Bilayers of chiral spin states

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We study the behavior of two planes of a quantum Heisenberg antiferromagnet in the regime in which a chiral spin liquid is stabilized in each plane. The planes are coupled by an exchange interaction of strength J_3. We show that in the regime of small J_3 (for both ferromagnetic *and* antiferromagnetic coupling), the system dynamically selects an *antiferromagnetic* ordering of the ground state *chiralities* of the planes. For the case of an antiferromagnetic interaction between the planes, we find that, at some critical value J_3^c of the interlayer coupling, there is a phase transition to a valence-bond state on the interlayer links. We derive an effective Landau-Ginzburg theory for this phase transition. It contains two U(1) gauge fields coupled to the order parameter field. We study the low-energy spectrum of each phase. In the condensed phase an "anti-Higgs-Anderson" mechanism occurs. It effectively restores time-reversal invariance by rendering massless one of the gauge fields while the other field locks the chiral degrees of freedom locally. There is no phase transition for ferromagnetic couplings.

I. INTRODUCTION

The discovery of superconductivity at high temperatures in the otherwise insulating copper oxides has motivated a thorough search for new physical mechanisms for both superconductivity and antiferromagnetism. This search has produced a host of new possible mechanisms, many of which are not yet established on solid ground. Among these new ideas, the anyon mechanism[1] stands as, perhaps, the most novel of them. For this reason, it has attracted a lot of attention. At a microscopic level, the anyon state requires that the underlying insulating state, known as the chiral spin liquid[2] (CSL), should necessarily break time-reversal (T) invariance and parity (P). An experimental signature of a state with broken T and P invariance is optical dichroism.[3] So far, however, there is no experimental evidence in support of the spontaneous breaking of either T or P in the copper oxides.[4] Clearly, the simplest option is that these symmetries are not broken in the copper oxides and that the insulating states are unrelated to the CSL. At the present time this appears to be the case.

In this paper we will explore the possibility that T and P may be broken in one individual plane but not on the system as a whole. Individual isolated planes may still be in states which break T and P but the *sign* of this breaking may not be the same from plane to plane. The simplest case is to imagine that the copper oxide planes are coupled by some interaction and that this coupling is responsible for the selection of the state. A version of this problem has been studied by Rojo and Leggett.[5] They considered two planes with a *doped* CSL on each plane and, hence, had an *anyon superconductor* on *each plane*. They further assumed that the planes were coupled together only by a direct Coulomb interaction between the anyons on each plane. They did not fix *a priori* the relative sign of the statistics of the anyons on each plane but, instead, asked which *relative sign* was preferred by the Coulomb interactions. They found that the Coulomb interactions prefer the relative statistics to be *antiferromagnetic* ordered, namely, opposite signs. The Rojo-Leggett result is due to a rather subtle edge effect. In fact, they found no effect in the bulk.

In many copper oxides, the physical situation is such that the planes come in groups in which the planes are closer together than among nearby groups. This is rather common in the bismuth-based copper oxides. Because in these materials the *interlayer* exchange constant which couples the copper spins can be comparable to the *intralayer* exchange constant, there is a competition between intralayer and interlayer types of ordering. Quite generally, one expects to find to distinct regimes in the phase diagram for bilayers. At weak interlayer coupling, the ground state of the individual layers may be stable. However, if the interlayer exchange coupling dominates, the likely ground state should be a valence-bond state on the interlayer links. The case of two coupled Néel states was considered recently by Uhbens and Lee,[6] by Millis and Monien,[7] and by Sandvik and Scalapino.[8] These authors considered the effects of an interlayer exchange interaction on the Néel ground states of the planes.

In this paper we will reconsider the problem of a bilayer of quantum antiferromagnets in a regime in which there is enough frustration to drive each plane separately into a chiral spin liquid. The planes will be assumed to be coupled by an antiferromagnetic exchange interaction of strength J_3. The problems that we want to address are the following: (a) Does the interlayer exchange interaction select the relative ordering of the chiralities and (b) what is the phase diagram for this system as a function of the interlayer interaction? We consider a situation in which there is a CSL ground state on each plane, with fixed chirality but arbitrary sign. We find that quantum fluctuations around this state select an *antiferromagnetic* ordering of the chiralities. This is a rather interesting result. It means that even if on each plane the system was allowed to break P and T, the dynamics selects the state which is on the whole *P and T* invariant. We also find that, as J_3 increases, there is phase transition to a state that we identify as a valence-bond state on the interlayer links,

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namely, a T- and P-invariant spin gap state very similar to the one found by Ubbens and Lee,[6] Millis and Monien,[7] and Sandvik and Scalapino.[8] The problem of the ordering of chiralities by an interlayer exchange interaction was considered previously by Gaitonde *et al.*[9] By means of a perturbative expansion in powers of J_3 they concluded that the chiralities order *ferromagnetically*. The results that we report here disagree with those of Gaitonde *et al.*

As is by now well known,[2,10–12] the CSL state and its low-lying excitations can be described in terms of an effective continuum field theory which is very much analogous to a set of Dirac self-interacting fermions in two space and one time dimensions. We find that the essential physics of this system can be understood in terms of the properties of an effective continuum theory of Dirac fermions on each plane provided that a physically sensible cutoff is introduced. The effective model contains two sets of massive Dirac fermions on each plane. The chirality of the state is given by the sign of the mass term. As in Ref. 2, the fluctuations around the CSL of each plane are represented by gauge fields (one for each plane). By a detailed microscopic analysis we find that the interlayer exchange fluctuations are represented by a *complex* order parameter field. The effective theory is controlled by three parameters: (1) the magnitude of the fermion mass on each plane (i.e., the fermion gap in the CSL), (2) the interlayer exchange constant (which determines the energy gap for fluctuations of the order parameter), and (3) the number of fermionic species (which we take to be N). In this picture, the phase transition to the valence-bond state becomes the phase transition to a state in which the complex order parameter acquires a nonvanishing expectation value. Our basic strategy is to first derive this effective theory and then use it to address the issues of the ordering of chiralities and of the nature of the phase diagram.

Mean-field theories (MFT's) of frustrated antiferromagnets on a single plane have yielded a host of possible nonmagnetic variational ground states. The actual phase diagram is not known in detail although it is generally accepted that nonchiral states are somewhat favored by variational calculations. In this paper we will not consider how interlayer couplings may alter this competition among possible single-layer variational states. Rather, we will describe how interlayer interactions disrupt the CSL in favor of an interlayer valence-bond state, which is clearly favored at strong coupling. The determination of the global phase diagram for bilayers is an interesting problem which is, however, still outside the reach of present theoretical tools and beyond the scope of this article.

The effective field theory of fermions can be studied within a $1/N$ expansion. We use this expansion for two different purposes. First we look at the quantum corrections to the ground state energy of a system in which the two CSL's are decoupled. We find that, at leading order in the $1/N$ expansion, the state with antiferromagnetic (opposite) chiralities is degenerate with the state with ferromagnetic chiralities. However, we find that for the leading corrections, due to fluctuations of interlayer exchange processes, the state with antiferromagnetic ordering of chiralities is selected. In addition to the spontaneous breaking of this discrete symmetry (the relative chirality), the fermionic theory for the bilayers undergoes a dynamical breaking of the interlayer (out-of-phase) gauge symmetry at a critical value of the interlayer coupling constant. This phenomenon is strongly reminiscent of the breaking of chiral symmetry in the related (but not equal) field theoretic Gross-Neveu and Nambu-Jona Lasinio models.[14] Also, within this $1/N$ expansion, we find a phase transition from a regime in which the two planes have CSL ground states with opposite signs, to a state in which the interlayer order parameter field condenses. We further investigate the physics of this phase transition by deriving an effective Landau-Ginzburg-type field theory, valid in the vicinity of the phase transition, i.e., for $J_3 \sim J_3^c$.

The degrees of freedom of the Landau theory, which is fully quantum mechanical, are the interlayer order parameter field and the gauge fields of the two planes. We present a qualitative study of the fluctuation spectrum of the two phases. The weak coupling phase has (almost) the same spectrum as that of two CSL's with opposite chiralities: semions with opposite chiralities and gapped gauge fluctuations. However, the phase with broken symmetry (in which the interlayer field condenses) displays an interesting "anti-Higgs-Anderson" mechanism: The condensation of the order parameter field causes a gauge fluctuation, which is massive in the unbroken phase due to the Chern-Simons terms, to become massless. This, in turn, implies that any excitation which couples to the gauge fields (the semions, in particular) to become confined by strong, long-range, logarithmic interactions. The resulting spectrum of the condensed phase is *equivalent* to the low-lying spectrum of a ground state of local singlets, i.e., a valence-bond state on the interlayer links. The interlayer gauge field remains massive and it effectively disappears from the spectrum. Thus, the "anti-Higgs-Anderson" mechanism wipes out all trace of broken time-reversal-invariance in the system. Unexpectedly, in this phase the system is actually more symmetric than in the noncondensed state.

The paper is organized as follows. In Sec. II we introduce the model for the bilayer and develop the mean-field theory and briefly discuss the phase diagram. In Sec. III we address the problem of the dynamical selection of chiralities. In Sec. IV we derive a gradient expansion for the low-energy modes of the (two) gauge fields and the relevant (scalar) channel of the field coupling the planes. In Sec. V we discuss the properties of the symmetric phase where the field coupling the planes does not condense, and an effective action for the gauge fields is derived and studied. Section VI deals with the broken-symmetry phase. Section VII is devoted to the conclusions. We also include appendixes which contain technical details of the mapping onto the effective continuum theory and the computation of Feynman diagrams relevant for the phase transition, the ordering of the chiralities, and the gradient and $1/N$ expansions.

II. MEAN-FIELD THEORY FOR TWO COUPLED CHIRAL SPIN STATES

Our model consists of two square-lattice spin-1/2 Heisenberg antiferromagnets coupled through an exchange interaction of nearest-neighbor spins between planes with strength J_3, and nearest-neighbor (NN) (J_1) and next-nearest-neighbor (NNN) (J_2) interactions on each plane. The lattice Hamiltonian reads

$$H=H_L+H_U+J_3\sum_{\vec{x}} \vec{S}_L(\vec{x})\cdot\vec{S}_U(\vec{x}+\vec{e}_z), \tag{2.1}$$

where $H_{L,U}$ is the usual Heisenberg Hamiltonian,

$$H_{L,U}=J_1\sum_{\vec{x},j=1,2} \vec{S}_{L,U}(\vec{x})\cdot\vec{S}_{L,U}(\vec{x}+\vec{e}_j) + J_2\sum_{\vec{x},j=+,-} \vec{S}_{L,U}(\vec{x})\cdot\vec{S}_{L,U}(\vec{x}+\vec{e}_1+j\vec{e}_2). \tag{2.2}$$

Using the slave fermion approach, the spin operator can be written in terms of fermionic creation and anihilation operators $\vec{S}(\vec{x})\equiv c^\dagger_\alpha(\vec{x})\vec{\sigma}^{\alpha\beta}c_\beta(\vec{x})$ with the usual constraint of single occupancy. We decouple the quartic terms by using a standard Hubbard-Stratonovich (HS) transformation. Up to an integration over the HS fields, the original theory is equivalent to the one that follows from the action given by the Lagrangian

$$\mathscr{L}=\mathscr{L}_L+\mathscr{L}_U-\frac{1}{J_3}\sum_{\vec{x}} |\chi_z(\vec{x})|^2 + \sum_{\vec{x}} [c^*_L(\vec{x})\chi_z(\vec{x})c_U(\vec{x}+\vec{e}_z)+\text{H.c.}], \tag{2.3}$$

where

$$\mathscr{L}_L=\sum_{\vec{x}} c^*_L(\vec{x})(i\partial_t+\mu)c_L(\vec{x})+\sum_{\vec{x}} \varphi_L(\vec{x})[c^*_L(\vec{x})c_L(\vec{x})-1]-\frac{1}{J_1}\sum_{\vec{x};j=1,2}|\chi_{j,L}(\vec{x})|^2-\frac{1}{J_2}\sum_{\vec{x};j=+,-}|\chi_{j,L}(\vec{x})|^2 + \sum_{\vec{x};j=1,2}[c^*_L(\vec{x})\chi_{j,L}(\vec{x})c_L(\vec{x}+\vec{e}_j)+\text{H.c.}]+\sum_{\vec{x};j=+,-}[c^*_L(\vec{x})\chi_{j,L}(\vec{x})c_L(\vec{x}+\vec{e}_1+j\vec{e}_2)+\text{H.c.}], \tag{2.4}$$

where we have dropped the spin indices α,β to simplify the notation, with a similar definition for $\mathscr{L}_U$. Here μ is the chemical potential and $\vec{x}$ means $(\vec{r},t)$. The constraint of single occupancy is enforced by the bosonic Lagrange multiplier field $\varphi(\vec{x})$. This type of factorization was originally proposed by Affleck and Marston[10] and by Kotliar.[11] The HS fields can be parametrized in terms of an amplitude $\rho_j(\vec{x})$ and a phase $A_j(\vec{x})$. This Lagrangian has a local symmetry if the Lagrange multiplier field φ transforms as the A_0 component of a U(1) gauge field.

