Soliton in an inhomogeneous weak ferromagnet with the Dzialoshinski-Moriya interaction

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We have identified an integrable model of the inhomogeneous radially symmetric weak Heisenberg ferromagnet in arbitrary n dimensions with Dzialoshinski-Moriya antisymmetric spin coupling. The elementary spin excitations of the magnet are found to be governed by soliton modes under small-angle oscillation of the antisymmetric spin coupling.

In ferromagnets the exchange interaction (spin-spin coupling), the single-ion anisotropy due to the crystal-field effect, and Zeeman energies have been treated as common but simple magnetic couplings. When the symmetry around the magnetic ions is not high enough an unfamiliar but important antisymmetrical coupling results due to the combined effect of spin-orbit coupling and the exchange interaction leading to the mechanism of weak ferromagnetism. Though the microscopic theory on the above mechanism was proposed by Dzialoshinski[1] and developed by Moriya[2] more than three decades ago, not much is known about the macroscopic properties of these ferromagnets including the elementary spin excitations. However, very recently there has been a lot of interest in the studies of the weak ferromagnets with Dzialoshinski-Moriya (DM) interaction because of its important role in insulators, spin glasses, and the low-temperature phase of copper oxide superconductors and also in the phase transition studies,[3] the quantum aspects,[4] and the statistical mechanics.[5] In a different platform, in recent years integrable nonlinear dynamical models of Heisenberg ferromagnets exhibiting an interesting class of localized nonlinear elementary spin excitations such as solitons, domain walls, etc.,[6–11] have been identified. In this context soliton spin excitations have also been identified very recently in one-dimensional weak Heisenberg ferromagnetic spin chains with DM interaction.[12] Motivated by this in the present paper, we investigate the nature of nonlinear spin excitations in an inhomogeneous radially symmetric Heisenberg ferromagnet with DM interaction in arbitrary n dimensions in the classical continuum limit and identify soliton modes.

In order to understand the underlying nonlinear dynamics of the above weak ferromagnetic systems, we formulate the dynamical equations starting from the Heisenberg model of the Hamiltonian expressed in terms of the ionic spin operators,

$$H=-\sum_j \sum_\delta f_j[J(\mathbf{S}_j\cdot\mathbf{S}_{j+\delta})+\mathbf{D}\cdot(\mathbf{S}_j\times\mathbf{S}_{j+\delta})]. \quad (1)$$

In Eq. (1), the first term represents the exchange interaction between adjacent spins with J, the exchange integral and the second term corresponds to the DM interaction due to antisymmetrical spin coupling and the vector δ represents all possible nearest neighbors. This antisymmetrical coupling acts to cant the spins because the coupling energy is minimized when the two spins are perpendicular to each other. The direction of the constant vector $\mathbf{D}$ can be related to the symmetry of the ferromagnetic crystal and when a center of inversion is located at the point half way between the spins coupled,[2] then $D=0$. We choose the vector $\mathbf{D}$ parallel to $\mathbf{m}=(1,1,1)$ (i.e., $\mathbf{D}=D\mathbf{m}$). In Hamiltonian (1) the site-dependent function f_j introduces inhomogeneity in the ferromagnetic lattice. In the case of ferromagnets when the spin angular momentum value is large, $\mathbf{S}_j$ represents the classical three-dimensional vector (S_j^x,S_j^y,S_j^z) in spin space at the site j of an n-dimensional lattice and $\mathbf{S}_{j+\delta}$ are its nearest neighbors. In the low-temperature long-wavelength limit the spin vector $\mathbf{S}_j$ and the site-dependent function f_j vary very slowly over the lattice distance a and we go to the continuum limit by introducing Taylor expansions for $\mathbf{S}_{j\pm\delta}$ and for $f_{j\pm\delta}$ up to $O(a^2)$ (a: lattice parameter) and replace $\mathbf{S}_j$ and f_j respectively by the continuous vector function $\mathbf{S}(\mathbf{r},t)$ and by the continuous scalar function $f(\mathbf{r},t)$ where $r=(r_1,r_2,\ldots,\ r_n)$. Also in order to reduce the mathematical complexity in the case of DM interaction we consider only those contributions during spin evolution that lies only within a small angle (θ) cone whose axis lies parallel to $\mathbf{m}$ ($\mathbf{m}\cdot\mathbf{S}\sim1$) (see Fig. 1). In view of the above, the equation of motion representing the spin dynamics [after suitable rescaling of time and redefinition of the parameter D ($d=D/Ja$)] in the radially symmetric case in arbitrary n dimensions can be written as

