Thermodynamical properties of an antiferromagnetic Heisenberg spin system on a fractal lattice of dimension between one and two

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(Received 10 October 1995)

The Suzuki-Takano quantum decimation technique is applied to the antiferromagnetic, nearest-neighbor, frustrated Heisenberg spin-$\frac{1}{2}$ system attached to a lattice with dimension d=ln3/ln2. Some thermodynamical functions are calculated. The temperature dependence of the specific heat is very similar to that obtained for the Heisenberg spin system on a *kagomé* lattice.

Considerable progress in understanding of the ground state and thermodynamical properties of Heisenberg antiferromagnets on different lattices has now been reached. It was stimulated mainly by the relation of this model with the phenomenon of high-T_c superconductivity. The purpose of this paper is to report an investigation of thermodynamical properties of the Heisenberg antiferromagnet

$$H=K\sum_{\langle i,j\rangle} \vec{S}_i\cdot\vec{S}_j, \tag{1}$$

on a Sierpiński fractal lattice. The sum in Eq. (1) runs over nearest-neighbor pairs and K is the coupling constant.

The Sierpiński fractal lattice is created in the following way. If one takes three equilateral triangles with sides a and attains mutual covering of three of their corners, one obtains a larger triangle of sides $2a$. Connecting three such larger triangles one gets a triangle of sides $4a$. Repeating such a process N times one obtains a fractal Sierpiński lattice of the order N and dimension $d=$ln3/ln2, see Fig. 1(a).

The present approach is based on the Suzuki-Takano[1,2] quantum decimation scheme applied lately to the analysis of the properties of a t-J model.[3] Because of the noncommutativity of the spin operators the quantum decimation cannot be carried out exactly (even in one dimension). The approximation,

$$\exp\left(\sum_i H_i\right)\approx\prod_i \exp(H_i)\approx\exp\left(\sum_i H_i'\right), \tag{2}$$

is made twice in opposite directions with (hopefully) mutual compensation. H and H' stand here for the Hamiltonian of the system before and after the decimation transformation, respectively.

To proceed, let us describe in more detail the decimation technique[1,2] applied to the antiferromagnetic Heisenberg spin system attached to a Sierpiński gasket lattice. The Hamiltonian H of the infinite system is split into *six-spin* parts H_i as seen in Fig. 1(b). The decimation relies on summing up over spins 2, 4, and 5. The renormalized coupling constant K' is determined from two equations:

$$\mathrm{Tr}_{123456}\exp(-KS)=\mathrm{Tr}_{136}\exp(-g-K'S'), \tag{3}$$

and

$$\mathrm{Tr}_{123456}[S'\exp(-KS)]=\mathrm{Tr}_{136}[S'\exp(-g-K'S')], \tag{4}$$

where

$$S=\vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{S}_3+\vec{S}_1\cdot\vec{S}_4+\vec{S}_2\cdot\vec{S}_4+\vec{S}_2\cdot\vec{S}_5+\vec{S}_3\cdot\vec{S}_5+\vec{S}_4\cdot\vec{S}_5+\vec{S}_5\cdot\vec{S}_6+\vec{S}_4\cdot\vec{S}_6, \tag{5}$$

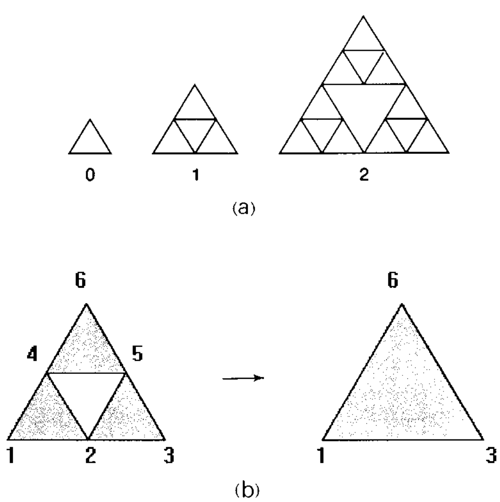

FIG. 1. (a) The first three steps in the construction of the Sierpiński gasket lattice having the fractal dimension ln3/ln2≈1.58. (b) The decimation transformation for the Sierpiński gasket lattice. The partial trace is taken over the Heisenberg spins labeled 2, 4, 5.

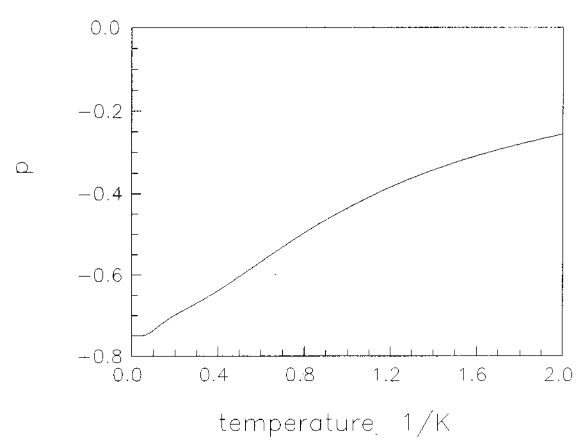

FIG. 2. The quantity p, defined in Eq. (8), versus temperature.