The MFT consists in integrating out the fermions, at a fixed density, and treating the fields $\chi_j(\vec{x})$ within a saddle-point expansion. As is well known, one serious problem with this mean-field theory is that there is no small parameter in powers of which to organize the semiclassical expansion. Following Affleck and Marston,[10] we will allow the number of spin species to run to N instead of 2, which is the case for the spin-1/2 Heisenberg model. After rescaling the coupling constant strengths J's and the fluctuating part of the fields, a one-loop expansion of the fermionic determinant around the $N\to\infty$ mean-field solution can be performed by keeping the diagrams up to order $1/N$. We have $\mathscr{S}_{\text{eff}}[\varphi,\chi_j]=N\bar{\mathscr{S}}[[\varphi,\chi_j]$, and the quantum partition function is $\mathscr{Z}=\int\mathscr{D}\chi\mathscr{D}\chi^*\mathscr{D}\varphi e^{iN\bar{\mathscr{S}}}$.

There exists a whole family of solutions of the saddle-point equations. The simplest solutions are the valence-bond states and the flux phases. These may or may not be chiral. In this work we consider the problem of the selection of the relative chirality of a state in which there is a chiral spin liquid on each plane. Thus, we *choose* a saddle point which represents chiral spin states on each plane and we will investigate which configuration of chiralities is chosen dynamically.

Wen, Wilczek, and Zee[2] (WWZ) have given a construction of the chiral spin state, which was first proposed by Kalmeyer and Laughlin.[1] WWZ begin with the flux phases, which have a uniform value for the amplitude of the NN HS fields, say, $\rho(\vec{x})=\bar{\rho}$. This amplitude, however, can fluctuate. The phases of the Bose fields on the NN links of an elementary plaquette have a circulation equal to π or $-\pi$ in mean-field value. This feature produces a collapse of the Fermi surface into four discrete points of the Brillouin zone $(\pm\pi/2a,\pm\pi/2a)$ at which two bands of states (positive and negative energy, "conduction" and "valence" bands) become degenerate. At these points, the excitation spectrum is linear and gapless. This allows for a mapping onto a discrete version of the Dirac theory with two massless fermion species of two-component spinors, with the "speed of light" equal to the Fermi velocity $v_F=2a\bar{\rho}$. This gapless state can become unstable due to the effects of fluctuations. Several channels are known to be possible. If the staggered part of the fluctuations of the amplitude of the Bose fields on the NN links picks up a nonzero expectation value, gaps will open up in the elementary excitation spectrum and they will provide masses (or gaps) to the Dirac-like fermionic excitations. These fluctuations can be seen to drive the flux phase into a dimer or Peierls state and do not break time-reversal invariance or parity.

A mass term in a Dirac equation for a *single* two-component spinor Fermi field in $2+1$ dimensions generally breaks T and P since the Hamiltonian, while Hermitian, becomes complex. Since all three Pauli matrices are involved (two for the gradient terms and the third one for the mass term), there is no basis in which the Hamiltonian could be real. Therefore, the Hamiltonian is not self-conjugate and T is broken. However, in the case in which two species of fermions are present, the presence of such mass terms does not necessarily break P and T since they may have opposite signs for the different species. This is the case of the so-called Peierls mass, which occurs in dimer phases. It is here where frustration comes to play a crucial role. By turning on NNN interactions, WWZ allowed for additional HS fields on the diagonals of the elementary plaquettes. The MF configuration for the phases can be arranged so that each triangle in an elementary plaquette is pierced by a flux equal to $\pi/2$. In this way, a time-reversal and parity-breaking mass can be generated; i.e., one can provide a mass with the *same sign* to both fermion species in the plane. In order to perform the mapping onto the Dirac theory it is necessary to introduce four different field amplitudes at each unitary cell of four sites. This procedure can be done on the real space lattice by defining four sublattices and assigning an independent field amplitude to each one and expanding in gradients of the field amplitudes,[12] or on the reciprocal lattice[18] by expanding the lattice amplitude at each point as a linear combination of four independent Fourier component amplitudes. On the reciprocal lattice these fields are the Fourier components of the lattice amplitude centered at the four Fermi points. The low-energy physics of the system is determined by the scattering processes among these four amplitudes. Any of these procedures is equivalent to a folding of the first Brillouin zone.

In the chiral spin state (CSS), the mean-field ansatz for the amplitudes and phases of the HS fields on the NN and NNN links is given by[2,18]

$$\begin{aligned}
\bar{\chi}_1(e,e)&=-\bar{\chi}_1(o,e)=\bar{\chi}_1(e,o)=-\bar{\chi}_1(o,o)=i\bar{\rho},\\
\bar{\chi}_2(e,e)&=-\bar{\chi}_2(o,e)=-\bar{\chi}_2(e,o)=\bar{\chi}(o,o)=-i\bar{\rho},\\
\bar{\chi}_+(e,e)&=\bar{\chi}_+(o,e)=-\bar{\chi}_+(e,o)=-\bar{\chi}_+(o,o)=i\chi,\\
\bar{\chi}_-(e,e)&=\bar{\chi}_-(o,e)=-\bar{\chi}_-(e,o)=-\bar{\chi}_-(o,o)=-i\chi.
\end{aligned} \tag{2.5}$$

The fields $\bar{\chi}_j$, with $j=1,2$ or $j=+,-$ are the HS fields sitting on the NN and NNN links, respectively. The four different sublattices are denoted by (e,e), (o,e), (e,o), (o,o), where e and o mean even or odd site, respectively.

Once the mean-field HS ansatz has been used into the Hamiltonian for one plane, a convenient linear combination of the four field amplitudes can be arranged in the form of two *two-component* spinors and one can rewrite the Lagrangian for a single plane in the form of a lattice Dirac Lagrangian with two massive fermion species. So far we did not include any fluctuations of the HS fields. We will be interested in the fluctuating part of the *phase* of the HS fields.

In order to capture the physics of the system in the regime of long-wavelength, low-energy of the spectrum, we do not need the full lattice theory, but a linearized version around the Fermi points that keeps all the scattering processes that are responsible for the behavior of the low-energy excitations of the system. In the case of only one square lattice bearing a chiral spin state, we arrive at a $(2+1)$-dimensional effective action involving two massive relativistic fermions coupled to a gauge field.[2,12,18] The form of this action is given by

$$\mathscr{S}=\int dx_0\int dx^2\{\bar{\psi}_1(i\partial\!\!\!/-A\!\!\!/-m_1)\psi_1+\bar{\psi}_2(i\partial\!\!\!/-A\!\!\!/-m_2)\psi_2\}. \tag{2.6}$$

The continuum field ψ_a is related to the lattice amplitude Ψ_a by $\psi_a(\vec{x})\equiv\Psi_a(\vec{x})/a$. We use a representation of Dirac gamma matrices in which $\gamma_0=\sigma_3$, $\gamma_1=-i\sigma_2$, and $\gamma_2=-i\sigma_1$, where σ_j, $j=1,2,3$, are the usual Pauli matrices. The coupling to the gauge field (the *statistical vector potential*) A_μ comes through the covariant derivative $D\!\!\!/\equiv\partial\!\!\!/-iA\!\!\!/$. The statistical vector potential is given by $A_j\equiv\tilde{\phi}_j/a=2\bar{\rho}\tilde{\phi}_j/v_F$ and $A_0=\varphi/v_F$, where $\tilde{\phi}_j$ is the fluctuating part of the phase of the Hubbard-Stratonovich fields on the NN links, φ is the Lagrange multiplier field,[2,12,18] and $x_0\equiv v_F t$.

The masses of the fermions come from the amplitude of the HS field on the NNN links and give a measure of the amount of frustration present in the system. These masses, although not necessarily equal in magnitude, have the same sign for both species. We assume that these amplitudes are fixed at their mean-field values, since we are interested only in the effects of interlayer fluctuations.

In what follows we adapt the methods of Refs. 2 and 12 to the bilayer problem.[18] We have a duplication of terms due to the inclusion of the second plane and new terms arising from the interplanar interaction. In the continuum limit, the action for the fermions in the low-energy theory has two species of Dirac fermions *on each plane* coupled to both the *intralayer* and *interlayer* Hubbard-Stratonovich fields which mediate the interactions among the fermionic degrees of freedom. For simplicity we will assume that the degree of chiral breaking is fixed and parametrized by two nonfluctuating masses m_U and m_L. These *masses* are given by $m_{L,U}\equiv 4\chi_{L,U}/v_F$, being $\chi_{L,U}$ the mean-field amplitude of the Hubbard-Stratonovich fields on the NNN links. We assume that the mean-field approximation amplitude of the HS fields on the NN links $\bar{\rho}$ is the same for both planes. Consequently the Fermi velocity is also the same. The only low-energy intralayer bosonic degrees of freedom left are the gauge fields of the upper and lower planes A_U and A_L and the interlayer fields χ_z.

The continuum action for the bilayer consists essentially of Eq. (2.6) written twice with labels L and U for lower and upper planes and an interlayer part given by the coupling between planes,

$$\begin{aligned}
\mathscr{S}_{\text{interlayer}}=&\int dx_0\int dx^2\{\bar{\psi}_L(\varphi_0\gamma_0\mathbf{1}+\varphi_1\gamma_1\tau_1+\varphi_2\gamma_2\tau_2+\varphi_3\mathbf{1}\tau_3)\psi_U+\text{H.c.}\}\\
&-\frac{1}{g_3}\int dx_0\int dx^2[U(|\varphi_0|^2)+U(|\varphi_1|^2)+U(|\varphi_2|^2)+U(|\varphi_3|^2)].
\end{aligned} \tag{2.7}$$

In this expression ψ_L and ψ_U represent the two Dirac *flavors* $\psi^{1,2}_{L,U}$ that live on the lower and upper planes of the bilayer. The τ matrices mix Dirac flavors inside each plane.

The intralayer gauge fields, which represent intralayer phase fluctuations on NN links, have to be kept since they enter at the leading order in the continuum limit. There are other operators, with the form of fermion mass terms, that have not been included which do not contain any derivatives but they describe other types of intralayer ordering which compete with the CSL. To include such effects would require a theory of the full phase diagram which is beyond the scope of this paper.

The bosonic part of the interlayer action shown in the second line of Eq. (2.7) comes from the corresponding bosonic terms in Eq. (2.3),

$$\mathscr{S}_b=-\frac{1}{J_3}\int dx_0\sum_{\vec{x}=(e,e)}\{|\chi_z(\vec{x})|^2+|\chi_z(\vec{x}+\vec{e}_1)|^2+|\chi_z(\vec{x}+\vec{e}_2)|^2+|\chi_z(\vec{x}+\vec{e}_1+\vec{e}_2)|^2\}, \tag{2.8}$$

where $\vec{x}$ is an even-even site on the lattice at, say, the lower plane. However, in going to the continuum limit it proves more convenient to introduce the rotation given by the linear combinations of the four HS fields $\chi_z(\vec{x})$ which link corresponding *plaquettes* of the planes,

$$\varphi_0(\vec{x})\approx\frac{1}{4}[\chi_z(\vec{x})+\chi_z(\vec{x}+\vec{e}_1)+\chi_z(\vec{x}+\vec{e}_2)+\chi_z(\vec{x}+\vec{e}_1+\vec{e}_2)], \tag{2.9}$$

$$\varphi_1(\vec{x})\approx\frac{1}{4}[\chi_z(\vec{x})-\chi_z(\vec{x}+\vec{e}_1)+\chi_z(\vec{x}+\vec{e}_2)-\chi_z(\vec{x}+\vec{e}_1+\vec{e}_2)], \tag{2.10}$$

$$\varphi_2(\vec{x})\approx\frac{1}{4}[\chi_z(\vec{x})+\chi_z(\vec{x}+\vec{e}_1)-\chi_z(\vec{x}+\vec{e}_2)-\chi_z(\vec{x}+\vec{e}_1+\vec{e}_2)], \tag{2.11}$$

$$\varphi_3(\vec{x})\approx\frac{1}{4}[\chi_z(\vec{x})-\chi_z(\vec{x}+\vec{e}_1)-\chi_z(\vec{x}+\vec{e}_2)+\chi_z(\vec{x}+\vec{e}_1+\vec{e}_2)]. \tag{2.12}$$

In terms of the rotated fields, and after taking the continnum limit, the bosonic part of the action takes the form

$$\begin{aligned}\mathscr{S}_b&=-\frac{1}{g_3}\int dx^3[\varphi_0^*(\vec{x})\varphi_0(\vec{x})+\varphi_1^*(\vec{x})\varphi_1(\vec{x})+\varphi_2^*(\vec{x})\varphi_2(\vec{x})+\varphi_3^*(\vec{x})\varphi_3(\vec{x})]\\&=-N\int\frac{dq^3}{(2\pi)^3}[\lambda\varphi_0^*(\vec{q})\varphi_0(\vec{q})+\lambda\varphi_1^*(\vec{q})\varphi_1(\vec{q})+\lambda\varphi_2^*(\vec{q})\varphi_2(\vec{q})+\lambda\varphi_3^*(\vec{q})\varphi_3(\vec{q})].\end{aligned} \tag{2.13}$$

In the second line of Eq. (2.13), Fourier transforms have been taken and the coupling constant g_3 has been rescaled by $1/N$ in order to allow a $1/N$ expansion (see below). In other words $\lambda\equiv 1/g_3'$, where $g_3'\equiv g_3/N$. The fields φ_j, $j=0,1,2,3$, also have been rescaled to φ/v_F. As a result, the effective coupling constant that controls the interlayer fluctuations is $g_3\equiv 2aJ_3/\bar{\rho}=J_3(2a)^2/v_F$ and has units of length. Throughout this work we use dimensions such that $[h]=[e]=[v_F]=1$ where h, e, and v_F are the Planck's constant, the unit of charge, and the Fermi velocity, respectively. We have a natural scale in our theory, which is the lattice constant a_0, or the inverse lattice constant which we shall call Λ, and characterizes the momentum cutoff.

