Landauer resistivity dipole in a strong magnetic field

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We consider a localized single impurity in a two-dimensional electron gas with a perpendicular magnetic field at filling factor 1. A scattering theory calculation is performed for the local electron density around the impurity in the presence of a finite transport current. It is shown that the current-induced Landauer resistivity dipole is rotated by 90° compared with the situation in the absence of a magnetic field. This behavior naturally reflects the structure of the resistivity tensor, with a finite value of ρ_{xy} but vanishing longitudinal resistivity ρ_l due to the absence of backscattering. We briefly discuss the implications of our results on the potential distribution in the quantum Hall effect, and for scanning tunneling microscopy measurements of the local density of states.

Quantum transport theory is usually based on Kubo's formulation of linear response, where one calculates the current resulting from an applied electric field. A quite different approach to determine transport coefficients, however, was suggested at about the same time by Landauer.[1] In Landauer's approach one starts from a given incident *current* and then calculates the resulting electric field in response to that. It turns out that this field is highly nonlocal, being caused by the nonuniform electron density associated with the scattering at localized impurities. Indeed, with carriers impinging on one side of the barrier, one expects the local density to be enhanced before and depleted after the barrier, thus leading to a dipolar density and potential distribution.[2] A pure scattering theoretic calculation of this prediction was performed recently.[3] It was shown that the asymptotic density at finite transport current $\vec{j}=-ne\vec{v}$ indeed has a dipolar distribution to linear order in $\vec{v}$ which is superimposed, however, by additional Friedel oscillations. Specifically, for a two-dimensional electron gas at zero temperature one finds that asymptotically[3]

$$n(\vec{x})=n(\vec{x})|_{v=0}-e\frac{\partial n}{\partial\mu}\frac{\vec{p}(r)\cdot\hat{x}}{r}, \tag{1}$$

where $\hat{x}$ is a unit vector in the direction of $\vec{x}$, and

$$\vec{p}(r)=\frac{\hbar k_F}{2\pi e}\left[\sigma_{\rm tr}+\frac{(8\pi)^{1/2}}{k_F}\,\mathrm{Re}(f_{k_F}(\pi)e^{2ik_Fr})\right]\vec{v} \tag{2}$$

the current-induced dipole moment. Here $\sigma_{\rm tr}$ is the usual transport cross section of the impurity and $f_{k_F}(\pi)$ the associated backscattering amplitude. Obviously for $f_{k_F}(\pi)\neq 0$ the effective dipole moment contains a Friedel-like spatially oscillating contribution in addition to the constant term proportional to $\sigma_{\rm tr}\cdot\vec{v}$ which represents the Landauer resistivity dipole. In the context of phase-sensitive voltage fluctuations in quasi-one-dimensional conductors, these oscillations had previously been discussed by Büttiker.[4] In the presence of a finite concentration n_i of *independent*, identical impurities, the spatially *averaged* electric field is insensitive to the Friedel oscillations and is given by

$$\langle\vec{E}\rangle=-2\pi n_i\langle\vec{p}\rangle=\rho(-ne\vec{v}) \tag{3}$$

(the factor 2π is specific for a two-dimensional system). Now, requiring that the average field $\langle\vec{E}\rangle$ obeys Ohm's law, Eq. (3) immediately leads to the standard Boltzmann-Drude result $\rho=m/ne^2\tau_{\rm tr}$ for the (longitudinal) residual resistivity, with $\tau_{\rm tr}^{-1}=n_iv_F\sigma_{\rm tr}$ as the corresponding scattering rate. This derivation explicitly shows that the standard Boltzmann-Drude theory of transport is equivalent to simply adding up the polarization field associated with each impurity to give a homogeneous average field $\langle\vec{E}\rangle$. Going beyond this rather crude approximation is an interesting but difficult problem.[5] In the present work this is avoided by considering a *single* impurity only, extending our earlier calculation[3] to the case of a finite magnetic field. This problem is of interest for several reasons. First of all it is important from a conceptual point of view, in particular in the context of local fluctuations of the potential distribution in the quantum Hall effect. In addition to that, recent measurements with a scanning tunneling microscope (STM) have been able to map out the local electronic density of states around impurities in a two-dimensional electron gas formed at the surface of copper.[6] We will comment on both problems in our discussion.

