Tilt instabilities and multiple coexisting vortex orientations in flux-line liquids

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Phase diagrams for various tilt instabilities, as well as phase diagrams for possible coexistence of vortex lines with different orientations in a tilted magnetic field of anisotropic type-II superconductors, are mapped out and compared within anisotropic London theory. Double minima in the Gibbs energy as a function of vortex orientation in the case of noninteracting vortex liquids may be found in a regime of mass anisotropy where long-wavelength tilt instabilities do not occur, and are associated with coexisting species of flux lines of different orientations with respect to the crystal $\hat{c}$ axis. At larger mass-anisotropies, long-wavelength tilt instabilities do occur for tilt modes perpendicular to the ab planes. For tilt modes parallel to the ab planes, instabilities only occur for very large wave numbers, beyond the regime of applicability of London theory. The cutoff dependence on the phase diagrams for the isolated flux-line limit is investigated. It is essential that the elliptic shape of the vortex core is accounted for. When interactions are accounted for in a dense flux-line liquid, the tilt instabilities are removed.

I. INTRODUCTION

Theoretical investigations of the stability of vortex structures in anisotropic superconductors have recently been carried out quite intensively.[1–7] This followed, for instance, from Bitter pattern decoration studies of magnetic-flux lines emerging from surfaces of the high-T_c superconductor $Bi_{2.2}Sr_2Ca_{0.8}Cu_2O_8$,[8] where chains of flux lines embedded in an approximately regular hexagonal Abrikosov vortex lattice were seen when the external magnetic field was tilted with respect to the $\hat{c}$ axis. This finding was unexpected: earlier work had predicted the chains of vortices, but not their coexistence with the background of an approximately regular hexagonal vortex lattice.[9]

Various scenarios of how such a situation could occur, were recently proposed. Basically, the main idea is due to Huse,[1] who suggested the possibility of an admixture of two species of vortices in anisotropic superconductors in tilted magnetic fields, one oriented almost parallel to the $\hat{c}$ axis, and another oriented almost parallel to the ab plane.

More recently, it was actually demonstrated rigorously within the framework of anisotropic London theory that such a situation could occur, at least at very low inductions, provided that the mass anisotropy of the material is large enough.[5] The hallmark of such a highly unusual effect was established as the observation of two degenerate minima at value zero of the Gibbs energy, for two distinct flux-line orientations with respect to the $\hat{c}$ axis. Finally, it has been demonstrated, also within anisotropic London theory,[6] that certain tilt instabilities (also to be discussed in this paper) could occur for bending modes of the flux lines at large wave numbers k_z, and it was speculated that there could be a connection between the above-mentioned coexistence of flux-line orientations, and such tilt instabilities.

Recent experimental work has claimed evidence for the coexistence of flux lines of different orientations,[10] but where neither vortex orientation is parallel to the direction of the applied field. This is consistent with the predictions of Ref. 5.

To our knowledge, a unique interpretation of the double minimum in the Gibbs energy has not yet been conclusively established. The issue is whether such double minima can be uniquely interpreted as coexistence of flux lines with different orientations, or whether they can be interpreted as tilt instabilities of flux lines oriented at oblique angles with respect to the $\hat{c}$ axis in very anisotropic and extreme type-II superconductors. Such tilt instabilities would imply that an assumption of a ground state consisting of *straight flux lines* would be incorrect, and that the correct ground state rather would consist of flux lines with spontaneously created kinks (most probably on the length scale of interlayer distance). In this paper, we clarify the connection between tilt instabilities and coexistence of vortex "species," characterized by different orientations relative to the uniaxial symmetry axis in anisotropic superconductors, in flux-line liquids. We emphasize that we are not considering entropic contributions to the free energy; thermal effects are ignored. Nonetheless, the phrase flux-line liquid will be used frequently, a term usually reserved for cases where thermal effects *are* important. In this paper, it will simply be used to denote a flux-line ensemble with zero shear stiffness.

This paper is organized as follows. In Sec. II, we introduce the anisotropic London theory and the tensorial interaction between flux lines of arbitrary shape in tilted magnetic fields, with a derivation of the result given in Appendix A. The elastic matrix is then given, along with cutoff procedures described in detail in Appendix B. In Sec. III, we give the Gibbs energy for noninteracting flux lines and study its properties, with a derivation of the self-energy given in Appendix C. In Secs. IV and V we discuss elastic eigenmodes of flux-line liquids, i.e., flux-line ensembles without a shear modulus, both for the noninteracting and interacting cases. The stability of tilted flux-line liquids are discussed for the dilute and dense cases.