From the free part of the action, and the fact that we are working in $2+1$ dimensions, it is clear that the dimension of the fermion operators must be $\Lambda\approx(\text{length})^{-1}$. The dimension of the operator $\hat{\varphi}$ is also that of Λ. The coupling constant g_3 is dimensional with $[\lambda]\equiv[1/g_3]=\Lambda$. This dimensional analysis tells us that the effective four-fermion operator which represents the interactions between the fermions of the two planes is irrelevant at the weak coupling fixed point and that, if a phase transition exists, it should happen at some finite value of the interlayer coupling. We will see that this is indeed the case.

Now we integrate out the fermions and obtain the effective action

$$\mathscr{S}_{\text{eff}}\equiv -iN\ \text{Tr}\ \ln\begin{bmatrix} i\not{D}_L-m_L & \hat{\varphi}\\ \hat{\varphi}^* & i\not{D}_U-m_U\end{bmatrix}+\mathscr{S}_b, \tag{2.14}$$

where we have defined

$$\hat{\varphi}\equiv\varphi_3\tau_3+\varphi_0\gamma_0+\varphi_1\gamma_1\tau_1+\varphi_2\gamma_2\tau_2. \tag{2.15}$$

The saddle-point equations are

$$\frac{1}{g_3}\varphi_j^*(0)=-i\int\frac{dk^3}{(2\pi)^3}\text{tr}\left[\begin{pmatrix}\not{k}-\not{A}_L-m_L & \hat{\varphi}(k)\\ \hat{\varphi}^*(k) & \not{k}-\not{A}_U-m_U\end{pmatrix}^{-1}\begin{pmatrix}0 & \dfrac{\delta\hat{\varphi}(k)}{\delta\varphi(0)}\\ 0 & 0\end{pmatrix}\right]. \tag{2.16}$$

Formally, this integral diverges linearly with the momentum cutoff scale Λ. As in all theories of critical phenomena, we will absorb the singular dependence on the microscopic scale in a renormalization of the coupling constant. We can define a critical coupling constant g_c as the value of the coupling constant at which the expectation values for the fields coupling the planes first become different from zero. Clearly the solution with $\langle\varphi_j^*\rangle=0$ is allowed for any finite value of the cutoff, no matter how large. This is the phase where the interplane field is not condensed. The nontrivial solution will first occur at the value of the coupling constant g_j^c given by

$$\frac{1}{g_j^c}\equiv -i\left(\frac{\delta}{\delta\varphi_j^*}\right)\int\frac{dk^3}{(2\pi)^3}\text{tr}\left[\begin{pmatrix}\not{k}-m_L & \hat{\varphi}(k)\\ \hat{\varphi}^*(k) & \not{k}-m_U\end{pmatrix}^{-1}\begin{pmatrix}0 & \dfrac{\delta\hat{\varphi}(k)}{\delta\varphi(0)}\\ 0 & 0\end{pmatrix}\right], \tag{2.17}$$

evaluated at the point where the φ's vanish.

Notice that although the bare value of the coupling constants are originally the same and equal to g_3, they are associated with operators which do not scale in the same way. Their critical values are different as well. As an abuse of notation, from now on we are calling "scalar" the interaction channel given by the field φ_3, "frequency-vector channel" the field φ_0, and "spatial-vector channels" the ones given by φ_1 and φ_2.

Without coupling between the planes we have a degenerate situation between a state in which both planes have the same amount of frustration (i.e., the fermion masses are the same in magnitude) but their relative sign could be the same or opposite. We are going to call these two states ferromagnetic (FM) or antiferromagnetic (AFM) ordered, respectively, understanding that we refer to the relative ordering of the sign of the chiralities. We want to investigate how the degeneracy between the FM and the AFM arrangement of masses is removed. For simplicity, we give the results for the case of $|m_L|=|m_U|=m>0$. They may carry any sign. We define the variable $s=\text{sgn}(m_L)\text{sgn}(m_U)$, which takes values ± 1. The critical values for the coupling constants are given by

$$\frac{1}{g_3^c}=i\ \text{tr}[\hat{\mathbf{S}}_L(k)\tau_3\hat{\mathbf{S}}_U(k)\tau_3]=\frac{1}{2\sqrt{\pi}}\Lambda-\frac{1}{2\pi}m(1+s), \tag{2.18}$$

$$\frac{1}{g_0^c}=i\ \text{tr}[\hat{\mathbf{S}}_L(k)\gamma_0\hat{\mathbf{S}}_U(k)\gamma_0]=\frac{1}{2\pi}m(1-s), \tag{2.19}$$

$$\frac{1}{g_j^c}=i\ \text{tr}[\hat{\mathbf{S}}_L(k)\gamma_j\tau_j\hat{\mathbf{S}}_U(k)\gamma_j\tau_j]=\frac{1}{4\sqrt{\pi}}\Lambda-\frac{1}{2\pi}m(1-s), \tag{2.20}$$

where $\hat{\mathbf{S}}_a(k)\equiv 1/(\not{k}-m_a)$, with $a=L,U$ and $j=1,2$.

When the interaction between the planes is antiferromagnetic (i.e., $J_3>0$) the physical coupling constants remain positive. We are interested in the regime where $m<<\Lambda$. For the case of an AFM relative ordering of chiralities (i.e., for $s=-1$) we obtain

$$\frac{1}{g_3^c}=\frac{\Lambda}{2\sqrt{\pi}},\quad \frac{1}{g_0^c}=\frac{m}{\pi},\quad \frac{1}{g_j^c}=\frac{\Lambda}{4\sqrt{\pi}}-\frac{m}{\pi}. \tag{2.21}$$

For $m<\Lambda$, we have $g_3^c<g_j^c<g_0^c$ hence the channel which will first undergo a transition within the mean-field approximation is the scalar channel, given by the field φ_3.

On the other hand, for the case of FM relative ordering of chiralities, we obtain

$$\frac{1}{g_3^c}=\frac{\Lambda}{2\sqrt{\pi}}-\frac{m}{\pi},\quad \frac{1}{g_0^c}=0,\quad \frac{1}{g_j^c}=\frac{\Lambda}{4\sqrt{\pi}}. \tag{2.22}$$

For $m<\Lambda$, again we have $0<g_3^c<g_j^c<g_0^c$. Again the channel which will first undergo a transition, if any, will be the scalar one.

In the case of ferromagnetic interplane coupling (i.e., in the case $J_3<0$) there is no transition, since the critical coupling constants always remain positive. The exact values of the critical coupling constants are not universal and they depend on the cutoff procedure that is being used. Our continuum approximation is not very sensitive to these short-distance features. However, the theory has a natural built-in regulator since the model comes from a lattice theory. In other words, the qualitative feature of the existence of critical values for the coupling constants is independent of the type of cutoff procedure, although their precise value is not. The question of whether these critical values can be physically reachable is a different issue that needs a more detailed specification of the short-distance properties of the model. We do not attempt to address this point here. We obtain the regularized saddle-point equations by subtracting the value of $1/g_c$ on both sides of Eq. (2.16),

$$\left(\frac{1}{g_3}-\frac{1}{g_j^c}\right)\varphi_j^*=-i\left\{\text{tr}\left[\hat{\mathbf{S}}_L\frac{\delta\hat{\varphi}}{\delta\varphi_j}\hat{\mathbf{S}}\hat{\varphi}^*(1-\hat{\mathbf{S}}_L\hat{\varphi}\hat{\mathbf{S}}_U\hat{\varphi}^*)^{-1}\right]\right.$$
$$\left.-\varphi_j^*\text{tr}\left[\hat{\mathbf{S}}_L\frac{\delta\hat{\varphi}}{\delta\varphi_j}\hat{\mathbf{S}}_U\frac{\delta\hat{\varphi}^*}{\delta\varphi_j^*}\right]\right\}. \tag{2.23}$$

The simplest nontrivial solution is the one where only the scalar channel φ_3 is condensed. This channel has the lowest critical coupling, and it will be the first to pick a nonvanishing expectation value. For an antiferromagnetic relative ordering of the chiralities, which we will show it is favored in the case of antiferromagnetic Heisenberg exchange between the planes, we find

$$\left(\frac{1}{g_3}-\frac{1}{g_3^c}\right)\varphi_3^*=4i\varphi_3^*\int\frac{dk^3}{(2\pi)^3}\left\{\frac{1}{(k_\mu k^\mu-m^2-|\varphi_3|^2)}\right.$$
$$\left.-\frac{1}{k_\mu k^\mu-m^2}\right\}. \tag{2.24}$$

When solving Eq. (2.24) one gets

$$\lambda_3=[m-\sqrt{m^2+|\varphi_3|^2}]\ . \tag{2.25}$$

In Eq. (2.25), $\lambda_3\equiv\pi/g_3-\pi/g_3^c$ is the distance to the critical point. This is our equation of state. The nontrivial solution is

$$|\varphi_3|^2=\lambda_3(\lambda_3-2m). \tag{2.26}$$

It is clear from Eq. (2.25) that $\lambda_3\leqslant 0$. When $\lambda_3<0$, i.e., when $g_3>g_3^c$, we find a phase where the scalar channel field has a nonvanishing expectation value given by Eq. (2.26).

The physics of this state is the following. The fact that φ_3 acquires an expectation value means that, on average, the interlayer Hubbard-Stratonovich field is different from zero. Thus, it appears that in this state the fermions from one layer are free to go onto the other layer. However, the corrections to this mean-field picture should, among other things, enforce the constraint of single-occupancy at each site of each layer. The only state which is compatible with the single-occupancy constraint *and* with interlayer fermion hopping is a state in which, on *each* link between the two layers, there is a *spin-singlet* or *valence-bond* state. Thus, the phase transition that we found is a transition between two CSL states on each layer (with antiferromagnetic ordering of the relative chiralities) and a *spin gap* state with spin singlets on the interlayer links. A number of recent works[6–8] have predicted a similar phase transition in bilayers but between Néel states and spin gap state with properties which are virtually indistinguishable from ours.

III. RELATIVE ORDERING OF CHIRALITIES

In this section we show that there exists a dynamical way in which the physical system selects a particular ordering of the chiralities in the planes. We assume that in each plane a CSS is stabilized. Thus, at each plane both Dirac fermion species are coupled to the mass term with the same sign. We assume that the mass is the same for both fermionic flavors in each particular plane, say, m_L and m_U, respectively. This is consistent with the fact that there is no explicit anisotropy present. As in Sec. II, the magnitudes of the masses are the same but their signs could be either the same or opposite. We neglect fluctuations of the NN amplitude of the HS fields inside the planes, which can generate a difference between the masses of the Dirac species inside each plane, and even drive the CSS into a dimer phase (see, for example, Ref. 12).

