ_0 is the flux quantum. Thus, the vortex number density differs from the applied flux density only if the screening magnetic field is taken into account, so that the local and applied fields are different.[4]

Some years ago, Kardar[5] suggested that in a ladder of Josephson junctions, the ground state should actually include *no* vortices below a certain critical applied field. Above this field, vortices do penetrate the ladder, just as flux penetrates a homogeneous type-II superconductor only above the lower critical field H_{c1}. This picture was recently confirmed by Denniston and Tang,[6] who calculated the critical field at zero applied current. Using a transfer matrix approach, they showed that the ground state of the ladder contains no vortices for an applied field below a critical value f_{c1}. They also showed that f_{c1} depends on the anisotropy of the ladder, i.e., the ratio of Josephson couplings perpendicular and parallel to the edges of the ladder. When this ratio is unity, they found $f_{c1}\approx0.28$ flux quanta per plaquette. They also found other plateaus at other vortex densities. At each such plateau, the vortex number was found to be stable against small changes in applied flux density. Thus, the ladder seems to exhibit a kind of screening in which the induced currents (flowing around the edges of the ladder) can fix the number of vortices in the ladder, even though the *flux density* is uniform.

In this paper, we extend the work of Kardar and of Denniston and Tang to treat the *IV* characteristics of a ladder array. We consider currents both perpendicular and parallel to the ladder edges. In the perpendicular case, we find that both the critical currents, and the Shapiro steps[7] show striking effects of screening, just as does the ground state. The critical current retains its zero-field value up to a critical "vortex penetration" field $f_{c1\perp}<f_{c1}$. Similarly, there are only integer Shapiro steps up to a critical field $f^*_{c1\perp}$, above which smaller Shapiro steps begin to appear, corresponding to the penetration of individual vortices. Another regime near $f=\frac{1}{2}$ has half-integer Shapiro steps, corresponding to the coherent motion of an alternating lattice of vortices and vacancies in the ladder,[8] and possibly by a narrower screening plateau.

For a parallel applied current, there appears to be *no* plateau in the critical current at low values of f, and hence, no evidence of a nonzero $f_{c1\parallel}$ for current applied in this direction. Instead, the critical current seems to vary smoothly with f. The response of a ladder in this geometry to a combined dc and ac current has been previously considered.[9] The resulting *IV* characteristics show integer and half-integer steps whose width varies also smoothly with f.

We turn now to our method of calculation. We consider an $N\times2$ ladder of Josephson junctions. The $2N$ superconducting grains are located at $(n,0)$ and $(n,1)$, with $n=1,2,\ldots,N$. The junctions parallel to the x and y axes have critical currents I_{cx} and $I_{cy}\equiv\alpha I_{cx}$, where α is the anisotropy. We assume that each junction is an overdamped resistively shunted junction. Thus each junction, in parallel with the supercurrent, has a normal shunt current flowing through a resistance R, which is taken to be the same for x and y junctions. For current injection perpendicular to the ladder edges, we use boundary conditions such that current I is injected into each grain at $y=1$ and extracted from each grain at $y=0$, with periodic boundary conditions in the x direction. For parallel current injection, we introduce current I into each grain at $x=0$, and extract it from each grain at $x=N$, with free boundary conditions in the y direction.

In the presence of a transverse magnetic field $B=f\Phi_0/a^2$, the phase difference $\phi_{\mathbf{R}}-\phi_{\mathbf{R}'}\rightarrow\phi_{\mathbf{R}}-\phi'_{\mathbf{R}}-A_{\mathbf{RR}'}$, where $A_{\mathbf{RR}'}=(2\pi/\Phi_0)\int_{\mathbf{R}}^{\mathbf{R}'}\mathbf{A}\cdot d\mathbf{l}$ and $\mathbf{A}$ is the vector potential. We use the transverse gauge $\mathbf{A}=-Bx\hat{\mathbf{y}}$, so that $A_{\mathbf{RR}'}$ vanishes for junctions parallel to the x axis and equals $\mathcal{A}_x\equiv-2\pi fx$ for junctions in the y direction $(x=1,2,...,N)$.