II. ANISOTROPIC LONDON THEORY

In the London theory, the total energy of an ensemble of interacting flux lines of arbitrary shape, including vortex loops, is given by

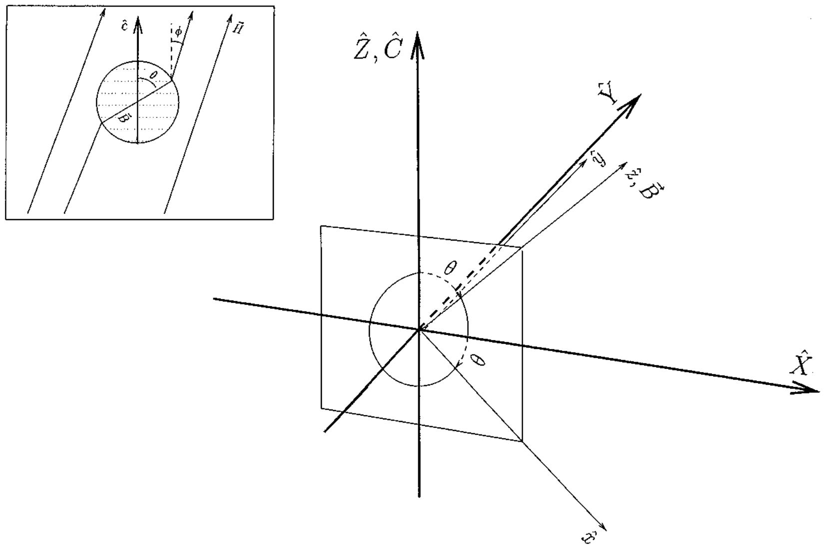

FIG. 1. Geometry of tilted vortex lines considered in this paper. $\hat{X},\hat{Y},\hat{Z}$ are the symmetry axis of the crystal and $\hat{x},\hat{y},\hat{z}$ are the axis for the vortex lines. The vortex frame can be obtained from the crystal frame by rotating it an angle θ around $\hat{Y}$ axis. The inset shows that the cylindrical system in the calculation of the double minima in Gibbs energy (coexistence).

$$E=\frac{\Phi_0^2}{8\pi}\sum_{i,j}\int\int dr_i^{\alpha}dr_j^{\beta}V_{\alpha\beta}(\mathbf{r}_i-\mathbf{r}_j) \tag{1}$$

where a summation over repeated indices α,β is understood and $(\alpha,\beta)\in(x,y,z)$. The expression for the tensor $V_{\alpha\beta}(\mathbf{r})$ when the average induction is tilted an angle θ away from the symmetry $\hat{c}$ axis is given by[11]

$$V_{\alpha\beta}(\mathbf{r})=\frac{1}{4\pi\lambda_a^2(\mathbf{r}\times\hat{c})^2}\left[G_1(\mathbf{r})(\delta_{\alpha\beta}-c_\alpha c_\beta)-G_2(\mathbf{r})\frac{(\mathbf{r}\times\hat{c})_\alpha(\mathbf{r}\times\hat{c})_\beta}{(\mathbf{r}\times\hat{c})^2}\right],$$

$$G_1(\mathbf{r})=e^{-r/\lambda_a}-e^{-\tilde{r}/\lambda_c},$$

$$G_2(\mathbf{r})=\left[2+\frac{(\mathbf{r}\times\hat{c})^2}{\lambda_a r}\right]e^{-\lambda_a/r}-\left[2+\frac{(\mathbf{r}\times\hat{c})^2}{\lambda_c\tilde{r}}\right]e^{-\tilde{r}/\lambda_c}, \tag{2}$$

$$\tilde{r}\equiv\sqrt{(\mathbf{r}\times\hat{c})^2+\Gamma^{-2}(\mathbf{r}\cdot\hat{c})^2},$$

where c_α is the projection of $\hat{c}$ onto the αth (x,y,z) axis, λ_a and λ_c are London penetration depths along the ab plane and $\hat{c}$ axis, respectively, and $\Gamma\equiv\sqrt{M_z/M}$ is the mass anisotropy in the uniaxially anisotropic superconductor. Within this effective-mass model, we have $\Gamma=\lambda_c/\lambda_a$. For the derivation of this expression, see Appendix A.

Expanding the total energy E to lowest order in vortex fluctuations around a ground-state Abrikosov vortex lattice solution, which has an energy denoted by E_0, we get

$$E=E_0+\mathscr{F}(\{\mathbf{u}\})=E_0+\frac{1}{2}\sum_{\mathbf{k}}u_\alpha(-\mathbf{k})\Phi_{\alpha\beta}(\mathbf{k})u_\beta(\mathbf{k}), \tag{3}$$

where now $(\alpha,\beta)\in(x,y)$, since displacements along the vortex lines have no physical meaning, and $\{\mathbf{u}\}$ are the Fourier transforms of the local distortions of the flux-line system around its ground state of rigid flux lines arranged in a well defined (super)lattice, and which we assume to be small. $\mathscr{F}(\{\mathbf{u}\})$ is the linear elastic energy associated with the Gaussian fluctuations of the flux lines around the ground state, and for a flux-line *liquid* in a tilted field in a uniaxial anisotropic superconductor, this elastic energy is given by[12]