Our goal is to compute the correction to the energy of the ground state of the bilayer system, due to the quantum fluctuations of the fields coupling the planes. We work in the phase where no field is condensed. Thus, the effective action derived in Sec. II will describe the fluctuating part of these bosonic fields with zero expectation value. The strategy is, therefore, to expand this action in powers of (the small fluctuating part of) the fields φ_0 to φ_3 and keep up to the Gaussian terms, then integrate the bosonic fields out, and, after reexponentiating the expression, obtain the desired correction to the ground-state energy density. This correction will contain a divergent part which is symmetric in the sign of the masses of the fermions in different planes and a finite contribution which is a function of the fermion masses of both planes, $m_{L,U}$, with their signs. At this point, we look for the configuration of masses which minimizes the energy. The case of zero mass at any plane is excluded since we assumed beforehand that a CSS is stabilized at each plane. This is important since these masses provide the energy gap which is necessary for our saddle-point approximation to be stable and to allow for a semiclassical expansion.

The integration over the fermionic degrees of freedom gives the following contribution to the effective action [see Eq. (2.14)]. We have

$$-iN\ \mathrm{Tr}\ \ln\begin{bmatrix} i\partial\!\!\!/ - m_L & \hat{\varphi} \\ \hat{\varphi}^* & i\partial\!\!\!/ - m_U \end{bmatrix} = -iN\ \mathrm{Tr}\ \ln\begin{pmatrix} i\partial\!\!\!/ - m_L & 0 \\ 0 & i\partial\!\!\!/ - m_U \end{pmatrix} + iN\sum_{n=1}^{\infty}\frac{1}{n}\ \mathrm{Tr}\{(\hat{\mathbf{S}}\hat{\mathbf{Q}})^n\}. \tag{3.1}$$

Here

$$\hat{\mathbf{S}} = \begin{pmatrix} i\partial\!\!\!/ - m_L & 0 \\ 0 & i\partial\!\!\!/ - m_U \end{pmatrix}^{-1} \quad \text{and} \quad \hat{\mathbf{Q}} = \begin{pmatrix} 0 & -\hat{\varphi} \\ -\hat{\varphi}^* & 0 \end{pmatrix}.$$

At this point it is convenient to rescale the fluctuating fields by $1/\sqrt{N}$. Under this transformation all the terms in Eq. (3.1) that are quadratic in the fields φ's and $\mathscr{S}_b$ as expanded in Eq. (2.13) become contributions of $\mathscr{O}(1)$, being the classical energy of the ground state (i.e., the classical part of the euclidean action) of $\mathscr{O}(N)$. To study the selection of the ordering of chiralities we need to compute this $\mathscr{O}(1)$ correction to the ground-state energy due to the effect of the fluctuations of the fields coupling the planes. We first need to calculate the one-loop contribution to the fermion determinant. There is only one diagram to this order, which has two external bosonic legs and two internal fermion propagators,

$$\begin{aligned} i\frac{1}{2}\int dx^3(\hat{\mathbf{S}}\hat{\mathbf{Q}}\hat{\mathbf{S}}\hat{\mathbf{Q}}) &= i\frac{1}{2}\int\frac{dk^3}{(2\pi)^3}\int\frac{dq^3}{(2\pi)^3} \\ &\quad\times \mathrm{Tr}[\hat{\mathbf{S}}(k)\hat{\mathbf{Q}}(q)\hat{\mathbf{S}}(k-q)\hat{\mathbf{Q}}(-q)] \\ &\equiv \frac{1}{2}\int\frac{dq^3}{(2\pi)^3}\mathscr{K}^{(j)}(q)\varphi_j^*(q)\varphi_j(q). \end{aligned} \tag{3.2}$$

As we saw before, this diagram has an ultraviolet divergence which will be absorbed in a renormalization of the coupling constants. So the kernels $\mathscr{K}^{(j)}(q)$ in Eq. (3.2) include both the finite part of the diagram and a contribution linearly divergent in the integration momentum. The computation of $\mathscr{K}(q)$, although rather cumbersome, is fairly straightforward. Let us recall that we have four channels: φ_3 can be regarded as a scalarlike coupling to the Dirac fermions; the other three, φ_0–φ_2, resemble a gauge-field-like coupling. This is not the case, however, since Lorentz invariance is broken by the presence of the τ matrices in the expression for $\hat{\mathbf{Q}}$. This point is crucial. Since we do not have to preserve Lorentz invariance when regulating the divergent diagrams, time and space components do not enter on equal grounds. Our theory is in fact the continuum limit of a lattice theory. At that level it is very clear that the only physically sensible cutoff at hand is the inverse lattice spacing. As a result, our regulating procedure consists of integrating over frequency first and then using an isotropic Gaussian cutoff for the spatial part of the momentum. In this way we expect to recover the qualitative features of the (finite) lattice theory in the continuum limit. Let us also mention that the only two spatially symmetric combinations of the interlayer amplitudes within a plaquette [see Eq. (2.12)] are given by φ_3 and φ_0.

From now on, the expressions will be given in their Wick-rotated (i.e., imaginary time) form. Consequently $q^2 \equiv q_0^2 + q_1^2 + q_2^2$, where $q_0 = -i\omega$. We obtain (see Appendix A)

$$\mathscr{K}^{(3)}(q)=\frac{\Lambda}{2\sqrt{\pi}}-\frac{1}{2\pi}\left\{\frac{1}{2}[q^2+m^2(1+s)]\mathscr{I}_0\right\}, \tag{3.3}$$

$$\mathscr{K}^{(0)}(q)=\frac{1}{2\pi}\left\{m(1+2\kappa_0^2+3\kappa_0^4)+\frac{q^2}{2m}(\kappa_0^2-\kappa_0^4)+\frac{1}{2}\mathscr{I}_0\left[2m^2s+4m^2\left(\kappa_0^2+\frac{3}{2}\kappa_0^4\right)-\frac{q^2}{2}(1+3\kappa_0^4)\right]\right\}, \tag{3.4}$$

$$\mathscr{K}^{(j)}(q)=\frac{\Lambda}{4\sqrt{\pi}}-\frac{1}{2\pi}\left\{m(1+2\kappa_j^2+3\kappa_j^4)+\frac{q^2}{2m}(\kappa_j^2-\kappa_j^4)+\frac{1}{2}\mathscr{I}_0\left[2m^2s+4m^2\left(\kappa_j^2+\frac{3}{2}\kappa_j^4\right)-\frac{q^2}{2}(1+3\kappa_j^4)\right]\right\}. \tag{3.5}$$

In Eq. (3.4) and Eq. (3.5) $\kappa_j^2\equiv q_j^2/q^2$, with $j=0,1,2$. Notice that the expression corresponding to the channel given by φ_0 (loosely speaking, the frequency channel) has an overall opposite sign to the expression for the channels given by φ_1 and φ_2 for the finite part of the diagrams. However, the frequency channel does not have a divergent contribution. This sign will turn out to be quite important for the phase diagram.

On the other hand, $\mathscr{I}_0$ is

$$\mathscr{I}_0=\frac{2}{|q|}\left\{\sin^{-1}\left(\frac{|q|}{\sqrt{4m^2+q^2}}\right)\right\}. \tag{3.6}$$

At this point, in Euclidean space, we have

$$\begin{aligned}\mathscr{Z}&=\int\mathscr{D}b\ \exp\left(-E_0-\sum_{j=0}^{3}\int\frac{d^3q}{(2\pi)^3}\varphi_j^*(q)[\lambda-\mathscr{K}^{(j)}(q)]\varphi_j(q)\right)\\ &=e^{-E_0}\prod_{j=0}^{3}\int\mathscr{D}\varphi_j^*\mathscr{D}\varphi_j\exp\left\{-\int\frac{d^3q_E}{(2\pi)^3}\varphi_j^*(q)[\lambda-\mathscr{K}_E^j(q)]\varphi_j(q)\right\}\\ &=\#e^{-E_0}\exp\left\{-\sum_{j=0}^{3}\int_q\ln[\lambda-\mathscr{K}^{(j)}(q)]\right\}.\end{aligned} \tag{3.7}$$

From Eq. (3.7), the correction to the energy of the ground state due to the fluctuations of the fields φ's is given by

$$\begin{aligned}\Delta E&=\int\frac{d^3q}{(2\pi)^3}\ln\left[\lambda-\frac{\Lambda}{2\sqrt{\pi}}+\frac{1}{2\pi}\left(2m+\frac{1}{2}[q^2+m^2(1+s)]\mathscr{I}_0\right)\right]+\int\frac{d^3q}{(2\pi)^3}\ln\left(\lambda-\frac{1}{2\pi}\left\{m(1+2\kappa_0^2+3\kappa_0^4)+\frac{q^2}{2m}(\kappa_0^2-\kappa_0^4)\right.\right.\\ &\quad\left.\left.+\frac{1}{2}\mathscr{I}_0\left[2m^2s+4m^2\left(\kappa_0^2+\frac{3}{2}\kappa_0^4\right)-\frac{q^2}{2}(1+3\kappa_0^4)\right]\right\}\right)+\sum_{j=1}^{2}\int\frac{d^3q}{(2\pi)^3}\ln\left[\lambda-\frac{\Lambda}{4\sqrt{\pi}}-\frac{1}{2\pi}\left\{m(1+2\kappa_j^2+3\kappa_j^4)\right.\right.\\ &\quad\left.\left.+\frac{q^2}{2m}(\kappa_j^2-\kappa_j^4)+\frac{1}{2}\mathscr{I}_0\left[2m^2s+4m^2\left(\kappa_j^2+\frac{3}{2}\kappa_j^4\right)-\frac{q^2}{2}(1+3\kappa_j^4)\right]\right\}\right).\end{aligned} \tag{3.8}$$

We want to study the weak coupling regime, which corresponds to the case of large λ in Eq. (3.8). Moreover, this is presumably the only regime for which Eq. (3.8) is valid, since as we show later, there is a critical value of the coupling constant at which there is an onset of condensation for some of the interaction channels between the planes.

By expanding in powers of $1/\lambda$, to first order we obtain that the energy correction does not depend on the relative sign of the masses s and it is completely symmetric with respect to the exchange m_L into m_U. This result remains true even when the magnitude of the masses are different. To second order we get

$$\Delta E^{(2)}=f_{\text{symm}}+\frac{1}{\lambda^2}\int\frac{d^3q}{(2\pi)^3}\left(\frac{m^2}{4\pi^2}\mathscr{I}_0\right)s\left[4m-\frac{4}{3}\frac{mq^2}{4m^2+q^2}+2q^2\left(1-\frac{4m^2}{q^2}\right)\mathscr{I}_0\right]. \tag{3.9}$$

The coefficient of s, where s is the relative sign of the masses (i.e., of the chiralities), is a function always positive. Thus, a minimum in the energy is obtained when $s=-1$, which indicates that the chiralities of the planes have opposite sign. This is the main result of this section. Recently, Gaitonde, Jajktar, and Rao[9] studied the problem of the selection of the relative chirality by means of a perturbation theory in the interlayer exchange coupling. They found that the ferromagnetic ordering was selected and that this result only appeared in third order in J_3. This result disagrees with ours [see Eq. (3.9)]. It is unclear to us what is the origin of this discrepancy. The work by Gaitonde *et al.* relies on a rather complex lattice perturbation theory calculation of the interlayer correlation effects. In our work we have evaluated the same correlation effects but within a continuum approximation which makes the computation more transparent and easy to check. We have used a cutoff only for the space components of the momentum transfers in our Feynman diagrams. The form of the cutoff that we chose closely mimics the effects of the lattice. Thus, it is unlikely that the discrepancy could be due to different choices of cutoffs. Similarly, the discrepancy appears at very weak interlayer coupling where $J_3 \ll |m|$, where $|m|$ is the magnitude of the mass of the chiral excitations on each layer. Although it is conceivable that this discrepancy could be due to highly energetic processes which may be treated differently by both cutoff procedures, this appears to be unlikely since the mass $|m|$ is very large in this regime. Barring some numerical difficulty (which is possible in such involved calculations), the absence of a correction which depends on the relative sign of the mass in the lattice calculation [to the same order as the one given by Eq. (3.9)] points to the occurrence of a special cancellation which we do not see in the effective continuum theory. We have also checked our result with other choices of cutoff on the space components and we have always found the same effect. Only in one instance, when we used a relativistic form of the cutoff, isotropic in both space and time, we found it necessary to go to third order in J_3, which resembles the result reported by Gaitonde *et al.*, but even in that case we found that the *antiferromagnetic* ordering of chiralities is the one energetically favored. However, the relativistic cutoff is certainly the one which is most unlike the lattice cutoff. In view of these considerations, we strongly believe that our treatment is robust and reliable.

IV. LANDAU-GINZBURG EFFECTIVE THEORY

We want to study the behavior of the low-energy modes for this system. The approach we are taking here is to derive an effective theory for the fluctuations of the φ fields and the gauge fields. We want to study and characterize the phase diagram at the tree level approximation or Landau-Ginzburg approximation, and further on, investigate the effects of the fluctuations. We showed that there exist critical values for the coupling constants which possibly mark a transition between a symmetric or noncondensed phase for the φ fields and a phase in which at least the scalar channel acquires an expectation value. The Landau-Ginzburg theory to be derived in this section will allow us to study the actual nature of this phase transition. We expand the fermionic determinant in a gradient expansion for slow varying modes of the fields in which we are interested.