$$\frac{\partial\mathbf{S}}{\partial t}(\mathbf{r},t)=f(r)\mathbf{S}\times\left(\frac{\partial^2\mathbf{S}}{\partial r^2}+\frac{(n-1)}{r}\frac{\partial\mathbf{S}}{\partial r}\right)+\frac{\partial f}{\partial r}\left(\mathbf{S}\times\frac{\partial\mathbf{S}}{\partial r}\right)-d\left[2f-a\frac{\partial f}{\partial r}\right]\frac{\partial\mathbf{S}}{\partial r}, \quad (2)$$

$$\mathbf{S}^2(r,t)=1,\quad r^2=r_1^2+r_2^2+\ldots+r_n^2,\quad 0\leqslant r\leqslant\infty.$$

Equation (2) contains several integrable spin models exhibiting soliton excitations for different n and f in the absence of DM interaction ($d=0$) (see, e.g., Refs. 6, 10, 11). Now the natural question arises to see whether the weak ferromag-

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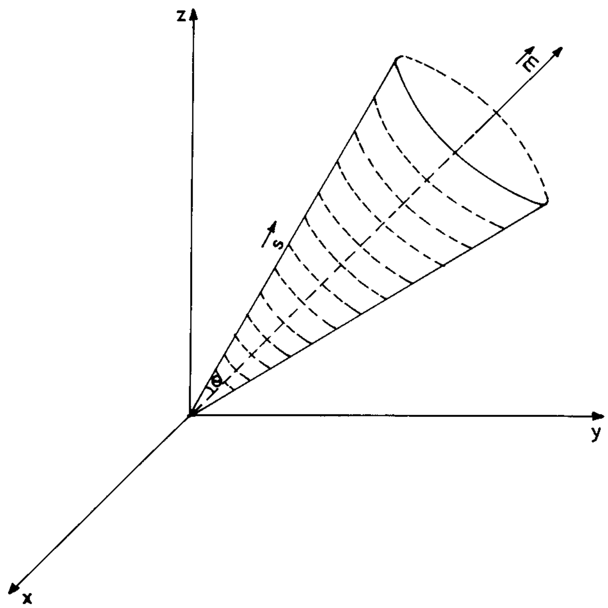

FIG. 1. Small-angle cone (θ) representing allowed region of spin oscillations due to DM interaction.

netic system (2) when $d\neq 0$ can also admit soliton spin excitations for arbitrary n and f and if not for specific choices of n and f. To find an answer to this question, we first carry out the Painlevé singularity structure analysis[13,14] on Eq. (2).

However, as Eq. (2) in its present form is not convenient for the Painlevé analysis, we map[15] the inhomogeneous weak ferromagnetic system onto a moving helical space curve in E^3 by identifying the spin vector with the tangent vector $\mathbf{e}_1$ of the space curve with curvature $\kappa(r,t)=(\mathbf{e}_{1r}\cdot\mathbf{e}_{1r})^{1/2}$ and torsion $\tau(r,t)=\kappa^{-2}\mathbf{e}_1\cdot(\mathbf{e}_{1r}\times\mathbf{e}_{1rr})$ and defining two unit normal vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ in the usual way. The change in orientation of the orthogonal trihedral $\mathbf{e}_i$, $i=1,2,3$, which defines the space curve uniquely within rigid motion is determined by the Serret-Frenet equations $\mathbf{e}_{ir}=\mathbf{d}\times\mathbf{e}_i$, $i=1,2,3$, where $\mathbf{d}=\tau\mathbf{e}_1+\kappa\mathbf{e}_3$ (suffix r: partial derivative). Now using Eq. (2) (after replacing $\mathbf{S}$ by $\mathbf{e}_1$) and the Serret-Frenet equations the time evolution of the orthogonal trihedral $\mathbf{e}_i$ can be written as $\mathbf{e}_{it}=\mathbf{e}_i\times\mathbf{\Omega}$, $i=1,2,3$ (suffix t: partial derivative), where $\mathbf{\Omega}=\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3$ and

$$\omega_1=\frac{1}{\kappa}\left[\left\{(f\kappa)_r+\frac{(n-1)}{r}f\kappa\right\}_r-f\kappa\tau^2-d(2f-af_r)\kappa\tau\right], \tag{3a}$$