The phases of the superconducting order parameters are denoted ϕ_{xy} and satisfy the equations of motion (for perpendicular current injection)

$$I = I_{cx}[\sin(\phi_{x1} - \phi_{x+1,1}) + \sin(\phi_{x1} - \phi_{x-1,1})] + I_{cy}\sin(\phi_{x1} - \phi_{x0} + 2\pi f x) + \frac{\hbar}{2eR}(3\dot{\phi}_{x1} - \dot{\phi}_{x+1,1} - \dot{\phi}_{x-1,1} - \dot{\phi}_{x0}); \tag{1}$$

and

$$-I = I_{cx}[\sin(\phi_{x0} - \phi_{x+1,0}) + \sin(\phi_{x0} - \phi_{x-1,0})] + I_{cy}\sin(\phi_{x0} - \phi_{x1} - 2\pi f x) + \frac{\hbar}{2eR}(3\dot{\phi}_{x0} - \dot{\phi}_{x+1,0} - \dot{\phi}_{x-1,0} - \dot{\phi}_{x1}) \tag{2}$$

These equations are appropriately modified for parallel injection, as described in Ref. 9. We solve these coupled equations numerically by a standard algorithm for a variety of ladders up to $N=40$, as discussed, for example in Ref. 10. Equations (1) and (2) constitute a set of $2N$ coupled first-order nonlinear ordinary differential equations. For each current of interest, we solve these iteratively in time, using a fourth-order Runge-Kutta procedure and time intervals of order $0.04t_0$, where $t_0 = \hbar/(2eRI_c)$ is the characteristic time of the problem.

In the perpendicular geometry, we first consider critical currents $I_{c\perp}(f)$ as a function of frustration, or applied field, f. At each f, we determine the critical current by ramping up the applied current, usually in units of $0.01I_{cx}$, starting from a very small current and a random initial phase configuration. As we increase the current, we generally take the final phase configuration for the previous current as the starting configuration of the new current. At each new *magnetic field*, however, we always began with a new random initial phase configuration. For each current we determined the time-averaged voltage $\langle V\rangle_t$ by averaging over a time interval at least $1000t_0$. The critical current is, of course, defined as the value of I at which a nonzero $\langle V\rangle_t$ is first detected.

Figure 1 shows our calculated $I_{c\perp}(f)$ for an isotropic ladder [$\alpha=1$; Fig. 1(a)] as well as for one anisotropic ladder [$\alpha=0.7$; Fig. 1(b)]. The most striking result for the isotropic ladder is that the critical current is *unchanged* up to a "vortex penetration field" $f_{c1\perp}$ of about 0.12. The inset to Fig. 1(b) shows parts of the *IV* characteristics at fields below and above f_{c1}. The value of 0.12 depends little on ladder size, as can be seen from the figure. We attribute the nonzero $f_{c1\perp}$ to the ability of the ladder to screen out vortices (as opposed to actual magnetic flux) by means of currents flowing in opposite directions along the edges of the ladder. The calculated $f_{c1\perp} < f_{c1}$ value f_{c1}, the value required for vortices to penetrate in the ground state.[6]

There is also a suggestion of a plateau near $f=\frac{1}{2}$. If one plots the *IV* characteristic near this field, one typically finds that $I_{c\perp}$ depends on the initial conditions (for each current, we reinitialize with a random initial phase configuration). For $f=\frac{1}{2}$, this procedure usually gives one of two critical currents, both of which are shown as dots in the figure. The lower such $I_{c\perp}$ is a smooth continuation of $I_{c\perp}(f)$ for f more distant from $\frac{1}{2}$. It probably represents the depinning of an additional vortex or vacancy inserted into the $f=\frac{1}{2}$ ground state. The larger $I_{c\perp}$ is the critical current for depinning the entire $f=\frac{1}{2}$ ground state. This larger I_c may occasionally persist for $f\neq\frac{1}{2}$, presumably because additional vortices are screened out for small $|f-\frac{1}{2}|$. Thus, it appears that there may be a plateau at $f=\frac{1}{2}$, but it is much more dependent on initial conditions than at $f=0$. This plateau would be the analog of the flat regions which occur in the ground state at $f=\frac{1}{2}$.[6]