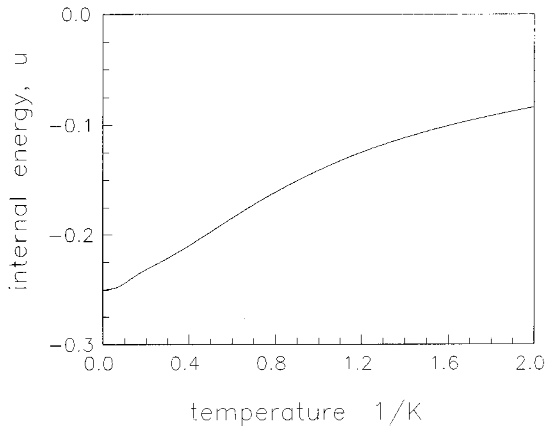

FIG. 3. The internal energy (per bond) of the Heisenberg spin system on the Sierpiński gasket lattice versus temperature.

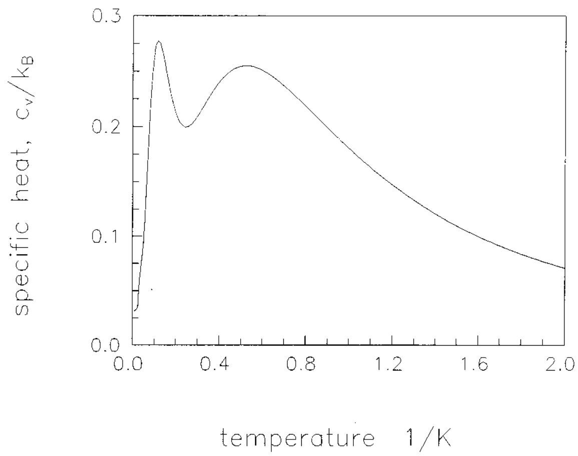

FIG. 4. The specific heat (per spin) of the Heisenberg spin system on the Sierpiński gasket lattice versus temperature.

and

$$S' = \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_6 + \vec{S}_6 \cdot \vec{S}_1 . \tag{6}$$

Accordingly, the renormalized coupling constant K' is given by

$$K' = \frac{2}{3} \ln \frac{3+4p}{3-4p}. \tag{7}$$

The quantity p is obtained by direct diagonalization of the relevant matrices, defined in Eqs. (5) and (6), for the system consisting of 6 spins:

$$p = \frac{\mathrm{Tr}_{123456}[S' \exp(-KS)]}{\mathrm{Tr}_{123456} \exp(-KS)}. \tag{8}$$

The dependence of p on temperature for the antiferromagnetic couplings present in the considered model is plotted in Fig. 2. Restricting the attention to this case one finds two fixed points of Eq. (7), namely the zero-temperature fixed point ($K=-\infty$, $p=-\frac{3}{4}$) and the paramagnetic fixed point ($K=0$, $p=0$). The former is nonstable, the latter — stable. It means (as one could expect) that the system under consideration does not undergo any phase transition at nonzero temperatures.

The free energy per spin is related to the renormalization of the constant term g in Eqs. (3) and (4). One has

$$-f/k_B T = \sum_i \frac{1}{3^i} g(K^{(i)}) \tag{9}$$

with $K^{(i)}$ representing the i-times transformed coupling constant K and k_B being the Boltzmann constant. In the ground state $f/k_B T \propto -K$ whereas in the high-temperature limit $f/k_B T = \ln 2$, as it should be. The internal energy u (per bond) is plotted in Fig. 3. Notice that in the ground state $u=-0.25$ which should be compared with the value of $u \approx -0.21$ obtained[4] within the variational approach.

In the dependence of the specific heat on temperature (Fig. 4) one can notice a further structure present in the low-temperature region. A very similar feature was observed in the antiferromagnetic Heisenberg spin system attached to the *kagomé* lattice.[5–8] It is worth notice that both those lattices have the same coordination number $z=4$ and the same basic unit (i.e., the smallest triangle) responsible for the frustration of the system. Finaly both can be divided into three interpenetrating, equivalent sublattices (A,B,C) to have a site (A) surrounded by two neighbors of remaining sublattices (two B and two C). This quite similar relation suggests that quantum $S=\frac{1}{2}$ *Sierpiński* antiferromagnet may also have exponentially decaying spin-correlation functions in its ground state and a gap for spin triplet excitations.

The author would like to thank Professor Masuo Suzuki for critical reading of the manuscript and his comments. Thanks are due also for Professor Ryszard Ferchmin for discussions. Support from the Polish Committee for Scientific Research under Project No. 2 2439 92 03 is acknowledged.

[1] M. Suzuki and H. Takano, Phys. Lett. **69A**, 426 (1979).
[2] H. Takano and M. Suzuki, J. Stat. Phys. **26**, 426 (1979).
[3] A. Falicov and A. N. Berker, Phys. Rev. B **51**, 12 458 (1995).
[4] P. Tomczak, A.R. Ferchmin, and J. Richter, in Proceedings of the Hayashibara Forum '95, Coherent Approaches to Fluctuations, edited by M. Suzuki and N. Kawashima (World Scientific, Singapore, in press).
[5] V. Elstner, Phys. Rev. Lett. **62**, 2405 (1989).
[6] N. Elstern and A. P. Young, Phys. Rev. Lett. **50**, 6871 (1995).
[7] Chen Zeng and V. Elser, Phys. Rev. B **51**, 8318 (1995).
[8] T. Nakamura and S. Miyashita, Phys. Rev. B **52**, 9174 (1995).