$$\omega_2=-\left[(f\kappa)_r+\frac{(n-1)}{r}f\kappa\right], \tag{3b}$$

$$\omega_3=[f\kappa\tau+d(2f-af_r)\kappa]. \tag{3c}$$

The compatibility $(\mathbf{e}_{ir})_t=(\mathbf{e}_{it})_r$, $i=1,2,3$, leads to a set of coupled nonlinear evolution equations for the curvature and torsion of the space curve given by

$$\kappa_t=-\left\{(f\kappa\tau)_r+\tau(f\kappa)_r+\frac{(n-1)}{r}f\kappa\tau+d[\kappa(2f-af_r)]_r\right\}, \tag{4a}$$

$$\tau_t=\left[\frac{1}{\kappa}\left\{\left([f\kappa]_r+\left[\frac{(n-1)}{r}f\kappa\right]\right)_r-f\kappa\tau^2-d(2f-af_r)\kappa\tau\right\}\right]_r +\kappa\left[(f\kappa)_r+\frac{(n-1)}{r}f\kappa\right]. \tag{4b}$$

Upon suitable identification of the curvature and torsion of the space curve with the energy and current densities of the ferromagnetic system Eq. (4) can be equivalently written as a set of coupled nonlinear evolution equations for the energy and current densities of the ferromagnet. By making the complex transformation $q=(\kappa/2)\exp\{i\int_0^r\tau(r',t)dr'\}$ we rewrite Eqs. (4) as the following generalized inhomogeneous nonlinear Schrödinger equation:

$$iq_t+2f|q|^2q+\left[(fq)_r+\left\{\frac{(n-1)}{r}fq\right\}+id(2f-af_r)q\right]_r +2\left[\int\left\{f_r|q|^2+2(n-1)\frac{f}{r}|q|^2\right\}dr\right]=0. \tag{5}$$

Thus Eq. (5) equivalently represents the spin dynamics of an inhomogeneous radially symmetric (in arbitrary n dimensions) weak ferromagnet and contains few integrable models with soliton solutions corresponding to the same values of n, f when $d=0$ as for Eq. (2).

Now in order to carry out the singularity structure analysis we rewrite Eq. (5) (within an arbitrary function of t which can be removed by a simple time-dependent gauge transformation) as

$$iq_t+2Wq+\left[(fq)_r+\left\{\frac{(n-1)}{r}fq\right\}+id(2f-af_r)q\right]_r=0, \tag{6a}$$

$$W_r-f(|q|^2)_r-2\left\{f_r+\frac{(n-1)}{r}f\right\}|q|^2=0. \tag{6b}$$

Further, it is required to rewrite Eqs. (6) and the complex conjugate of (6a) by replacing $q(r,t)$ by $P(r,t)$ and $q^*(r,t)$ by $Q(r,t)$. For isolating those cases for which the system of Eqs. (6) are free from movable critical manifolds so that the general solution will be single valued around the noncharacteristic movable singular manifold $\phi(r,t)=0$, we express the functions P, Q, and W locally in the form of the Laurent series with at least one of the leading function coefficients different from zero. An analysis on the leading order behavior of the Laurent series solutions in Eqs. (6) (after rewriting it in terms of P and Q) shows that $P\sim P_0\phi^{-1}$, $Q\sim Q_0\phi^{-1}$, $W\sim W_0\phi^{-2}$ with $P_0Q_0=-\phi_r^2$ and $W_0=-f\phi_r^2$ and the resonances, namely the powers at which the arbitrary functions can enter into the Laurent series, are found to be $-1,0,2,3,4$. The resonances -1 and 0 correspond to the arbitrariness of the singular manifold and the coefficient P_0 or Q_0, respectively. For proving the existence of a sufficient number of arbitrary functions without the introduction of movable critical manifolds at the resonance values 2, 3, and 4, we substitute the full Laurent series solutions in Eqs. (6) and collect the coefficients of different powers of ϕ. A detailed analysis of the resultant equations shows that there exist arbitrary functions without the introduction of movable critical manifolds at resonance values 2, 3, and 4 when the inhomogeneous function $f(r)$ assumes the specific form

$$f(r)=\alpha r^{-(n-2)}+\beta r^{-2(n-1)}, \tag{7}$$

where α and β are arbitrary constants. Thus we conclude that the inhomogeneous radially symmetric Heisenberg weak ferromagnet in arbitrary n dimensions is expected to be an integrable nonlinear dynamical model and hence the elementary spin excitations may be expressed in terms of solitons when the inhomogeneity is of the form (7).