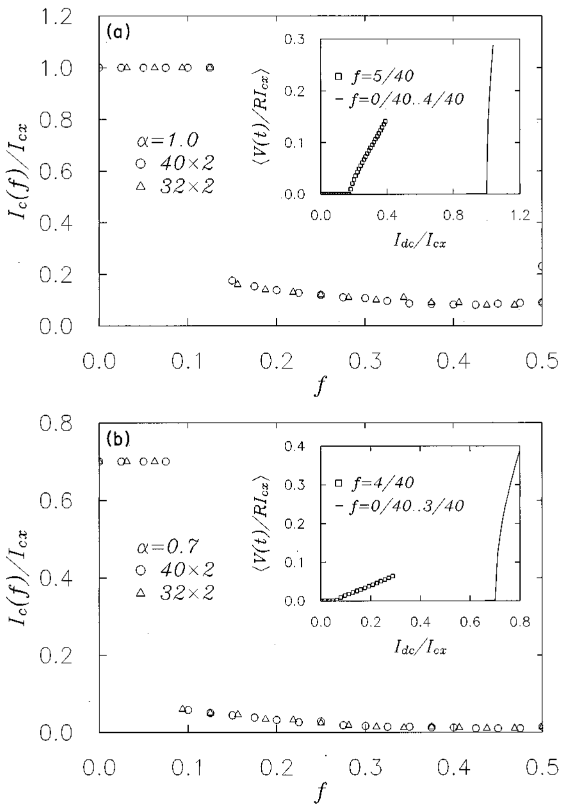

FIG. 1. Critical current $I_{c\perp}(f)$ as a function of frustration f in Josephson ladders of two different sizes, with current injected perpendicular to the edges of the ladder, (a) $\alpha \equiv I_{cy}/I_{cx} = 1.0$; (b) $\alpha=0.7$. Insets: *IV* characteristics for small f in a 40×2 array. The two points shown at $f=\frac{1}{2}$ represent results of different random initial conditions.

As is evident from Fig. 1, $f_{c1\perp}$ drops rapidly in the anisotropic ladders, i.e., with decreasing α. This is again consistent with the trend observed in Ref. 6, although critical current drops more sharply than does the ground state vortex penetration field, f_{c1}. Indeed, at $\alpha=0.3$, the smallest value we considered (not plotted), we found that $f_{c1}\approx 0$ to within our accuracy of 0.025. At f just above $f_{c1\perp}$, there is usually only a single vortex in the ladder. Thus, at these fields, $I_{c\perp}(f)$ can be considered as the *depinning current* for a single vortex, which we denote I_d. In the isotropic ladder, $I\approx 0.17I_{cx}$, somewhat larger than the value $I_d \approx 0.1I_{cx}$ found for a square array.[11] As the ladder becomes more anisotropic, however, I_d rapidly diminishes, as would be expected since such a junction represents a smaller energy barrier than in the isotropic case. We estimate $I_d \approx 0.06I_{cx}$ for $\alpha=0.7$. The depinning current for a vortex in the $f=\frac{1}{2}$ ground state is much smaller than at $f=0$.

A similar kind of vortex exclusion can also be seen in calculations of Shapiro steps in the same ladder geometry. We assume a combined dc and ac current, $I = I_{dc} + I_{ac}\sin(\omega t)$, in an isotropic ladder. The resulting *IV* characteristics are shown in Figs. 2 and 3 for a representative

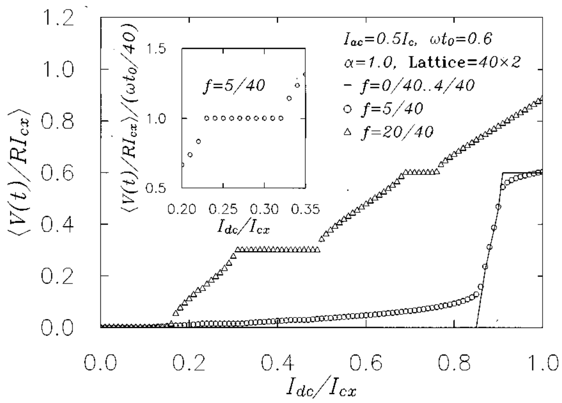

FIG. 2. *IV* characteristics for isotropic ladders in the presence of a current $I = I_{dc} + I_{ac}\sin(\omega t)$, with $I_{ac}/I_{dc} = 0.5$, $\omega t_0 = 0.6$, and several values of f as indicated, for a 40×2 ladder.

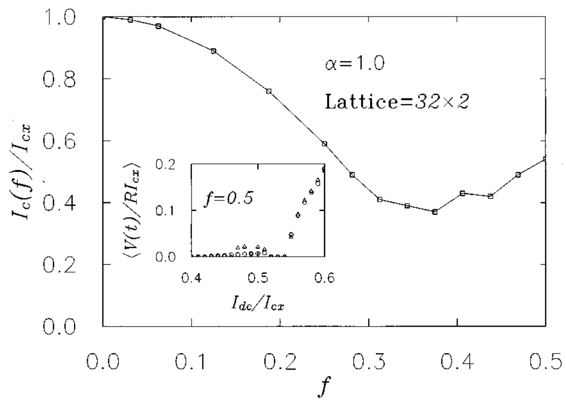

FIG. 4. Critical current $I_{c\parallel}(f)$ versus frustration in Josephson ladder with current injected parallel to the ladder edges, assuming $\alpha = 1$. Inset: *IV* characteristics at $f = \frac{1}{2}$, for two different sets of random initial conditions (denoted by different symbols).

choice of ac current amplitude and frequency, and at several values of f, in a 40×2 array. For $f < f^*_{c1\perp} \approx 0.12$, vortex exclusion is suggested by the occurrence of only integer Shapiro steps.[7] This indicates that each junction experiences the same voltage drop of $\langle V\rangle_t = n\hbar\omega/2e$, $n = 1,2,\ldots$. For example, in a 40×2 array, at $f = 5/40$, we find (weak) steps at multiples of a fundamental voltage $\langle V\rangle_t = \hbar\omega/(2e \times 40)$ (see inset to Fig. 2). By an extension of the argument of Ref. 8, these steps correspond to the motion of *one* vortex in the array, moving with a velocity of n plaquettes per ac cycle. As seen in Fig. 2, the voltage at $f = 5/40$ shows steps corresponding to the presence of *one* vortex in the ladder. Thus, the vortex density is not only zero below $f^*_{c1\perp}$; it remains substantially lower than f even above $f^*_{c1\perp}$.