We derive an effective action only for the scalar channel. This particular channel is the one that first undergoes a condensation, for the case of an antiferromagnetic ordering of the chiralities, since it has the lowest critical coupling constant with a positive value. The other three channels will remain massive modes and consequently they can be integrated out of the theory. This process will involve renormalization of the parameters of the system but it will not affect dramatically the underlying physics. On the contrary, the scalar channel effectively undergoes a transition as the critical value of the coupling constant is approached and crossed. The bosonic excitations become massless at the transition point and we want to study the physics on both sides of this transition. We use the definitions $A_{+}^{\mu}(x) \equiv A_{L}^{\mu}(x) + A_{U}^{\mu}(x)$ and $A_{-}^{\mu}(x) \equiv A_{L}^{\mu}(x) - A_{U}^{\mu}(x)$ for the in-phase and out-of-phase gauge fields, respectively. The covariant derivative is defined as $\mathscr{D}_{\mu} \equiv \partial_{\mu} - iA_{\mu}^{-}$.

The details of the calculation are described roughly in Appendix B. The following effective action is obtained by Fourier antitransforming the contributions of the one-loop diagrams up to order $1/N$, where N is the fermion species number. This includes bubble diagrams with up to four legs, since each of these legs represents the fluctuating part of either a matter or a gauge field, which has been previously rescaled by a factor $1/\sqrt{N}$. The loop integration adds a factor of N coming from the number of fermions propagating in the loop. From these diagrams we keep terms up to second order in the external momenta. In real space we find various terms; we get a contribution involving only the gauge fields which we call $\mathscr{S}_{\text{gauge}}^{(0)}$. This arises from the fermion loops corresponding to the propagation of spinon-hole pairs inside each plane, without mixing. It contains the usual square of the field strength tensor and the induced Chern-Simons term. In the $A_{(+)}$- $A_{(-)}$ coordinates this term is off diagonal, since the sign of time-reversal invariance is opposite between the planes,

$$\mathscr{S}_{\text{gauge}}^{(0)}(x) = \frac{1}{16\pi}\int dx^3 \epsilon_{\mu\nu\lambda}[F_{(+)}^{\mu\nu}(x)A_{(-)}^{\lambda}(x) + F_{(-)}^{\mu\nu}(x)A_{(+)}^{\lambda}(x)] - \frac{1}{64\pi|m|}\int dx^3[F_{(+)}^{\mu\nu}(x)F_{\mu\nu}^{(+)}(x) + F_{(-)}^{\mu\nu}(x)F_{\mu\nu}^{(-)}(x)]. \tag{4.1}$$

The following term has a free part for the field φ and another part coupling this field to the gauge fields. A term coupling the gauge-invariant current for the matter field φ to the field strength tensor of the in phase gauge field is also present,

$$\mathscr{S}_{\varphi}^{(1)}(x) = \frac{1}{4\pi|m|}\int dx^3\left[\left(\partial_{\mu} + \frac{i}{\sqrt{N}}A_{\mu}^{-}(x)\right)\varphi^*(x)\left(\partial^{\mu} - \frac{i}{\sqrt{N}}A_{-}^{\mu}(x)\right)\varphi(x)\right] - \frac{1}{32\pi}\frac{1}{|m|^2}\int dx^3 \epsilon_{\mu\nu\lambda}F_{(+)}^{\mu\nu}(x)\mathbf{J}_{(-)}^{\lambda}(x). \tag{4.2}$$

In Eq. (4.2) we have defined the current operator for the field φ as

$$\mathbf{J}_{\lambda}^{(-)}(x) = i[\varphi^*(x)\partial_{\lambda}\varphi(x) - \varphi(x)\partial_{\lambda}\varphi^*(x)] + \frac{2}{\sqrt{N}}A_{\lambda}^{(-)}(x)|\varphi(x)|^2. \tag{4.3}$$

Notice that all the terms are manifestly gauge invariant as it should be, since this symmetry was present before we integrated out the fermions. Notice also that the matter field couples only to the out-of-phase or relative gauge field. This is consistent with the symmetry of plane exchange which remains intact if the magnitude of the fermion mass is the same on both planes. In other words, the original theory was invariant under the exchange of A_L and A_U and the sign of the masses. This invariance should remain at this level for our approximation to be consistent. However, $A_{(-)}$ changes sign under this operation. This amounts to a reversal of the sign of the charge, or charge conjugation, and consequently φ has to be conjugated. This renders the covariant derivative term and the gauge-invariant current unchanged. On the other hand, $F^{\mu\nu}_{(+)}$ is invariant under plane exchange. All the other terms are even on $A_{(-)}$ and our effective action verifies the plane interchange symmetry. Finally, from the contributions coming from the four-leg diagrams which are of second order in the external momenta we can derive the higher-derivative terms

$$\mathscr{S}^{(2)}(x)=\frac{1}{16\pi}\frac{1}{|m|^3}\left\{\int dx^3|\mathscr{D}^2\varphi(x)|^2-\frac{2}{3}\int dx^3[\{\mathscr{D}_\mu,\mathscr{D}_\nu\}\varphi(x)]^*[\{\mathscr{D}^\mu,\mathscr{D}^\nu\}\varphi(x)]+\frac{1}{2}\int dx^3F^{(-)}_{\mu\nu}(x)F^{\mu\nu}_{(-)}(x)|\varphi(x)|^2\right.$$
$$\left.+\frac{1}{6}\int dx^3F^{(+)}_{\mu\nu}(x)F^{\mu\nu}_{(+)}(x)|\varphi(x)|^2\right\}. \tag{4.4}$$

We also get a self-interacting term for φ given by

$$\mathscr{S}_{\rm self}(x)=-\frac{1}{N}\frac{1}{4\pi|m|}\int dx^3|\varphi(x)|^4-\left(\frac{1}{g}-\frac{1}{g_c}\right)\int dx^3|\varphi(x)|^2. \tag{4.5}$$

In Eq. (4.5) above, we use the definition for g_c introduced in Sec. II, i.e., $1/g_c=\Lambda/2\sqrt{\pi}$. In order to rewrite this effective action in a simplified way we introduce some field rescaling and define the coupling costants

$$\varphi(x)\equiv\sqrt{4\pi|m|}\,\phi(x),\quad A^\mu_{(-)}(x)\equiv\sqrt{N}A^\mu_{(-)},\quad m_0^2\equiv4\pi|m|\left(\frac{1}{g}-\frac{1}{g_c}\right),\quad \lambda\equiv\frac{4\pi|m|}{N},$$

$$\theta\equiv\frac{N}{2\pi},\quad \mathbf{e}^2\equiv\frac{16\pi|m|}{N},\quad G_A\equiv\frac{1}{8|m|},\ \text{and}\ \bar{G}\equiv\frac{1}{4m^2}.$$

By plugging all of these in, we obtain

$$\mathscr{L}_{\rm gauge}=\frac{\theta}{8}[\tilde{F}^{(+)}_\lambda(x)A^\lambda_{(-)}(x)+\tilde{F}^{(-)}_\lambda(x)A^\lambda_{(+)}(x)]-\frac{1}{4\mathbf{e}^2}[F^2_{(+)}(x)+F^2_{(-)}(x)], \tag{4.6}$$

where $\tilde{F}_\lambda\equiv\epsilon_{\mu\nu\lambda}F^{\mu\nu}$ is the dual of the field strength tensor. We also get a Lagrangian density for the field ϕ given by

$$\mathscr{L}_\phi=\partial^\mu\phi^*\partial_\mu\phi-m_0^2|\phi|^2-\lambda|\phi|^4+\mathscr{L}_I, \tag{4.7}$$

where

$$\mathscr{L}_I\equiv i[\phi^*(x)\partial_\mu\phi(x)-\phi(x)\partial_\mu\phi^*(x)][A^\mu_{(-)}(x)-G_A\tilde{F}^\mu_{(+)}(x)]$$
$$+|\phi(x)|^2\left[A^2_{(-)}(x)-2G_A\tilde{F}^\mu_{(+)}(x)A^{(-)}_\mu(x)+\bar{G}\left(\frac{1}{2}F^2_{(-)}(x)+\frac{1}{6}F^2_{(+)}(x)\right)\right]. \tag{4.8}$$

In Eq. (4.8) we dropped the higher-derivative terms which appear in Eq. (4.4), except for the antisymmetric parts which involve renormalizations of the (different) effective charges for the in-phase and out-of-phase gauge fields.

V. SYMMETRIC PHASE

In this section we want to study the physics of the regime in which there is no condensation of the field ϕ for the effective theory derived in the previous section, i.e., where ϕ has a vanishing vacuum expectation value. This phase consists of the bilayer system with relatively opposite broken time-reversal invariance between both planes, but the difference with the case of decoupled planes is that they are now linked through the fluctuations of the field ϕ. This field represents a massive boson-like mode with mass given by m_0 defined in the previous section by

$$m_0^2=\frac{1}{g}-\frac{1}{g_c}=\frac{v_F}{(2a)^2J_3}-\frac{\Lambda}{2\sqrt{\pi}}=\frac{\rho}{2aJ_3}-\frac{\Lambda}{2\sqrt{\pi}}. \tag{5.1}$$

The magnitude of this mass measures the distance to the critical point. In this phase we are on the side of the transition in which $m_0^2>0$. It clearly corresponds to a weakly interplane coupling regime, i.e., the limit of small J_3. The ϕ field can be integrated out to get an effective action for the gauge fields only. However, for our approximations to be consistent we need to assume that m_0^2 is much smaller that the fermion mass m. In other words, our results are valid on a window not to close to the phase transition (where ϕ becomes massless as $g\to g_c$ and $m_0\to 0$) but also not too far from the transition so that the mass of the collective mode represented by ϕ never becomes comparable to the fermion mass.

We are going to show that there is no renormalization of coefficient of the Chern-Simons terms that had been induced by the fermionic fluctuations on the planes, arising from the fluctuations of ϕ, at least to order $1/N$. There are "charge" renormalizations in the sense that the coefficients of the field strength tensor for both $A_{(+)}$ and $A_{(-)}$ get renormalized. Furthermore, we will show that the spectrum of low-energy excitations in this phase has two massive photons, whose masses do not violate the gauge symmetry but they break parity and time-reversal invariance, and are very effective in taming the fluctuations of the gauge fields. In a sense we still have pretty much the same physical picture corresponding to two decoupled chiral spin liquid with opposite relative breaking of time-reversal invariance. Consequently we will still have deconfined spinon as the elementary excitations of the system. The issue of the statistic of the quasiparticles is a little more involved as we discuss below.

Starting from the effective action derived in the previous section we can integrate out perturbatively the field ϕ. This gives a result valid within the region of applicability of the gradient expansion.

The integration over ϕ gives the effective action (for small A_μ) We have

$$\begin{aligned}\int \mathscr{D}\phi\mathscr{D}\phi^* \exp i\mathscr{S}(\phi,A_\mu) &= \exp[i\mathscr{S}_{\text{gauge}}(A_\mu)]\int \mathscr{D}\phi\mathscr{D}\phi^*\exp\left[i\int dx^3\mathscr{L}_\phi\left(1+i\mathscr{L}_I-\frac{1}{2}(\mathscr{L}_I)^2\right)\right]\\ &= Z_0\exp(i\mathscr{S}_{\text{gauge}})\Bigg\{1+i\langle i(\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^*)\rangle[A^\mu_{(-)}-G_A\tilde{F}^\mu_{(+)}]\\ &\quad +i\langle|\phi|^2\rangle\left[A^2_{(-)}-2G_A\tilde{F}^\mu_{(+)}A^{(-)}_\mu+\bar{G}\left(\frac{1}{2}F^2_{(-)}+\frac{1}{6}F^2_{(+)}\right)\right]\\ &\quad -\frac{1}{2}\langle i^2(\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^*)(\phi^*\partial_\nu\phi-\phi\partial_\nu\phi^*)\rangle[A^\mu_{(-)}-G_A\tilde{F}^\mu_{(+)}][A^\nu_{(-)}-G_A\tilde{F}^\nu_{(+)}]\Bigg\}. \qquad (5.2)\end{aligned}$$

The cumulant coefficients can be computed in the usual way[12] to find

$$i\langle|\phi(x)|^2\rangle=-\int dq^3\frac{1}{q^2-m_0^2}=-\#_{\text{sing}}-\frac{i}{4\pi}|m| \quad (5.3)$$

and

$$-\frac{1}{2}\langle[i(\phi^*\partial_\lambda\phi-\phi\partial_\lambda\phi^*)][i(\phi^*\partial_\eta\phi-\phi\partial_\eta\phi^*)]\rangle$$

$$=\left(\#_{\text{sing}}+\frac{i}{4\pi}|m|\right)g_{\lambda\eta}. \quad (5.4)$$

Both integrals in Eq. (5.3) and Eq. (5.4) have a linear ultraviolet divergence and need to be regularized. One can use any of the usual regulators, for example Pauli-Villars or minimal subtraction (which is equivalent to an analytical continuation of the negative argument γ function.) However, the finite part of both integrals after we treated them with the same regulating scheme is exactly the same but with opposite sign. This should be the case since it is required to preserve gauge invariance. In other words, we cannot generate a $A_\mu A^\mu$ term in the symmetric phase because such a term would manifestly break gauge invariance and we know this is not the case. Therefore, the term $A^\lambda A_\lambda\times\langle|\phi|^2\rangle$ in the right-hand side (RHS) of Eq. (5.2) should cancel exactly (and it does) the term $A^\lambda A^\eta\langle i^2(\phi^*\partial_\lambda\phi-\phi\partial_\lambda\phi^*)(\phi^*\partial_\eta\phi-\phi\partial_\eta\phi^*)\rangle$. Notice that by the same token the term which could have given a renormalization of the cross Chern-Simons terms gets canceled. In a sense, it is also gauge invariance which prevents the cross Chern-Simons terms from being renormalized.