$$\mathscr{F}(\{\mathbf{u}\})=\frac{1}{2}\sum_{\mathbf{k}}[c_{11}(\mathbf{k})(\mathbf{k}\cdot\mathbf{u})^2+c_{44}^{\perp}(\mathbf{k})(k_zu_x)^2+c_{44}^{\parallel}(\mathbf{k})(k_zu_y)^2+2c_{14}(\mathbf{k})k_zu_x(\mathbf{k}\cdot\mathbf{u})]. \tag{4}$$

In this continuum limit of a flux-line liquid, we have four different elastic moduli, due to the loss of symmetry in the problem when the induction is directed away from the $\hat{c}$ axis. c_{11} is the bulk modulus, $c_{44}^{\perp}$ is the tilt modulus associated with tilts in the $(\hat{c},\mathbf{B})$ plane, $c_{44}^{\parallel}(\mathbf{k})$ is a tilt modulus associated with tilts out of the $(\hat{c},\mathbf{B})$ plane, while $c_{14}(\mathbf{k})$ is a mixed bulk and tilt modulus which has no counterpart in the isotropic case. For the geometry of the problem, see Fig. 1. In the case of a very dilute flux-line liquid, the moduli associated with the interactions between flux lines, i.e., $c_{11}(\mathbf{k})$ and $c_{14}(\mathbf{k})$, become unimportant. Moreover, the components (k_x,k_y) of the Fourier mode $\mathbf{k}$ also have no physical significance any longer, and thus $\mathscr{F}(\{\mathbf{k}\})$ reduces to the much simpler expression

$$\mathscr{F}(\{\mathbf{u}\})=\frac{1}{2}\sum_{k_z}[c_{44}^{\perp}(k_z)(k_zu_x)^2+c_{44}^{\parallel}(k_z)(k_zu_y)^2]. \tag{5}$$

The above expansion, using Eqs. (1)–(3), gives the following expression for the elastic matrix $\Phi_{\alpha\beta}(\mathbf{k})$ of the flux-line system[3,13]

$$\Phi_{\alpha\beta}(\mathbf{k})=\frac{B^2}{4\pi}\sum_{Q}[f_{\alpha\beta}(\mathbf{K})-f_{\alpha\beta}(\mathbf{Q})],$$

$$f_{\alpha\beta}(\mathbf{k})=k_z^2\tilde{V}_{\alpha\beta}(\mathbf{k})+k_\alpha k_\beta\tilde{V}_{zz}(\mathbf{k})-k_zk_\alpha\tilde{V}_{z\beta}(\mathbf{k})-k_zk_\beta\tilde{V}_{z\alpha}(\mathbf{k}), \tag{6}$$

$$\tilde{V}_{\alpha\beta}(\mathbf{k})=\frac{1}{1+\Lambda_1k^2}\left[\delta_{\alpha\beta}-\frac{\Lambda_2q_\alpha q_\beta}{1+\Lambda_1k^2+\Lambda_2q^2}\right],$$

where $\mathbf{q}=\mathbf{k}\times\hat{c}$, $\Lambda_1=\lambda_a^2$, $\Lambda_2=\lambda_c^2-\lambda_a^2$. For the purposes of the calculations to be presented in this paper, it is convenient to express the elements of $\Phi_{\alpha\beta}(\mathbf{k})$ in the form given in Appendix B. In the above expressions, $\mathbf{Q}$ is a set of reciprocal-lattice vectors of an equilibrium lattice configuration, $\mathbf{K}=\mathbf{Q}+\mathbf{k}$. Such equilibrium configurations could, in principle, be quite general, including vortex lattices with a basis, depending on what the induction and orientation with respect to $\hat{c}$ axis are, and what material parameters such as mass anisotropy and Ginzburg-Landau parameter, are considered.

Note that all of the above lattice sums, which are two-dimensional, formally diverge logarithmically. The sums are therefore *dominated* by the region of $\mathbf{k}$ space corresponding to the *vortex core*, which is not explicitly described in London theory. A prescription for cutting off these sums has been described elsewhere,[14,15] and it turns out to be crucial to account for the deformation of the vortex core shape as the average induction is tilted away from the $\hat{c}$ axis.[3] The sum is given an upper cutoff on an ellipse in (k_x, k_y) space defined by

$$\xi_\theta^2 k_x^2 + \xi_a^2 k_y^2 = 1,$$
$$\xi_a^2 \cos^2\theta + \xi_c^2 \sin^2\theta \equiv \xi_\theta^2, \tag{7}$$

where ξ_a and ξ_c are the superconducting coherence lengths along the ab plane and $\hat{c}$ axis, respectively, and $\xi_a/\xi_c = \Gamma$. This may either be implemented by a sharp cutoff on the integrals (or sums) or by a Gaussian cutoff. For the results presented in this paper, we have used both procedures, in order to compare the results we obtain; it is always a matter of concern that results in London theory could be artifacts of the details of cutoff procedures. For the same reason, results that appear in the large-k_z limit within London theory should also be viewed with suspicion. We note however, that the elliptical cutoff in $\mathbf{k}$ space that we employ has been derived from Ginzburg-Landau theory.[16] We have found that the results do not depend on the details of the way the cutoff is implemented.