Having identified the integrable model in Eq. (5) and hence in Eq. (2), we now try to obtain the Lax pair of operators or the associated linear eigenvalue problem for the purpose of constructing soliton solutions. This can be achieved by using the Laurent series solutions of (6) truncated at the constant level term.[14] Thus we substitute the truncated Laurent series solutions $P=P_0\phi^{-1}+P_1$, $Q=Q_0\phi^{-1}+Q_1$, and $W=W_0\phi^{-2}+W_1\phi^{-1}+W_2$ in Eqs. (6) and collect the coefficients of ϕ^{-2} and ϕ^{-1} to obtain an overdetermined system of equations which upon identification of $P_0=i\psi_1^2$, $Q_0=i\psi_2^2$, $P_1=q$, $Q_1=q^*$, and $\phi_r=-i\psi_1\psi_2$, give the linear eigenvalue problem $\psi_r=U\psi$, $\psi_t=V\psi$, with $\psi=(\psi_1\psi_2)^T$ and the Lax pair of operators U and V given by

$$U=\begin{pmatrix} i\lambda r^{n-1} & q \\ -q^* & -i\lambda r^{n-1}\end{pmatrix}, \tag{8a}$$

$$V=\begin{pmatrix} A & B \\ -B^* & -A\end{pmatrix}, \tag{8b}$$

where

$$A=iW-2i\lambda^2 fr^{2(n-1)}, \tag{8c}$$

$$B=-2\lambda fqr^{n-1}+i\left\{(fq)_r+\frac{(n-1)}{r}fq\right\}d(2f-af_r)q. \tag{8d}$$

Here λ is the spectral parameter obtained as a constant of integration and f takes the specific form as given in Eq. (7). The consistency of the linear eigenvalue problem leads to the evolution equation (6), provided the spectral parameter evolves as $\lambda(t)\equiv\rho(t)+i\eta(t)=(2n\alpha t+\gamma)^{-1}$, where γ is the free parameter.

The explicit form of the soliton solution to Eq. (5) can now be obtained by constructing the Bäcklund transformation[11,16] from the linear eigenvalue problem. We obtain the Bäcklund transformation connecting the N and $N-1$ solitons as

$$(1+|\chi|^2)(q+q')=4\eta r^{n-1}\chi, \tag{9}$$

where $\chi=(\psi_1/\psi_2)$ satisfies the Riccati equations, $\chi_r=2ir^{n-1}\lambda\chi+q+q^*\chi^2$ and $\chi_t=B+2A\chi+B^*\chi^2$ with A and B as given in Eqs. 8(c) and 8(d). The one soliton solution $q(1)$ can be constructed using the Bäcklund transformation starting from the zero soliton solution $q(0)\equiv 0$ and with the knowledge of $\chi(0)=\psi_1(0)/\psi_2(0)$ which can be obtained by solving the linear eigenvalue problem for $q(0)\equiv 0$. Thus we obtain the one soliton solution to Eq. (5) as

$$q=2\eta(t)r^{n-1}\mathrm{sech}\left[2\eta(t)\left\{\frac{r^n}{n}-4\beta\rho(t)t-\delta_1\right\}\right]$$
$$\times\exp\left[2i\left\{\frac{\rho(t)r^n}{n}-2\beta\int_0^t\rho^2(t')-\eta^2(t')dt'+\delta_2\right\}\right], \tag{10}$$

with

$$\lambda(t)\equiv\rho(t)+i\eta(t)$$
$$=\frac{\{\rho(0)+i\eta(0)+2n\alpha[\rho^2(0)t+\eta^2(0)t]\}}{\{[1+2n\alpha t\rho(0)]^2+4n^2\alpha^2\eta^2(0)t^2\}}$$

and δ_1,δ_2 phase constants. Knowing q and, therefore, the curvature κ, torsion τ and the tangent vector $\mathbf{e}_1$ of the space curve, the spin-vector $\mathbf{S}$ can be constructed.[17,18] Thus, we obtain the elementary spin excitations of the