We consider noninteracting electrons in two dimensions which are subject to a perpendicular magnetic field $\vec{B}=B\cdot\vec{e}_z$. In the Landau gauge $\vec{A}=Bx\cdot\vec{e}_y$ the associated one-particle Hamiltonian is

$$H_0=\frac{p_x^2}{2m}+\frac{1}{2m}(p_y+eBx)^2. \tag{4}$$

Its eigenstates $|kn\rangle$ are products of shifted harmonic oscillator states $u_n(x)$ in the x direction, and plane waves in the y direction,

$$\langle\vec{x}|kn\rangle = e^{iky}u_n(x+kl_0^2), \tag{5}$$

with $l_0=(\hbar/eB)^{1/2}$ the magnetic length. Their energies are the infinitely degenerate discrete Landau levels $\varepsilon_n=\hbar\omega_c(n+\frac{1}{2})$, $n=0,1,2,\ldots$, with $\omega_c=eB/m$ the cyclotron frequency. The normalization of the states $|kn\rangle$ is chosen such that

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}|\langle\vec{x}|kn\rangle|^2=\frac{1}{2\pi l_0^2} \tag{6}$$

is the spatially constant density associated with each filled Landau level n. It is well known that the states $|kn\rangle$ do not carry a finite current, and in fact the discrete nature of the unperturbed spectrum prevents one from developing a straightforward scattering theory. However, for a given nonvanishing drift velocity $-\vec{v}$ of the electrons with respect to the fixed impurities, there is a finite Hall field $\vec{E}_H=\vec{v}\times\vec{B}$ which the electrons experience in the frame where the scattering potential is stationary. Since $\vec{A}$ is in the y direction for the Landau gauge, it is convenient to choose $\vec{v}=v\cdot\vec{e}_y$. Thus with $v>0$ the electrons are moving in the *negative-y* direction on average. The effective Hamiltonian in the absence of scattering is then

$$\bar{H}_0=H_0+eBvx. \tag{7}$$

It should be noted that the Hall field $\vec{E}_H$ is different from the actual electric field even in an ideal sample without scattering. Indeed, since edge charges generate a logarithmic potential in two dimensions, the field is concentrated near the edges while it is much smaller in the bulk of the sample[7] (see below). The true local field with impurities is then obtained from this nonuniform distribution by adding the effect induced by the density variations in the electron gas due to scattering, which is the problem addressed in the following.

The eigenstates $|\overline{kn}\rangle$ of $\bar{H}_0$ differ from $|kn\rangle$ only by a phase factor

$$\langle\vec{x}|\overline{kn}\rangle=e^{-imvy/\hbar}\cdot\langle\vec{x}|kn\rangle; \tag{8}$$

however, the spectrum is now no longer discrete but is continuous:

$$\varepsilon_n(k)=\varepsilon_n-\hbar kv+mv^2/2. \tag{9}$$

Physically the dependence of the energy on the momentum $k=-\langle x\rangle/l_0^2$ is a result of the tilting of the Landau levels due to the Hall field. We now introduce a static short-range scattering potential $V(\vec{x})$ such that

$$H=\bar{H}_0+V. \tag{10}$$

The eigenstates $|\overline{kn}+\rangle$ of the full Hamiltonian H may then be expressed in terms of the states $|\overline{kn}\rangle$ by the formal solution of the Lippmann-Schwinger equation,

$$|\overline{kn}+\rangle=\{1-\bar{G}_0[\varepsilon_n(k)+i0]V\}^{-1}|\overline{kn}\rangle. \tag{11}$$

Here $\bar{G}_0(z)=(z-\bar{H}_0)^{-1}$ is the resolvent of the ideal system including both the magnetic and Hall fields. Apart from the scattering states $|\overline{kn}+\rangle$ there are also localized bound states $|\psi_b\rangle$, which are present even for a purely repulsive potential.[8] A simple model where the scattering problem may be solved analytically to a large extent arises by choosing a separable potential[9]

$$V=V_0|\bar{\varphi}_0\rangle\langle\bar{\varphi}_0| \tag{12}$$

with $V_0>0$. The wave function $\langle\vec{x}|\bar{\varphi}_0\rangle$ differs from $\langle\vec{x}|\varphi_0\rangle$ by the same phase factor as in (8). For $\langle\vec{x}|\varphi_0\rangle$ we choose a Gaussian localized state whose overlap with $|kn\rangle$ is[9]

$$\langle kn|\varphi_0\rangle=(2\pi)^{-1/4}\left(\frac{l_0}{\hbar n!}\right)^{1/2}\left(\frac{kl_0}{\sqrt{2}}\right)^n\exp-(kl_0)^2/2. \tag{13}$$