A minimal subtraction procedure will consist in the complete removal of the singular part. In fact, any cutoff procedure which preserves gauge invariance would work as well. It can be shown that our regularization prescription is entirely equivalent to the introduction of a Gaussian spherical cutoff in the imaginary frequency (or Euclidean) reciprocal phase space. A term of the form $\exp[-(\pi/\Lambda^2)(q_E^2+m_0^2)]$ does the job for us. One should be aware, however, that this cutoff is not exactly the same used in Secs. II and III since there the cutoff was Gaussian isotropic on the spatial components of the momentum but the frequency range was unbounded. Here that cutoff procedure would not work because it breaks gauge invariance. In Secs. II and III gauge invariance was not at stake and we were trying to implement a regularization that resembles closely what happens on a lattice. It should also be noticed that, although aparently the same field φ is involved in both cases, we were dealing before with ultraviolet divergences of a fermion loop integral, while here the field propagating is the bosonic field φ itself. In other words, we were dealing in the previous sections with the self-energy of the field φ while here we are dealing with the self-energy of the photon or the gauge fields. Finally we do get renormalizations for the $F^2_{(+)}$ and $F^2_{(-)}$ terms.

After this procedure is applied, we are left with the regularized (finite) form of Eq. (5.2),

$$Z_{\rm reg} = \exp(i\mathscr{S}_{\rm gauge})\left\{1+\frac{i}{4\pi}|m|\left[\frac{\bar{G}}{2}F^2_{(-)}+\frac{\bar{G}}{6}F^2_{(+)}-2G_A^2F^2_{(+)}\right]\right\}$$
$$\approx \exp\left(i\int dx^3\mathscr{L}_{\rm eff}\right), \tag{5.5}$$

where

$$\mathscr{L}_{\rm eff}=\frac{\theta}{8}(\tilde{F}^{(+)}_{\lambda}A^{\lambda}_{(-)}+\tilde{F}^{(-)}_{\lambda}A^{\lambda}_{(+)})-\left(\frac{1}{4\mathbf{e}^2}-\frac{1}{4\pi}|m|\frac{\bar{G}}{2}\right)F^2_{(-)}-\left(\frac{1}{4\mathbf{e}^2}-\frac{1}{4\pi}|m|\left[\frac{\bar{G}}{6}-2G_A^2\right]\right)F^2_{(+)}. \tag{5.6}$$

In Eq. (5.6) we used that $\tilde{F}^{(+)}_{\lambda}\tilde{F}^{\lambda}_{(+)}=2F^2_{(+)}$.

We now explore the energy momentum dispersion relation. The low-energy collective modes are fluctuations of the gauge fields. We will show that there exists a photonlike mode but it is massive. This is of great importance for the survival of spinons in the energy spectrum (see for example Ref. 12). The regularized (finite) theory has the form

$$\mathscr{S}_{\rm eff}(A_\mu)=\int dx^3\left[-c_-F^2_{(-)}-c_+F^2_{(+)}+\frac{\theta}{8}\epsilon_{\mu\nu\lambda}(F^{\mu\nu}_{(+)}A^{\mu}_{(-)}+F^{\mu\nu}_{(-)}A^{\mu}_{(+)})\right], \tag{5.7}$$

where

$$c_-=\frac{1}{64\pi|m|}(N-2)$$

and

$$c_+=\frac{1}{64\pi|m|}\left(N-\frac{1}{6}\right).$$

In momentum space we have

$$\mathscr{S}_{\rm eff}(A_\mu)=-\int_p[A^{\mu}_{(-)}(p)A^{\nu}_{(+)}(p)]\begin{bmatrix}2c_-(p^2g_{\mu\nu}-p_\mu p_\nu) & i\frac{\theta}{4}\epsilon_{\mu\lambda\nu}p^\lambda\\ i\frac{\theta}{4}\epsilon_{\mu\lambda\nu}p^\lambda & 2c_+(p^2g_{\mu\nu}-p_\mu p_\nu)\end{bmatrix}\begin{pmatrix}A^{\mu}_{(-)}(-p)\\ A^{\nu}_{(+)}(-p)\end{pmatrix}$$
$$\equiv\int_p A^{\mu}_{a}(p)[c_0(p^2g_{\mu\nu}-p_\mu p_\nu)\mathbf{I}^{ab}+c_3(p^2g_{\mu\nu}-p_\mu p_\nu)\mathbf{T}^{ab}_3+\kappa_0\epsilon_{\mu\lambda\nu}p^\lambda\mathbf{T}^{ab}_1]A^{\nu}_b, \tag{5.8}$$

with $a,b=(-),(+)$ and μ,ν the usual Lorentz indices, $c_0=-(c_-+c_+)$, $c_3=-(c_--c_+)$, and $\kappa_0=-i\,\theta/4$.

This is a bilinear form in A^{μ}_a and the propagator for the gauge fields is just the inverse of the matrix shown in Eq. (5.8). However, this matrix is singular unless we fix a gauge. This is so because the gauge field propagator $\langle A^{\mu}_a A^{\nu}_b\rangle$ is not a gauge-invariant operator and does not have a physical meaning unless we are working in a particular gauge. We need to add gauge fixing terms in order get the propagator. We may add for example,

$$-\frac{1}{\alpha}(\partial_\mu A^{\mu}_{(-)})^2$$

and

$$-\frac{1}{\beta}(\partial_\mu A^{\mu}_{(+)})^2,$$

which in momentum space take the simple form

$$-\frac{1}{\alpha}p_\mu p_\nu$$

and

$$-\frac{1}{\beta}p_\mu p_\nu.$$

The three operators $\hat{P}_{\mu\nu}\equiv p_\mu p_\nu$, $\hat{G}_{\mu\nu}\equiv p^2 g_{\mu\nu}$, and $\hat{K}_{\mu\nu}\equiv\epsilon_{\mu\lambda\nu}p^\lambda$ satisfy a closed algebra, and now the matrix can be inverted. After some lengthy though fairly straightforward algebra one gets

$$D(c_-,c_+,\theta,\alpha,\beta)=\begin{pmatrix}\hat{D}^{(-)(-)}_{\mu\nu} & \hat{D}^{(-)(+)}_{\mu\nu}\\ \hat{D}^{(+)(-)}_{\mu\nu} & \hat{D}^{(+)(+)}_{\mu\nu}\end{pmatrix}, \tag{5.9}$$

given by

$$\hat{D}^{(-)(-)}_{\mu\nu}=-\frac{1}{2c_-}\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2-M^2_{\rm ph}}-\alpha\frac{p_\mu p_\nu}{p^2}, \tag{5.10}$$

$$\hat{D}^{(+)(+)}_{\mu\nu} = -\frac{1}{2c_+}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2 - M^2_{\rm ph}} - \beta\frac{p_\mu p_\nu}{p^2}, \tag{5.11}$$

$$\hat{D}^{(-)(+)}_{\mu\nu} = \hat{D}^{(+)(-)}_{\mu\nu} = i\frac{\theta}{4}\frac{1}{4c_- c_+}\frac{1}{p^2}\epsilon_{\mu\lambda\nu}p^\lambda\frac{1}{p^2 - M^2_{\rm ph}}, \tag{5.12}$$

where we defined

$$M_{\rm ph} = \sqrt{\frac{\theta^2}{64 c_- c_+}} = \frac{8\pi|m|\theta}{\sqrt{(N-2)(N-1/6)}} = \frac{4|m|}{\sqrt{\left(1-\frac{2}{N}\right)\left(1-\frac{1}{6N}\right)}} \tag{5.13}$$

as the "photon" mass.[13] To leading order in $1/N$ we can rotate back to the A_L, A_U coordinates to get (in the Lorentz gauge $\alpha=\beta=0$)

$$\hat{D}_{LL} = 64\pi\frac{1}{N}|m|\frac{1}{p^2 - M^2_{\rm ph}}\left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + 4i|m|\epsilon_{\mu\lambda\nu}\frac{p^\lambda}{p^2}\right] \tag{5.14}$$

and

$$\hat{D}_{UU} = 64\pi\frac{1}{N}|m|\frac{1}{p^2 - M^2_{\rm ph}}\left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - 4i|m|\epsilon_{\mu\lambda\nu}\frac{p^\lambda}{p^2}\right]. \tag{5.15}$$

To next order in $1/N$ corrections we find additional off-diagonal symmetric mixing terms.

VI. BROKEN-SYMMETRY PHASE

In this section we want to study the phase where the matter field ϕ condenses. Let us assume that we went through the critical point into the phase where m_0^2 in Eq. (5.1) becomes negative. From Eq. (4.7) we now have another possible solution with finite $\langle\phi\rangle$, which actually minimizes the energy. This is the usual nontrivial solution for a double-well effective potential of a ϕ^4 theory. When m_0^2 becomes negative, the solution $\phi=0$ now becomes a local maximum instead of a minimun. The value of the new local minimun can be obtained by minimizing Eq. (4.7) to be $\bar{\phi}_0^2 = -m_0^2/2\lambda$, where we are using the definitions given in Sec. IV. If we plug in this constant value of ϕ_0, Eq. (4.8) becomes

$$\mathscr{L}_I = \phi_0^2\left[A^2_{(-)}(x) - 2G_A\tilde{F}^\mu_{(+)}(x)A^{(-)}_\mu(x) + \bar{G}\left(\frac{1}{2}F^2_{(-)}(x) + \frac{1}{6}F^2_{(+)}(x)\right)\right]. \tag{6.1}$$

As in the symmetric case we have

$$\mathscr{L}_{\rm gauge} = \frac{\theta}{8}[\tilde{F}^{(+)}_\lambda(x)A^\lambda_{(-)}(x) + \tilde{F}^{(-)}_\lambda(x)A^\lambda_{(+)}(x)] - \frac{1}{4\mathbf{e}^2}[F^2_{(+)}(x) + F^2_{(-)}(x)]. \tag{6.2}$$

Now our effective action for the gauge fields reads

$$\mathscr{S}_{\rm eff}(A_\mu) = \int dx^3\left[\phi_0^2 A^2_{(-)} + \left(\frac{\theta}{8} - \phi_0^2 G_A\right)\times\epsilon_{\mu\nu\lambda}(F^{\mu\nu}_{(+)}A^\mu_{(-)} + F^{\mu\nu}_{(-)}A^\mu_{(+)})\right] - c_- F^2_{(-)} - c_+ F^2_{(+)}, \tag{6.3}$$

where the new coefficients are

$$\phi_0^2 = -\frac{m_0^2}{2\lambda} = -\frac{N}{8\pi|m|}m_0^2, \tag{6.4}$$

where now $m_0^2 < 0$

$$c_- = \frac{1}{4\mathbf{e}^2} - \phi_0^2\frac{\bar{G}}{2} = \frac{N}{64\pi|m|}\left(1 + \frac{m_0^2}{m^2}\right), \tag{6.5}$$

$$c_+ = \frac{1}{4\mathbf{e}^2} - \phi_0^2\frac{\bar{G}}{6} = \frac{N}{64\pi|m|}\left(1 + \frac{1}{3}\frac{m_0^2}{m^2}\right), \tag{6.6}$$

and we define κ_0 to be

$$\kappa_0 = -2i\left(\frac{\theta}{8} - \phi_0^2 G_A\right) = -i\frac{\theta}{4}\left(1 + \frac{1}{4}\frac{m_0^2}{m^2}\right) = -i\frac{N}{8\pi}\left(1 + \frac{1}{4}\frac{m_0^2}{m^2}\right). \tag{6.7}$$