At $f = \frac{1}{2}$, we generally obtain strong half-integer steps if we initialize from a random phase configuration at low currents and ramp up the current as described above. On the half-steps, there is a lattice of alternating vortices and vacancies, which moves an integer number of plaquettes during each cycle of the ac field. These half-steps can also be seen just off half-filling (e.g., at $f = 19/40$ and $f = 21/40$; see Fig. 3). However, at these values of f, the *IV* characteristics are highly sensitive to initial phase configurations. This can be seen in Fig. 3, where we show the voltages at $f = 19/40$; each point represents a different random initial phase configuration. Evidently, in this case one obtains a state with either 19 or 20 vortices (in a ladder of 40 plaquettes), depending on the initial phase configuration. In the 20-vortex case, there are clear half-integer steps, which are absent in the 19-vortex case. For other values of f, farther from $f = \frac{1}{2}$ or $f = 0$, there are usually no strong fractional steps other than those from the independent motion of individual vortices.

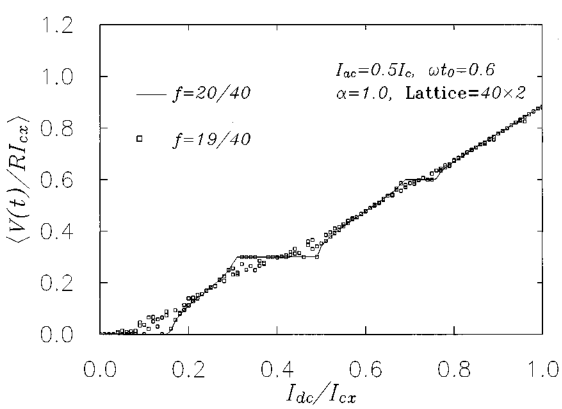

FIG. 3. *IV* characteristics for isotropic ladders in the presence of a current $I = I_{dc} + I_{ac}\sin(\omega t)$, with $I_{ac}/I_{dc} = 0.5$, $\omega t_0 = 0.6$, and $f = 19/40$. The points are calculated by reinitializing the phase in a random configuration at each new current; we show two random initial conditions for each current. Some such points correspond to a state containing 20 vortices (broad steps) while others involve states with fewer vortices (no broad steps). Result for $f = 20/40$ are shown for comparison.

The *IV* characteristics for the parallel geometry are shown in Fig. 4 in an isotropic 32×2 array ($\alpha = 1$). In contrast to the perpendicular case, there are *no* plateaus in the critical current. Instead, $I_{c\parallel}(f)$ seems to vary smoothly with f even for small f. As the *IV* characteristics are symmetric about $f = \frac{1}{2}$, we show the critical current only for $0 \leq f \leq \frac{1}{2}$. There is a strong maximum in $I_{c\parallel}(f)$ at $f = 0$ and a weak secondary maximum at $f = \frac{1}{2}$.[12]

Analogous calculations for combined dc and ac currents have been carried out previously by Yu *et al.*[9] There are both integer and half-integer Shapiro steps at $\langle V\rangle_t = Nn\hbar\omega/2e$, where n is integer or half-integer. The widths of both integer and half-integer steps vary continuously with f. The $n = 1$ step width exhibits the same primary and secondary maxima at $f = 0$ and $f = \frac{1}{2}$ as the critical current. But once again, there is no clear evidence that vortices are screened out at low f as they are in the perpendicular geometry.

To summarize, the present results provide further evidence that a Josephson ladder array can screen out *vortices* even though the *magnetic flux* is exactly given by the applied field. The screening is most conspicuous in the isotropic ladder, and, for this ladder, at small applied magnetic fields, although it is also evident near $f = \frac{1}{2}$. The screening is apparently produced by currents flowing in opposite directions along the edges of the ladder.

For the "parallel" geometry, in which current is injected parallel to the edges of the ladder, the screening appears to be absent. In particular, we find no evidence of plateaus in the critical current near $f=0$, as we find in the perpendicular geometry. Possibly, the screening is weaker in this geometry because the screening currents would have to act by flowing across a single rung of the ladder. This is not sufficient to allow vortices to be screened out.

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