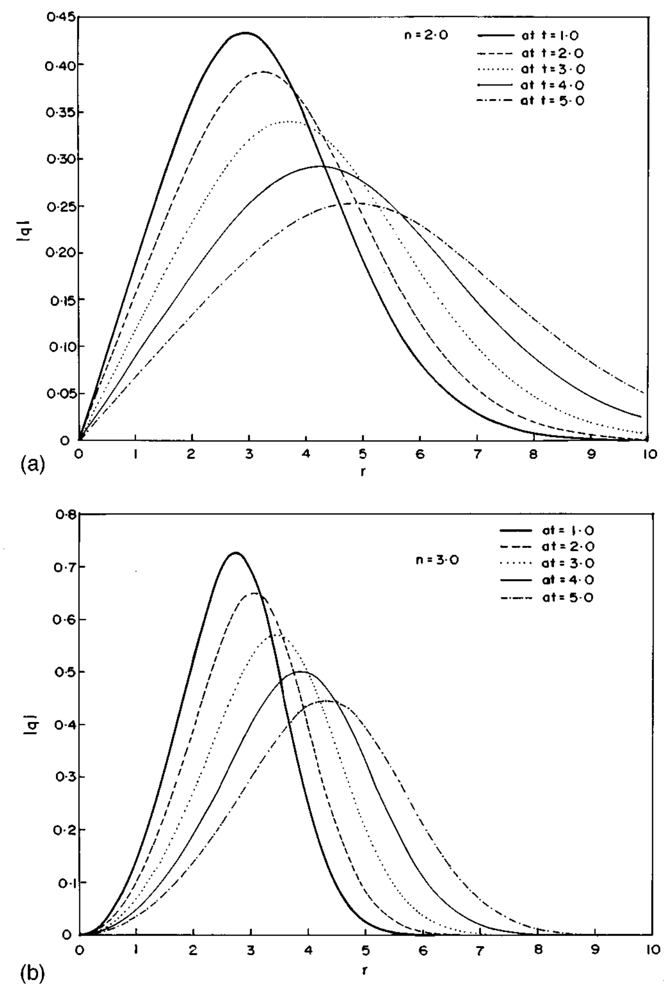

FIG. 2. (a) Evolution of soliton ($|q|$) in the circularly symmetric case. (b) Evolution of soliton ($|q|$) in the spherically symmetric case.

n-dimensional radially symmetric inhomogeneous weak ferromagnet in the form of one soliton solution corresponding to (10) as

$$S^{\pm}\equiv S^x \pm iS^y$$
$$=\frac{2\eta}{\rho^2+\eta^2}\left\{\rho \mp i\eta \tanh\left[2\left(\eta\frac{r^n}{n}-4\beta\int_0^t \eta\rho dt'\right)-\delta_1\right]\right\}$$
$$\times \operatorname{sech}\left[2\eta\left\{\frac{r^n}{n}-4\beta\int_0^t \rho dt'-\delta_1\right\}\right]$$
$$\times \exp\left[\pm 2i\left\{\rho\frac{r^n}{n}-2\beta\int_0^t(\rho^2-\eta^2)dt'+\delta_2\right\}\right], \tag{11a}$$

$$S^z=1-\left(\frac{2\eta^2}{\rho^2+\eta^2}\right)\operatorname{sech}^2\left[2\eta\left\{\frac{r^n}{n}-4\beta\int_0^t \rho dt'-\delta_1\right\}\right]. \tag{11b}$$

It may be noted that the structure of the one soliton solutions are similar to the case when $d=0$. However, by inspection we notice that the structure of multisolitons will differ. In Fig. 2 we have plotted the evolution of the one soliton ($|q|$) in the physically important circularly ($n=2$) and spherically ($n=3$) symmetric cases for the following specific values of the parameters: $\rho(0)=0.2$, $\eta(0)=0.2$, $\delta_1=0.3$, $\alpha=0.5$, $\beta=0.1$. On comparing the evolution of solitons in Figs. 2(a) and 2(b), we notice that as the dimension of the ferromagnet at a given time increases the soliton becomes sharp and more localized with increase of amplitude and decrease of width. Further, it is observed that the soliton in both the circularly and spherically symmetric cases spreads as time passes on, which is due to the form of the amplitude function $\eta(t)$ and which is common in soliton systems with nonisospectral flows.[11] Though energy is not a constant here (since $\eta(t)$ is not a constant) there exists an infinite number of constants of motion each of which are summed over the infinite number of constants of motion of the standard cubic nonlinear Schrödinger equation multiplied by time-dependent coefficients (for details see Refs. 11, 17, and 21 of Ref. 11 here). Finally, the complete integrability nature of the system ensures the stability of solitons under collisions.

In conclusion, the radially symmetric inhomogeneous weak Heisenberg ferromagnet with DM interaction in arbitrary n dimensions in the classical continuum limits is found to be an integrable nonlinear dynamical model under small-angle spin oscillation due to antisymmetrical spin coupling and when the inhomogeneity is of the form $f(r)=\alpha r^{-(n-2)}+\beta r^{-2(n-1)}$. The elementary spin excitations of this integrable ferromagnetic system is found to be governed by solitons.

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