We still need a gauge fixing term for $A_{(+)}$. In this way we recover the structure of Eq. (5.8) with minor changes. The photon mass has changed to

$$M_{\rm ph} = \sqrt{16m^2\left(1 - \frac{4}{3}\frac{m_0^2}{m^2}\right)} \approx 4|m|\left(1 - \frac{2}{3}\frac{m_0^2}{m^2}\right). \tag{6.8}$$

The propagator for the out of phase field $A_{(-)}$ still corresponds to a massive field, which also has a longitudinal component

$$\hat{D}^{(-)(-)}_{\mu\nu} = -\frac{1}{2c_+}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2 - M^2_{\rm ph}} - \frac{p_\mu p_\nu}{\phi_0^2}, \tag{6.9}$$

$$\hat{D}^{(+)(+)}_{\mu\nu} = -\frac{1}{2c_+}\left(1 - \frac{\phi_0^2}{2c_-}\frac{1}{M^2_{\rm ph}}\right)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2 - M^2_{\rm ph}} - \frac{\phi_0^2}{4c_- c_+ M^2_{\rm ph}}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2} - \beta\frac{p_\mu p_\nu}{p^2}, \tag{6.10}$$

$$\hat{D}^{(-)(+)}_{\mu\nu} = \hat{D}^{(+)(-)}_{\mu\nu} = i\frac{\theta}{4}\left(1 + \frac{m_0^2}{4m^2}\right)\frac{1}{4c_- c_+}\frac{1}{p^2}\times\epsilon_{\mu\lambda\nu}p^\lambda\frac{1}{p^2 - M^2_{\rm ph}}. \tag{6.11}$$

Equations (6.9),(6.10),(6.11) give the propagators of the gauge fields in the condensed phase. By assumption m_0^2 is a small parameter, since our approximation is valid for the vicinity of the phase transition where m_0^2 is small (in units of the fermion mass) and measures the distance to the critical point. Notice that, in this phase, the expansion in powers of $1/N$ has become an expansion in powers of this new parameter. This is consistent with our approximation because a gradient expansion amounts to an expansion in powers of an inverse (large) length scale, which in our case is set by the fermion mass or, in other words, by the spinon gap of the decoupled system. On the other hand, the ''photon'' mass is fairly large in this phase.

We conclude this section with a qualitative description of the excitation spectrum in the broken-symmetry phase. Let us look first at the gauge excitations. In looking at Eq. (6.10) we notice the existence of a *massless* gauge mode in the spectrum. The existence of this massless mode implies that any excitation which couples to the $A_{(+)}$ component of the gauge field experiences an effective *long-range* (logarithmic) force mediated by the massless mode. In particular, the *spinon* excitations of the individual planes (which are semions in the unbroken phase) become permanently confined by the massless gauge fields. Recall that the logarithmic force is actually replaced by a confinig potential due to the strong fluctuations of the gauge fields dominated by monopolelike configurations.[12,15,16] In a sense, this makes fractional statistics unobservable since the quasiparticles which were able to bear it are no longer present in the spectrum. This spectrum is consistent with the fact that the statistical parameter θ is not well defined anymore in this phase (it is no longer a topological number) since, as can be seen in Eq. (6.3), it is now modified by a term proportional to the magnitude of the order parameter ϕ_0. Also, in the broken-symmetry phase, the time-reversal-breaking mass coming from the coefficient of the Chern-Simons term is no longer effective in controling the fluctuations of this particular mode. Actually, in this phase, the Higgs mechanism that takes place conspires to give a mass to the gauge field $A_{(-)}$, breaking spontaneously its gauge symmetry, while leaving the in-phase field $A_{(+)}$ untouched. In some sense the breaking of the phase symmetry enables the in-phase gauge field to become massless.

This phenomenon is in striking contrast with the conventional Higgs-Anderson mechanism in which a spontaneously broken symmetry renders a gauge field massive. The remaining out-of-phase component is massive and its mass is huge [see Eq. (6.8)], i.e., of the order of the fermion mass. This huge mass supresses the fluctuations of the field $A_{(-)}$ and, in this manner, it restores the broken time-reversal invariance that was present in the decoupled bilayer system. In particular, this spectrum implies that the only allowed fluctuations of the bilayer system are such that the chiralities of the planes become rigidly locked *locally*. Only in-phase, long-wavelength fluctuations of the chiralities are allowed. Since the two chiralities have opposite sign, we conclude that, in this phase, there is a *local cancellation* of the chiralities of the planes. Hence, chiral fluctuations are eliminated from the physical spectrum. Recall[17] that if a Chern-Simons term were to be present, the monopole configurations would be suppressed and fractional statistics would become observable. This is precisely what happens in the symmetric phase.

The spectrum that results from our analysis of the phase with broken symmetry is strikingly similar to the spectrum of the bilayer system in the singlet phase discussed by Sandvik and Scalapino[8] and Millis and Monien.[7] In fact, we believe that the two phases are the same phase and that the broken-symmetry phase is a phase with spin singlets connecting the two layers.

VII. CONCLUSIONS

In this paper we have reconsidered the problem of the selection of the relative sign of the chiralities of two planes with chiral spin liquid states coupled via an exchange interaction. We found that the exchange coupling selects the *antiferromagnetic* ordering of chiralities and, thus, that T and P are not broken in bilayers. This result holds for both signs of the interlayer exchange constant J_3. Hence, even if each plane has a net chirality, the bilayer system does not. Such a system will not give rise to any unusual optical activity in light scattering experiments. We determined the phase diagram of the bilayer system and found a phase transition to a valence bond (or spin gap) state. Our analysis reveals the presence of an unusual ''anti-Higgs-Anderson'' mechanism which is responsible for wiping out all trace of broken time-reversal invariance in the valence-bond state. In a separate publication we will report on results on the quantum numbers of the excitations and on the form of the wave function for the bilayer system.

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APPENDIX A: COMPUTATION OF THE BUBBLE DIAGRAMS

In this appendix we go through the computation of the correction to the energy of the ground state. The expressions given by Eqs. (3.3)–(3.5) are obtained from a one-loop diagram. In momentum space,

$$\mathcal{K}(\vec{q}) = i\int \frac{dk^3}{(2\pi)^3}\,\frac{\mathbf{T}_r[(\not{k}+m_L)\hat{\varphi}(\vec{q})(\not{k}-\not{q}+m_U)\hat{\varphi}^*(-\vec{q})]}{(\not{k}\not{k}-m_L^2)[(\not{k}-\not{q})(\not{k}-\not{q})-m_U^2]}, \tag{A1}$$

with the definitions of Sec. II. In computing the expression given above the following identities involving (2+1)-dimensional Dirac γ matrices will be important:

$$\gamma_a\gamma_b = g_{ab} + i\epsilon_{abc}\gamma^c, \quad \mathrm{tr}(\gamma_a\gamma_b) = 2g_{ab},$$

$$\mathrm{tr}(\gamma_a\gamma_b\gamma_c) = 2i\epsilon_{abc},$$

$$\mathrm{tr}(\gamma_a\gamma_b\gamma_c\gamma_d) = 2(g_{ab}g_{cd} + g_{ad}g_{bc} - g_{ac}g_{bd}). \tag{A2}$$

The integral in Eq. (1.1) needs to be regularized; i.e., we need to cut off the unphysical ultraviolet divergence due to the integration over momentum $\vec{k}$. There is a natural cutoff in the original theory, which is the lattice spacing. However, in going to the continuum approximation we encounter the usual field theory divergences.

There are several methods for regulating this type of integral. The important point is that they should preserve the physical symmetries involved in the problem. For the case of a gauge field there are well-established procedures such as the Pauli-Villars or the dimensional regularization methods. It can be shown that they preserve transversality (i.e., gauge invariance). In the case of the φ fields, we do not have such a symmetry to preserve. In fact, not even Lorentz invariance is preserved. We have a length scale given by the lattice spacing, which in turn provides the momentum cutoff Λ that was mentioned in Sec. II. On the lattice there is no cutoff for the frequency integral. Thus, our regulating procedure is as follows. First we perform a subtraction at the level of the integrands. This is equivalent to writing the kernels in Eq. (1.1) in the following way:

$$\mathcal{K}_j(\vec{q})\equiv\mathcal{K}_j(0)+[\mathcal{K}_j(\vec{q})-\mathcal{K}_j(0)]. \tag{A3}$$

In this expression only the first term has an ultraviolet divergence. The term between brackets is convergent and can be calculated using standard methods. In the computation of $\mathcal{K}_j(0)$ we integrate first over frequency without any cutoff. This integral is still finite. After this step, we introduce an isotropic Gaussian cutoff for the space directions of order $\Lambda\approx 1/a_0$ where a_0 is the lattice spacing. However, once the integration over frequency has been performed, the finite contribution of the divergent term $\mathcal{K}_j(0)$ is independent of the particular form of the cutoff being used.

The computation of $\mathcal{K}_j(0)$ with $j=0,1,2,3$ is nothing but the calculation of the critical coupling constants performed in Sec. II. We have

$$\mathcal{K}_3(0)=i\ \mathrm{tr}[\hat{\mathbf{S}}_L(\vec{k})\tau_3\hat{\mathbf{S}}_U(\vec{k})\tau_3]=\frac{1}{2\sqrt{\pi}}\Lambda-\frac{1}{2\pi}m(1+s), \tag{A4a}$$

$$\mathcal{K}_0(0)=i\ \mathrm{tr}[\hat{\mathbf{S}}_L(\vec{k})\gamma_0\hat{\mathbf{S}}_U(\vec{k})\gamma_0]=\frac{1}{2\pi}m(1-s), \tag{A4b}$$

$$\mathcal{K}_j(0)=i\ \mathrm{tr}[\hat{\mathbf{S}}_L(\vec{k})\gamma_j\tau_j\hat{\mathbf{S}}_U(\vec{k})\gamma_j\tau_j] =-\frac{1}{4\sqrt{\pi}}\Lambda-\frac{1}{2\pi}m(1-s), \tag{A4c}$$

where $j=1,2$. To get $\mathcal{K}_3(\vec{q})-\mathcal{K}_3(0)$ we need to integrate

$$\mathcal{T}_3^E=-4\int\frac{dk^3}{(2\pi)^3}\times\frac{(k^2+m^2)\vec{q}\cdot(\vec{q}-\vec{k})+m^2(1+s)\vec{q}\cdot(2\vec{k}-\vec{q})}{(k^2+m^2)^2[(k-q)^2+m^2]}, \tag{A5}$$

where k^2, q^2, $\vec{k}$, and $\vec{q}$ refer to the imaginary frequency rotated form of the trivectors k_μ and q_μ, and $s=\mathrm{sgn}(m_L)\cdot\mathrm{sgn}(m_U)$.

For the frequency channel given by φ_0 we get after rotating to an imaginary frequency

$$\mathcal{T}_0^E=-4\int_k\left\{\frac{[2k_0q_0+\vec{q}\cdot(\vec{k}-\vec{q})](k^2+m^2)-\vec{q}\cdot(2\vec{k}-\vec{q})[2k_3^2+m^2(1-s)]}{(k^2+m^2)^2[(k-q)^2+m^2]}\right\}. \tag{A6}$$

For the channels given by φ_1 and φ_2 we obtain expressions similar to the one for φ_0 with k_0 exchanged by k_1 or k_2 in each case. The kernel for the spatial channels also have an opposite sign to $\mathcal{K}_0(\vec{q})-\mathcal{K}_0(0)$.