III. GIBBS ENERGY, RIGID NONINTERACTING FLUX LINES

We first briefly recapitulate the principle for detecting coexistence, a brief version of what is presented below has appeared elsewhere.[3] The present calculation represents a slight improvement in accuracy over previous results due to a more precise numerical calculation of the vortex self-energy of rigid flux lines tilted an arbitrary angle θ away from the $\hat{c}$ axis, and which is given in general by[5]

$$J(\theta) = \frac{\Phi_0^2}{8\pi} \int \frac{d^2\mathbf{k}}{4\pi^2} \frac{1+\lambda_\theta^2 k_\perp^2}{[1+\lambda_a^2 k_\perp^2][1+\lambda_\theta^2 k_x^2 + \lambda_c^2 k_y^2]},$$
$$\lambda_\theta^2 \equiv \lambda_a^2 \sin^2\theta + \lambda_c^2 \cos^2\theta, \tag{8}$$

which was previously approximated analytically in a manner described in Appendix C. A cutoff of the formally logarithmically divergent sum, is understood. One may either employ a sharp cutoff, where the upper limit in k_x, k_y space is given by an ellipse reflecting the elliptic shape of the vortex core when the flux line is tilted, or a Gaussian cutoff. The procedure is described in Appendix B.

We consider a cylindrical superconductor with the applied field perpendicular to the symmetry axis of the cylinder and tilted an angle ϕ away from the crystal $\hat{c}$ axis, see the inset of Fig. 1. The Gibbs energy for such a system is given by

$$G = nJ(\theta) - HB\cos(\phi - \theta), \tag{9}$$

where n is the areal density of flux lines, H is the applied field, and B is the induction. This expression is valid at $T=0$, where the entropic contribution to the free energy is zero. Previous analytical approximations to this expression[3] are in excellent agreement with the numerical results of Appendix C, particularly in the large-κ limit. The minimum criterion for seeing double minima (not necessarily degenerate) in the Gibbs energy is that

$$\min_\theta \frac{\partial^2 G}{\partial \theta^2} < 0. \tag{10}$$

Two precisely degenerate minima in the Gibbs energy at $G=0$ requires two distinct solutions to the coupled equations

$$G(\Gamma,\kappa;\theta) = 0,$$
$$\frac{\partial G(\Gamma,\kappa;\theta)}{\partial\theta} = 0. \tag{11}$$

When $\min_\theta \partial^2 G/\partial\theta^2 < 0$, it turns out that there are precisely three angles $\theta_1 < \theta_2 < \theta_3$ for which $\partial G/\partial\theta = 0$, corresponding to two minima (θ_1, θ_3) and one maximum (θ_2) in the Gibbs energy. The maximum at θ_2 corresponds to a vortex orientation of an unstable state and will not be considered further. In the isotropic case, or when a circular core cutoff is used in the anisotropic case, it is well known that one unique orientation of the flux lines exists, given by the condition

$$\tan(\phi) = \Gamma^{-2}\tan(\theta). \tag{12}$$

A circular cutoff is expected to be a reasonable and qualitatively correct approximation to the core cutoff for not too large mass anisotropies, due to the fact that the self-energy depends only weakly (logarithmically) on the details of the cutoff. For not too large mass anisotropies, we thus expect that the above expression essentially gives correct results for the vortex orientation at given mass anisotropy and orientation of external field. Note that the Ginzburg-Landau parameter does not enter into the above expression.

From the above, it is immediately clear that there must exist a *minimum* mass anisotropy at a given value of κ to see a double minimum in the Gibbs energy, if at all, at any value of the direction of the applied field, ϕ, and at any vortex orientation. It is also clear that the elliptical core cutoff is essential to bring about qualitatively new effects of the mass anisotropy. The orientation(s) for the penetrating flux line(s) inside the superconductor is determined from Eq. (11)

$$\tan(\phi) = \frac{\tan(\theta) + J'/J}{1 - J'/J\,\tan(\theta)}, \tag{13}$$

where $J' = \partial J/\partial\theta$. The right-hand side of Eq. (13) must be nonmonotonic in order for the system to exhibit multiple solutions θ for a given value of ϕ. Nonmonotonicity develops as a function of mass anisotropy due to the self-energy cost associated with deforming the vortex core as the flux-