For this model there is precisely *one* bound state between successive Landau levels (for $V_0<0$ there would be an additional one below the lowest Landau level). The bound-state energies E_b follow from[8]

$$V_0\langle\bar{\varphi}_0|\bar{G}_0(E_b)|\bar{\varphi}_0\rangle=1. \tag{14}$$

The associated state $|\psi_b\rangle$ is localized in space, and may be determined from

$$|\psi_b\rangle=\text{const}\bar{G}_0(E_b)|\bar{\varphi}_0\rangle \tag{15}$$

up to normalization.

With the solution of the scattering problem, it is now straightforward to evaluate the expectation value of any one particle operator. In particular the local density $n(\vec{x})$ is given by

$$n(\vec{x})=\sum_{n=0}^{\infty}f(\varepsilon_n)\int_{-\infty}^{\infty}\frac{dk}{2\pi}|\langle\vec{x}|\overline{kn}+\rangle|^2+\sum_b f(E_b)|\langle\vec{x}|\psi_b\rangle|^2. \tag{16}$$

Here $f(\varepsilon_n)=[\exp\beta(\varepsilon_n-\mu)+1]^{-1}$ is the usual Fermi function, and Σ_b the sum over all bound states. It is important to note that the relevant occupation probabilities are identical with those in the absence of the Hall field $\vec{E}_H$, since the effect of the average drift is already included in the wave functions. As a result, the transport velocity v of the electrons enters into (16) only through the v-dependent scattering and bound states $|\overline{kn}+\rangle$ and $|\psi_b\rangle$, but not through a transformation of the energies. By contrast, in the corresponding expression[3]

$$n(\vec{x})=\int\frac{dk}{(2\pi)^d}f(\varepsilon_{\vec{k}-m\vec{v}/\hbar})|\langle\vec{x}|\vec{k}+\rangle|^2, \tag{17}$$

valid in the *absence* of a magnetic field, the dependence of the scattering states on the transport velocity could easily be transformed into a shift $\varepsilon_{\vec{k}}\rightarrow\varepsilon_{\vec{k}-m\vec{v}/\hbar}$ of the energies, equivalent to a shifted Fermi sphere. Evidently this is not possible in the present case. As a second point we note that expression (16) is gauge invariant, as it should be. In fact a gauge transformation effectively leads to a shift of the momentum k of the extended states which is irrelevant after the integration $\int dk$ is done. Similarly the wave function of the localized state is multiplied by a phase factor which drops out in $|\langle\vec{x}|\psi_b\rangle|^2$.

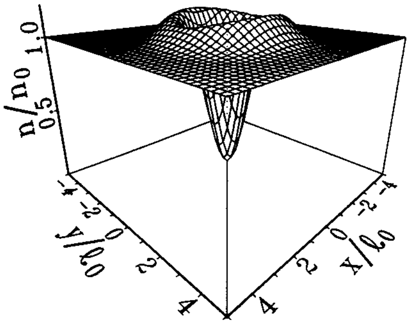

FIG. 1. The normalized local density around a repulsive impurity at the origin. The x and y coordinates are given in units of the magnetic length l_0.

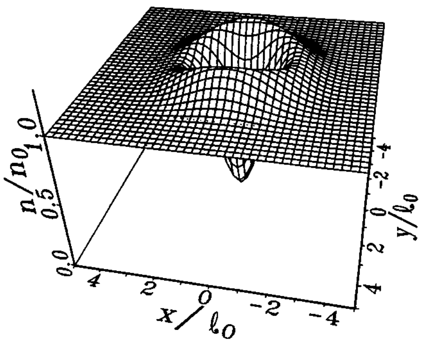

FIG. 2. The same as in Fig. 1, but now with a finite transport current flowing in the negative-y direction, inducing an asymmetry transverse to that.