We may write down the denominators in Eq. (1.5) and Eq. (1.6) in the form

$$\frac{1}{D_E[m^2]}=\int_0^1 du(1-u)\int_0^\infty \lambda^2 d\lambda\, e^{-\lambda\vec{l}^2}e^{-\lambda[m^2+u(1-u)\vec{q}^2]}. \tag{A7}$$

In Eq. (A7) we performed the change of variables $\lambda=\alpha+\beta$ and $u=\beta/(\alpha+\beta)$, we defined $\vec{l}\equiv\vec{k}-u\vec{q}$, and once again, for simplicity, we restricted to the case $|m_L|=|m_U|=m$. After integrating over k (or, rather, l) and λ and a simple change of variables, Eq. (A5) becomes

$$\mathcal{T}_3^E=\frac{1}{2\pi}\left\{m(1+s)-\frac{2}{|\vec{q}|}\sin^{-1}\left(\frac{|\vec{q}}{\sqrt{4m^2+\vec{q}^2}}\right)\left(m^2(1+s)+\frac{1}{2}\vec{q}^2\right)\right\}. \tag{A8}$$

For the vector channels we get

$$\mathscr{T}_j^E=\frac{1}{2\pi}g_{jj}\left\{m(s+2\kappa_j^2+3\kappa_j^4)+\frac{2m\vec{q}^{\,2}}{4m^2+\vec{q}^{\,2}}(\kappa_j^4-\kappa_j^2)\right\}$$
$$-\frac{1}{2\pi}g_{jj}\left[2m^2s+4m^2\left(\kappa_j^2+\frac{3}{2}\kappa_j^4\right)-\frac{\vec{q}^{\,2}}{2}(1-3\kappa^4)\right]\frac{1}{|\vec{q}|}\sin^{-1}\left(\frac{|\vec{q}|}{\sqrt{4m^2+\vec{q}^{\,2}}}\right). \tag{A9}$$

In Eq. (A9), g_{jj} only indicates that the channels given by φ_1 and φ_2 give a contribution with sign opposite to the one given by φ_0. Also, we have defined $\kappa_j\equiv q_j/|\vec{q}|$ and $c\equiv 1+4m^2/\vec{q}^2$. The expressions given by Eqs. (3.3)–(3.5) can now be obtained simply by combining Eq. (A4) with Eq. (A8) and Eq. (A9).

APPENDIX B: GRADIENT EXPANSION

In order to obtain an effective theory valid for long-wavelength excitations, in momentum space we only need an expansion to the few lowest orders in the external momenta of the diagrams, since each external momentum will generate a space derivative in the Fourier antitransformed expression. Here we pursue further the expansion indicated by Eq. (3.1). However, instead of computing the loop diagrams exactly as we did in order to calculate the correction to the ground-state energy in Sec. III, we expand up to second order in the external momenta. Notice that we are integrating out neither the gauge fields nor the φ fields in this case.

1. Diagrams with two external legs

We showed in Sec. III that the classical energy is of $\mathscr{O}(N)$. From the expansion of the logarithm of the determinant, to second order in powers of the (small) fluctuating fields contained in the operator $\hat{\mathbf{Q}}$ of Sec. III, which now also includes the gauge fields in the planes—or to $\mathscr{O}(1)$ in the $1/N$ expansion—we have

$$i\,\frac{1}{2}\int dx^3(\hat{\mathbf{S}}\hat{\mathbf{Q}}\hat{\mathbf{S}}\hat{\mathbf{Q}})=\frac{i}{2}\int_q\int_k \mathrm{tr}[\hat{\mathbf{S}}_L(\vec{k})\not{A}_L(\vec{q})\hat{\mathbf{S}}_L(\vec{k}+\vec{q})\not{A}_L(-\vec{q})]$$
$$+\frac{i}{2}\int_q\int_k \mathrm{tr}[\hat{\mathbf{S}}_U(\vec{k})\not{A}_U(\vec{q})\hat{\mathbf{S}}_U(\vec{k}+\vec{q})\not{A}_U(-\vec{q})]$$
$$+2\,\frac{i}{2}\int_q\int_k \mathrm{tr}[\hat{\mathbf{S}}_L(\vec{k})\hat{\varphi}(\vec{q})\hat{\mathbf{S}}_U(\vec{k}+\vec{q})\hat{\varphi}(-\vec{q})]. \tag{B1}$$

As a shorthand we have used the notation $\int dk^3/(2\pi)^3\equiv\int_k$.

For the gauge fields alone, the diagram has an ultraviolet divergence that needs to be treated. We use dimensional regularization to ensure transversality, i.e., to preserve gauge invariance. The calculation is similar to the one shown in Chap. VII of Ref. 12. The first two lines of the RHS of Eq. (B1) give $\int_q\Pi_{\mu\nu}^{LL}(q)A_L^\mu(q)A_L^\nu(-q)$ where

$$\Pi_{\mu\nu}^{LL}(\vec{q})=i\epsilon_{\mu\nu\lambda}q^\lambda\Pi_A^{LL}(q^2)+(q_\mu q_\nu-q^2g_{\mu\nu})\Pi_S^{LL}(q^2), \tag{B2}$$

with

$$\Pi_A^{LL}(q^2)=\frac{1}{2\pi}\frac{m}{\sqrt{q^2}}\sinh^{-1}\left(\frac{1}{\sqrt{\frac{4m^2}{q^2}-1}}\right) \tag{B3}$$

and

$$\Pi_S^{LL}(q^2)=\frac{1}{8\pi}\frac{1}{\sqrt{q^2}}\left[-\frac{2|m|}{\sqrt{q^2}}+\left(1+\frac{4m^2}{q^2}\right)\times\sinh^{-1}\left(\frac{1}{\sqrt{\frac{4m^2}{q^2}-1}}\right)\right], \tag{B4}$$

and the corresponding expression (UU) for the upper plane. This is the full expression for the polarization tensor for the gauge fields. The small momentum limit for Eq. (B2) is given by

$$\frac{1}{4\pi}\frac{m}{|m|}i\epsilon_{\mu\nu\lambda}q^\lambda+\frac{1}{16\pi}\frac{1}{|m|}(q_\mu q_\nu-q^2g_{\mu\nu}).$$

Notice that the fermion mass with its sign enters the antisymmetric part of $\Pi_{\mu\nu}^{LL}$. Therefore, for an antisymmetric ordering of chiralities in the ground state, Π_A^{LL} and Π_A^{UU} will bear opposite signs. In fact, the ratio $m/|m|$ in our case is actually always positive 1, since the signs of the masses have already been taken into account when defining $\hat{\mathbf{S}}_L$ and $\hat{\mathbf{S}}_U$ in Sec. III, and we are considering the case when the magnitude of the masses is the same on both planes. From now on, we give the details for the computation of the one-loop diagrams involving exclusively the scalar channel. The third line in Eq. (B1) can be rewritten as

$$i\int_q\int_k \mathrm{tr}[\hat{\mathbf{S}}_L(k)\tau_3\hat{\mathbf{S}}_U(k+q)\tau_3]|\varphi(q)|^2. \tag{B5}$$

This expression is similar to the ones we encountered in the previous appendix. Since we are interested in a small external momentum expansion, the exact expression shown above can be approximated by

$$2i\int_q\int_k \mathrm{tr}\left[\hat{\mathbf{S}}_L(k)\sum_{n=0}^{\infty}(-1)^n(\hat{\mathbf{S}}_U(k)\not{q})^n\hat{\mathbf{S}}_U(k)\right]|\varphi_3(q)|^2, \tag{B6}$$

which gives the result

$$\int_q|\varphi_3(q)|^2\left[\frac{1}{\pi}(\sqrt{\Lambda^2+m^2}-\sqrt{m^2})+\frac{1}{4\pi}\frac{1}{(m^2)^{1/2}}q_\mu q^\mu\right]. \tag{B7}$$

The above expressions are valid up to second order in the external momenta of the one-loop diagram, i.e., in the momenta of the field φ or in the momenta of the gauge fields.

2. Diagrams with three external legs

The next term in the expansion of the logarithm of the fermionic determinant is

$$\begin{aligned}-\frac{1}{\sqrt{N}}\frac{i}{3}\int dx^3(\hat{\mathbf{S}}\hat{\mathbf{Q}})^3=&-\frac{3}{\sqrt{N}}\times\frac{i}{3}\int_p\int_q\int_k \mathrm{Tr}\,\{\hat{\mathbf{S}}_L(k)\gamma_\mu\hat{\mathbf{S}}_L(k+p)\hat{\varphi}(q)\hat{\mathbf{S}}_U(k+p+q)[\hat{\varphi}(p+q)]^*\}A_L^\mu(p)\\ &-\frac{3}{\sqrt{N}}\times\frac{i}{3}\int_p\int_q\int_k \mathrm{Tr}\,\{\hat{\mathbf{S}}_U(k)\gamma_\mu\hat{\mathbf{S}}_U(k+p)[\hat{\varphi}(-q)]^*\hat{\mathbf{S}}_L(k+p+q)\hat{\varphi}(-p-q)\}A_U^\mu(p).\end{aligned} \tag{B8}$$

Both terms in the RHS of Eq. (2.8) are similar. It can easily be shown that the zeroth-order term in external momenta vanishes. For the scalar channel only, up to second order in the external momenta, the three-leg one-loop diagrams give the contribution

$$\begin{aligned}&-\frac{1}{\sqrt{N}}\frac{1}{4\pi}\frac{1}{|m|}\int_{q,s,p}(q_\mu-s_\mu)[A_L^\mu(p)-A_U^\mu(p)]\delta(p+s+p)\varphi(q)[\varphi(-s)]^*\\ &+\frac{i}{\sqrt{N}}\frac{1}{8\pi}\frac{1}{|m|^2}\epsilon_{\mu\nu\lambda}\int_{q,s,p}[A_L^\mu(p)+A_U^\mu(p)]p^\nu s^\lambda\delta(p+s+p)\varphi(q)[\varphi(-s)]^*.\end{aligned} \tag{B9}$$

3. Diagrams with four external legs

The fourth order in the expansion of the logarithm of the fermionic determinant gives

$$\begin{aligned}\frac{i}{4N}\int dx^3(\hat{\mathbf{S}}\hat{\mathbf{Q}})^4=&+\frac{i}{N}\int_{l,p,q,k}\mathrm{Tr}\,\{\hat{\mathbf{S}}_L(k)\gamma_\mu\hat{\mathbf{S}}_L(k+p)\gamma_\nu\hat{\mathbf{S}}_L(k+p+l)\hat{\varphi}(q)\hat{\mathbf{S}}_U(k-s)[\hat{\varphi}(-s)]^*\}A_L^\mu(p)A_L^\mu(l)\\ &+\frac{i}{N}\int_{l,p,q,k}\mathrm{Tr}\,\{\hat{\mathbf{S}}_U(k)\gamma_\mu\hat{\mathbf{S}}_U(k+p)\gamma_\nu\hat{\mathbf{S}}_U(k+p+l)\hat{\varphi}(q)\hat{\mathbf{S}}_L(k-s)[\hat{\varphi}(-s)]^*\}A_U^\mu(p)A_U^\mu(l)\\ &+\frac{i}{N}\int_{l,p,q,k}\mathrm{Tr}\,\{\hat{\mathbf{S}}_L(k)\gamma_\mu\hat{\mathbf{S}}_L(k+p)\hat{\varphi}(q)\hat{\mathbf{S}}_U(k+p+q)\gamma_\nu\hat{\mathbf{S}}_U(k-s)[\hat{\varphi}(-s)]^*\}A_L^\mu(p)A_U^\mu(l)\\ &+\frac{i}{2N}\int_{l,p,q,k}\mathrm{Tr}\,\{\hat{\mathbf{S}}_L(k)\hat{\varphi}(p)\hat{\mathbf{S}}_U(k+p)[\hat{\varphi}(q]^*\hat{\mathbf{S}}_L(k+p+q)\hat{\varphi}(l)\hat{\mathbf{S}}_U(k-s)[\hat{\varphi}(-s)]^*\}\\ &+\text{terms involving four gauge fields.}\end{aligned} \tag{B10}$$

From this expression we are going to consider only the four first lines on the RHS of Eq. (B10) as they will show to be the relevant terms for our gradient expansion. As we did before, we consider only the scalar channel. The third term on the RHS of Eq. (B10) gives a total contribution, valid to first order in the external momenta, that looks like

$$-\frac{1}{N}\frac{1}{16\pi}\frac{1}{(m^2)^{1/2}}\left\{8g_{\mu\nu}+\frac{m}{|m|}\,\mathrm{tr}[\gamma_\mu\gamma_\nu(\not p+\not l)]\right\}A_L^\mu(p)A_U^\nu(l)\varphi_3(q)[\varphi_3(-s)]^*. \tag{B11}$$

The contribution coming from the first term on the RHS of Eq. (B10) is

$$-\frac{1}{N}\frac{1}{16\pi}\frac{1}{(m^2)^{1/2}}\left\{-4g_{\mu\nu}+\frac{m}{|m|}\,\mathrm{tr}[\gamma_\mu\gamma_\nu(\not p+\not q)]\right\}A_L^\mu(p)A_L^\nu(l)\varphi_3(q)[\varphi_3(-s)]^*. \tag{B12}$$

The second term on the RHS of Eq. (B10) gives

$$-\frac{1}{N}\frac{1}{16\pi}\frac{1}{(m^2)^{1/2}}\left\{-4g_{\mu\nu}-\frac{m}{|m|}\,\mathrm{tr}[\gamma_\mu\gamma_\nu(\not p+\not s)]\right\}A_U^\mu(p)A_U^\nu(l)\varphi_3(q)[\varphi_3(-s)]^*. \tag{B13}$$

The origin of the relative sign between the antisymmetric parts of Eqs. (B12) and (B13) is the relative sign of the fermion masses on the planes.

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