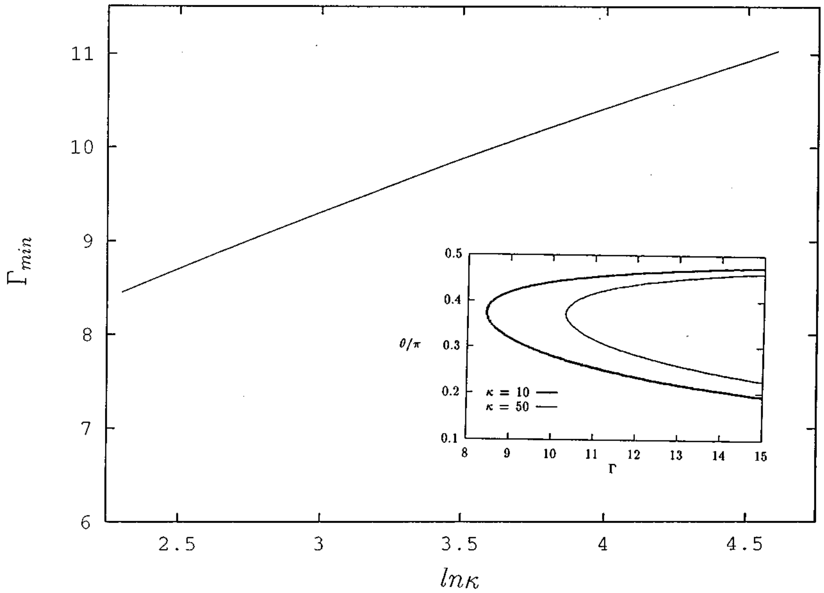

FIG. 2. Critical value of Γ as a function of $\ln\kappa$, for seeing double minima in the Gibbs energy, using a sharp cutoff in the self-energy $J(\theta)$. The inset shows the corresponding angles of orientations for two coexisting "species" of flux lines as a function of Γ, for two values of κ=10,50.

line orientation θ is varied. Figures 2 and 3 illustrate the situation: Shown is the minimum mass anisotropy $\Gamma_{\rm min}$ as a function of ln κ required to see nonmonotonicity in the right-hand side of Eq. (13). Figure 2 shows the result of implementing the cutoff Eq. (7) sharply, while Fig. 3 shows the result of implementing it smoothly, as described in Appendix B.

As can be seen, the results are quite insensitive to the details of the implementation of the core cutoff. This is a satisfactory feature, since the cutoff must be introduced in an *ad hoc* way within London theory due to the lack of an explicit description of the vortex core. The only minor difference in the results given the two cutoff procedures is that smooth cutoffs give a slightly smaller critical mass anisotropy, i.e., coexistence is possible at a smaller $\Gamma_{\rm min}$. The slight difference in the values of $\Gamma_{\rm min}$ compared to earlier results[3] is due to the more accurate, numerical, evaluation of the self-energy compared to earlier work, where an analytical approximation to $J(\theta)$ was used. The picture that emerges is however identical: As κ increases, a monotonic increase in $\Gamma_{\rm min}$ is required to see double minima in the Gibbs energy thus facilitating the coexistence of flux lines of different orientations at the lower critical field. The inset shows the angles at which precisely two degenerate minima at $G=0$ are found. Again, it is seen that the results are essentially independent of the way the cutoff is implemented.

A physical interpretation of this result has been given previously:[3] The two degenerate minima at H_{c1} indicate that the flux lines, upon entering the superconductor at the lower critical field, may enter as straight flux lines at two different orientations with respect to the $\hat{c}$ axis.

More recently, Sardella and Moore have detected certain tilt instabilities at very large wave numbers in systems of isolated flux lines,[13] and also speculated that this could be related to the results of Ref. 3. In particular they found critical angles at which tilt instabilities occurred which superfi-