In the absence of scattering each filled Landau level makes a contribution $(2\pi l_0^2)^{-1}$ to the density, which is constant in space, as is easily seen from (6). For a finite potential $V \neq 0$ the density becomes nonuniform and has a strong variation in the vicinity of the scattering center. An example of this distortion at zero temperature and vanishing transport velocity is shown in Fig. 1. It is based on a numerical evaluation of (16) for the particular case of a completely filled Landau level and potential strength $V_0 = \hbar\omega_c$. Apart from the strong depletion of the density around the repulsive scattering center there is an appreciable density maximum at intermediate distance which arises from the bound-state contribution. The fact that $n(\vec{x})$ is not radially symmetric is due to our choice of the scattering potential, where the width of the Gaussian function $\langle \vec{x} | \varphi_0 \rangle$ in the x direction is twice that in the y direction. In order to study the way in which the density approaches its asymptotically constant value, we have to determine the behavior of $\langle \vec{x} | \bar{G}_0(E) | \vec{x}' \rangle$. Similar to the situation at $v = 0$, where the propagator may be calculated analytically,[10] it may be shown[11] that $\langle \vec{x} | \bar{G}_0(E) | \vec{x}' \rangle$ decays like a Gaussian on the scale of a magnetic length l_0 as long as v is small enough that the coupling to higher Landau levels remains negligible. As a result, the density $n(\vec{x})$ also decays very quickly on scales of order l_0. This is in marked contrast to the situation at vanishing field $B = 0$, where the asymptotic behavior is a static Friedel oscillation decaying like r^{-2} in two dimensions.[3] Clearly the origin of this difference is the fact that the magnetic field destroys the Fermi surface and leads to a completely discrete spectrum.

Let us now study the case of a finite transport velocity $v \neq 0$. Choosing the same parameters as used in Fig. 1 and $v = 0.1(\hbar\omega_c/m)^{1/2}$, the resulting electron density is shown in Fig. 2. It is evident that $n(\vec{x})$ now exhibits an asymmetry in the x direction which is transverse to the current, while it remains symmetric along the y direction in which the current flows. Again, as in the field-free case, there are no Friedel oscillations, and the whole density variation decays exponentially on a scale of order l_0. Similar, but less detailed, results were found previously by Chaudhuri, Bandyopadhyay, and Cahey,[12] who used attractive δ impurities as scattering centers. Qualitatively these results may be understood as follows: For vanishing magnetic field the resistivity tensor is diagonal. Its relevant longitudinal component $\rho_{yy} \neq 0$ is a measure of the asymmetry in density in the direction of the incoming current.[3] For a strong magnetic field with one completely filled Landau level, however, the longitudinal resistivity ρ_l vanishes as long as excitation to higher Landau levels may be neglected. In terms of the local-density variation this property is reflected in the absence of a longitudinal asymmetry in density. For any system with $\rho_l = 0$ the usual Landauer resistivity dipole $\vec{p} \sim \rho_l \vec{v}$ in the direction of the incident current is therefore quenched. Since the Hall resistivity $\rho_{xy} = \rho_H$ is now finite, however, there is a corresponding asymmetry *transverse* to the incident current, which is the effect observed in Fig. 2. Instead of the dipolar contribution (1) with an oscillating dipole moment (2) for the case $B = 0$, however, the density variation is now a Gaussian, decaying on the scale of the magnetic length l_0. Moreover, scattering and bound states make opposite contributions to the current-induced density. This may be seen from a simple exactly soluble model,[11] where $|\varphi_0\rangle$ is just the zero angular momentum state in the lowest Landau level, i.e.,

$$\langle kn | \varphi_0 \rangle = \left(\frac{l_0}{\sqrt{\pi}\hbar} \right)^{1/2} \exp - (k l_0)^2 / 2 \delta_{n,0}. \tag{18}$$

To lowest order in v the associated bound state is given by

$$|\langle \vec{x} | \psi_b \rangle|^2 = |\langle \vec{x} | \varphi_0 \rangle|^2 \left(1 + \frac{\hbar v}{V_0 l_0^2} x + \cdots \right). \tag{19}$$

For a repulsive scattering potential $V_0 > 0$ the probability density of the bound state therefore shifts to the right, which is opposite to the direction of the force an electron experiences through the Hall field. On the other hand, scattering states turn out to behave just in the opposite way, and apparently overcompensate for the effect of the bound states.[11] It would be very interesting to see how this behavior of the local *density* is connected with the general result[8,13] that the introduction of a single impurity does not change the ideal value of ρ_{xy} because the missing *current* due to localized states is exactly compensated for by a corresponding increase in that carried by extended states.