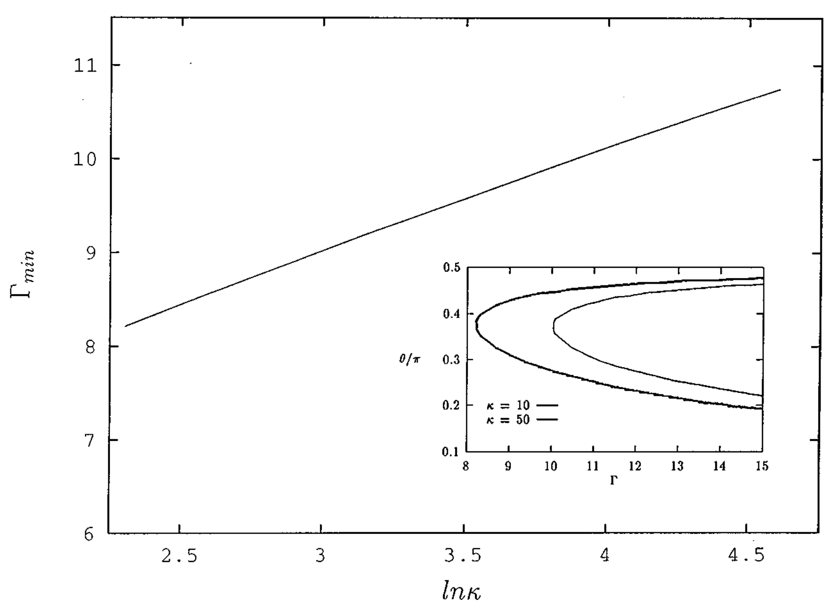

FIG. 3. Same as Fig. 2, using a smooth cutoff.

cially resemble the insets of Figs. 2 and 3; however we point out that, contrary to Figs. 2 and 3, Ref. 13 found that these critical angles depended on Γ, but not κ, in the limit where the wave number $k_z \to \infty$.

The issue thus arises whether the interpretation of the above result of two degenerate minima in the Gibbs energy as coexistence of two species of straight flux lines is in fact correct, or if they should rather be interpreted as an indication of the breakdown of the assumption of a ground state consisting of straight rigid flux lines. The result of this section relies on such an assumption, but if tilt instabilities were to *preempt* the onset of coexisting species of straight flux lines oriented at two different orientations with respect to the $\hat{c}$ axis, our above results and conclusions would be invalid.

In order to investigate this, we will in the following section consider in detail tilt instabilities of the noninteracting flux-line liquid. We shall find that the tilt mode considered in Ref. 13 is irrelevant for comparing with the results found in the present section, since it involves tilting of flux lines parallel to the *ab* planes, whereas the Gibbs energy of this section involves tilting perpendicular to the *ab* planes. We also consider in detail a tilt mode associated with tilting perpendicular to the *ab* plane, and find that this tilt mode has an instability, not in the limit $k_z \to \infty$ as the case was for the parallel tilt mode, but rather for the case $k_z \to 0$, which is also more physical within the London theory. Moreover, we map out the phase diagram where such instabilities occur and compare with Figs. 2 and 3. Again, we have found throughout that results do not depend on details of implementations of the core cutoff, and we thus only show results obtained by using a sharp cutoff procedure.

IV. EIGEN (TILT) MODES, NONINTERACTING FLUX-LINE LIQUID

The limit of a noninteracting flux-line liquid is obtained from the general expression for $\Phi_{\alpha\beta}(k)$ by converting the summations over the reciprocal-lattice vectors $\mathbf{Q}$ to an integration. This means that (i) all structure dependence of an underlying lattice disappears and therefore all shear stiffness vanishes, (ii) the limit $B \to 0$ is implied since $|\mathbf{Q}| \sim B^{1/2}$.

By comparing the expression for the linear elastic energy in the dilute limit, Eq. (5), with Eq. (3), it is seen that the elastic matrix in the noninteracting case takes the form

$$\Phi_{xx}(\mathbf{k}) = c_{44}^{\perp}(\mathbf{k}) k_z^2,$$
$$\Phi_{yy}(\mathbf{k}) = c_{44}^{\parallel}(\mathbf{k}) k_z^2, \tag{14}$$
$$\Phi_{xy}(\mathbf{k}) = 0.$$

For details of how to compute $c_{44}^{\perp}(k_z)$ and $c_{44}^{\parallel}(k_z)$, see Appendix B. In the isolated vortex liquid limit, the two tilt moduli only depend on the variable k_z, since the components k_x, k_y have no physical significance in this case. The eigenvalues of the elastic matrix for this case are given by

$$\lambda^{(+)} = k_z^2 c_{44}^{\perp},$$
$$\lambda^{(-)} = k_z^2 c_{44}^{\parallel}. \tag{15}$$

At $\theta=0$, we have $\Phi_{xx} = \Phi_{yy}$, due to the rotational symmetry about the $\hat{c}$ axis in this case, hence $c_{44}^{\perp}(k_z) = c_{44}^{\parallel}(k_z)$. For completeness, we give an analytical expression for $\Phi_{xx} = \Phi_{yy} \equiv \Phi$ in this case.

$$\Phi(k_z) = \frac{B^2 k_z^2}{32\pi^2 n \lambda_c^2} \ln Y(k_z) + \frac{B^2}{32\pi^2 n \lambda_a^4} \ln Z(k_z),$$
$$Y(k_z) = \frac{\xi_a^{-2} + a(k_z)/\lambda_c^2}{q_0^2 + a(k_z)/\lambda_c^2},$$
$$Z(k_z) = \frac{Y(0)}{Y(k_z)}, \tag{16}$$
$$a(k_z) = 1 + \lambda_a k_z^2; \quad q_0^2 = k_{\mathrm{BZ}}^2/2; \quad n = B/\Phi_0,$$

where $\Phi_0 = h/2e$ is the elementary flux quantum, $b = B/H_{c2}$, and k_{BZ} indicates the radius of a circularized Brillouin zone of the flux-line lattice, which however vanishes in the limit $B \to 0$. In principle, this expression shows the tendency towards a crossover from the Lorentzian contributions to the elastic moduli in the dense limit $b \sim 1$, and the logarithmically dispersive elastic moduli in the extreme dilute limit, $b \to 0$. The single vortex limit is obtained by setting $b=0$ in the factors Y and Z, and the logarithmically dispersive elastic modulus for the isolated flux line at $\theta=0$ is found by comparing Eqs. (14) and (16)

$$c_{44}(k_z) = \frac{B\Phi_0}{32\pi^2}\left[\frac{\ln Y}{\lambda_c^2} + \frac{\ln Z}{k_z^2 \lambda_a^4}\right]. \tag{17}$$

Note that the first contribution tends to 0 for $k_z=0$ in the extreme anisotropic limit $\Gamma \to \infty$, whereas the latter contribution remains finite. It is interesting that London theory correctly produces both logarithms in the tilt modulus. For general θ, we have $c_{44}^{\parallel} > c_{44}^{\perp}$.

The elastic moduli $c_{44}^{\perp}$ and $c_{44}^{\parallel}$ at arbitrary θ are obtained numerically from the procedure described in Appendix B for the dilute case. They are shown in Figs. 4–7.

The angular dependence of the the perpendicular tilt modulus $c_{44}^{\perp}$ and the parallel tilt modulus $c_{44}^{\parallel}$ for $k_z=0$ is shown in Figs. 4 and 5. The main difference between these two quantities at $k_z=0$ is that the parallel tilt modulus never goes negative for any value of θ, whereas the perpendicular does. Figure 4 shows that when Γ increases, $c_{44}^{\perp}(\theta)$ develops a sharp peak at $\theta=\pi/2$. The range $\theta_1^* < \theta < \theta_3^*$ where $c_{44}^{\perp}(\theta)$ changes sign, increases, and the lower critical tilt angle θ_1^* moves towards 0, as expected on physical grounds. Note also that θ_2^* which gives the minimum in the Gibbs energy, moves towards $\pi/2$. (For an explanation of the symbols $\theta_1^*, \theta_2^*, \theta_3^*$, see Fig. 4). In Fig. 4, a *sharp* cutoff on the $\mathbf{k}$-space integrals at large values of $\mathbf{k}$ have been used. Figure 5 shows results for the parallel tilt modulus, which is positive in the entire range $\theta \in [0, \pi/2]$ at $k_z=0$. The prominent peak at $\theta=\pi/2$ which was found for the perpendicular tilt modulus is not found for the parallel tilt modulus. This is because at $\theta=\pi/2$, the parallel tilt modulus $c_{44}^{\parallel}$ describes tilting *in the ab plane*, and the remnants of a lock-in transition found within Lawrence-Doniach theory is no longer felt for this geometry.

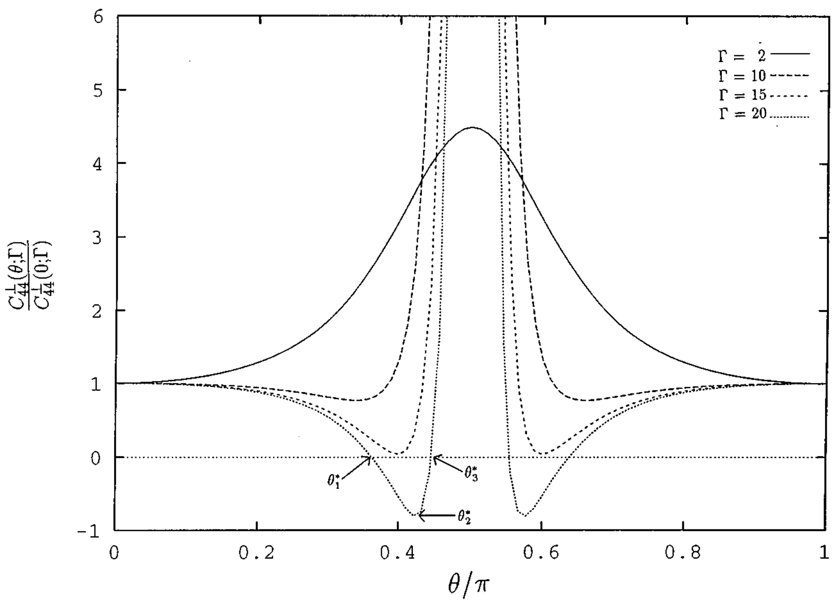

FIG. 4. The "perpendicular" tilt modulus $c_{44}^{\perp}(\theta)$ for isolated flux lines, for fixed $\kappa=20$, $k_z=0$ for various values of $\Gamma=2,10,15,20$. For each Γ the corresponding $c_{44}^{\perp}(\theta;\Gamma)$ is normalized such that $c_{44}^{\perp}(\theta=0;\Gamma)=1$. A sharp cutoff has been used.