In conclusion, we have calculated the local electron density around a single impurity in a two-dimensional electron gas at high magnetic fields both at vanishing and finite transport current. While the spatial variations are irrelevant for determining the overall conductance coefficients $G_{m,n}$,[14] they are of interest both conceptually[2] and in the context of local fluctuations of the potential distribution in the quantum Hall effect. As emphasized by Beenakker and van Houten,[7] its explanation in terms of the ideal transmission of edge states[15] is quite independent of the detailed spatial dependence of the currents. Nevertheless the local Hall field $E_H(x)$ is an interesting and measurable quantity.[16] As was mentioned above, this field is nonuniform even in the absence of impurity scattering, where $E_H(x)$ together with the local currents and densities can be calculated explicitly.[7,17,18] Since most of the voltage drop occurs near the edges, the Hall field in the bulk is smaller than its ideal value by a factor $2[1+\ln(W/\xi)]^{-1}$ at filling factor 1.[7] Here W is the width of the Hall bar and $\xi = l_0^2/\pi a^*$ a characteristic length determined by l_0 and the effective Bohr radius $a^* = \varepsilon\hbar^2/me^2$. Since W is typically much larger than ξ, the effective Hall field is reduced considerably in the bulk. As a result, the transverse asymmetry in the density and the local potential around scattering centers is smaller than its ideal value calculated here because the effective drift velocity v is reduced by the corresponding factor. The presence of a nonuniform density around short-range scattering centers for vanishing transport current will, however, lead to local fluctuations in the potential profile which would be interesting to observe. In this context a very attractive possibility would be provided by a two-dimensional electron gas at a *free* surface which allows us to measure directly the local electronic density of states $n(\vec{x},\varepsilon_F)$ (Ref. 19) at the Fermi energy with a tunneling microscope.[6] In the absence of a magnetic field and for a two-dimensional scattering potential which vanishes asymptotically, this may be written as an angular average of the wave function squared,

$$n(\vec{x},\varepsilon_F) = \frac{m}{(2\pi\hbar)^2}\int d\Omega_k |\psi_{k_F}(\vec{x})|^2. \tag{20}$$

Taking a single impurity with backscattering amplitude $f_{k_F}(\pi)$, the resulting behavior sufficiently far from the scattering center is a Friedel oscillation,

$$n(\vec{x},\varepsilon_F) = \frac{m}{2\pi\hbar^2}\left[1+\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{k_F r}\,\mathrm{Re}[f_{k_F}(\pi)e^{2ik_F r}]\right], \tag{21}$$

decaying like $1/r$, similar to result (2) for the *total* density at finite transport current. These oscillations have been observed for impurities in a two-dimensional electron gas at a copper surface.[6] At finite magnetic field where bound states appear even with repulsive impurities, the local density of states is given by

$$n(\vec{x},\varepsilon_F) = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}|\langle\vec{x}|\bar{k}n+\rangle|^2\delta(\varepsilon_n-\varepsilon_F) + \sum_b |\langle\vec{x}|\psi_b\rangle|^2\delta(E_b-\varepsilon_F) \tag{22}$$

which is again gauge invariant as required. In contrast to the total density which involves states at all energies up to ε_F, the local density of states is determined only by states at a fixed energy. Now in the presence of a finite concentration of impurities, the Landau levels are broadened into a continuum. For a finite sample, the states in the vicinity of a bare Landau level effectively behave like extended states, while those in the tails are localized.[20] Thus by varying the energy, a STM measurement of the local density of states should be able to detect the opposite shift of extended or localized states in a finite current discussed above.

Note added in proof. After submission of the manuscript Dr. R. Landauer has kindly pointed out to us that the rotation of the resistivity dipole in a magnetic field was anticipated by him some time ago [R. Landauer, J. Phys. F **8**, L245 (1978)]. Moreover, in unpublished results, Briner, Feenstra, Chin, and Woodall report on a direct experimental observation of the resistivity dipole in Bi films at high transport currents.

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