The dispersion of both tilt moduli are shown in Figs. 6 and 7, for various values of θ. In Fig. 6 it is seen that for $\theta=0$ only a weak logarithmic dispersion is found for $c_{44}^{\perp}(k_z)$, as previously found analytically in Ref. 17. It is also seen that $c_{44}^{\perp}(k_z)$ decreases with increasing θ. For $\theta_1^* < \theta < \theta_3^*$, the perpendicular tilt modulus $c_{44}^{\perp}(k_z)$ is negative for small values of k_z and obtains a minimum for $\theta=\theta_2^*$. For $\theta>\theta_3^*$ (not shown), $c_{44}^{\perp}(k_z)$ is again positive for all k_z, and for $\theta=\pi/2$ (not shown) $c_{44}^{\perp}(k_z)$ has a logarithmic dispersion, similar as for the case $\theta=0$. In Fig. 6, a sharp cutoff on **k**-space integrals have been used. Figure 7 shows the θ dependence for $c_{44}^{\parallel}(k_z)$, obtained by using a sharp cutoff. As θ increases, the parallel tilt modulus $c_{44}^{\parallel}(k_z)$ decreases and changes sign for *large* k_z. For increasing θ greater than a critical θ^* (not shown) $c_{44}^{\parallel}(k_z)$ increases, and for θ near $\pi/2$, $c_{44}^{\parallel}(k_z)$ is again positive for all k_z.

Based on the above discussions, we believe that the issue of the *ad hoc* introduction of a core cutoff within London theory is not a particularly serious one, provided only that a qualitatively correct θ-dependent shape of the core (elliptic contours of constant currents, with correct aspect ratios) are used.

A striking difference between $c_{44}^{\perp}(k_z)$ and $c_{44}^{\parallel}(k_z)$ is the range in k_z in which these moduli become negative. For $c_{44}^{\perp}$, it happens at intermediate values of θ, and for small values of k_z. For $c_{44}^{\parallel}$ it also happens at intermediate values of θ, but now only for large values of k_z. Results for large values of k_z must be viewed with a certain suspicion when obtained within London theory, since it is a description of layered superconductors in terms of an anisotropic continuum. For large values of Γ and large values of k_z, a description where the discreteness of the layers is accounted for is undoubtedly more appropriate, and the Lawrence-Doniach energy functional provides a considerably more reliable starting point.

Again, we may map out a phase diagram (Γ,κ) which shows the minimum mass-anisotropy versus $\ln \kappa$, required to

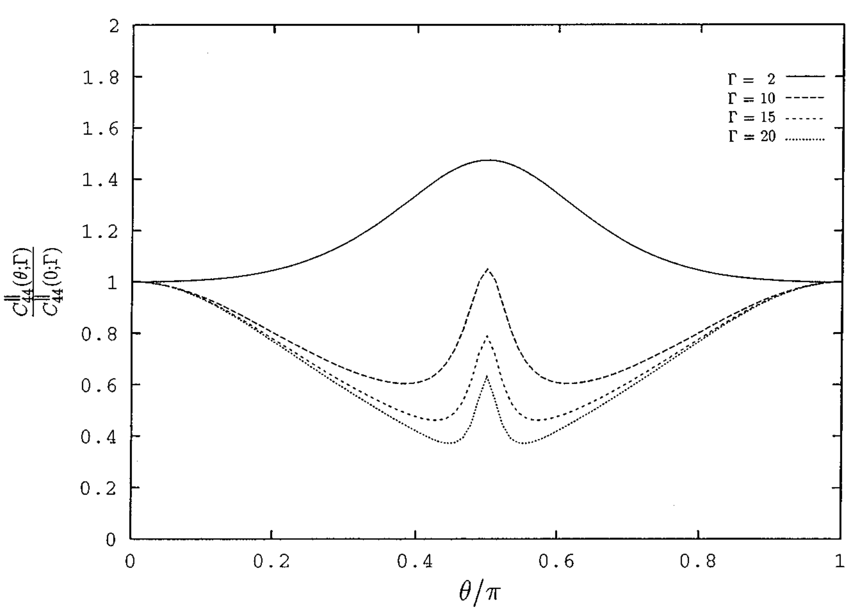

FIG. 5. Same as Fig. 4, for $c_{44}^{\parallel}(\theta)$, using a sharp cutoff.

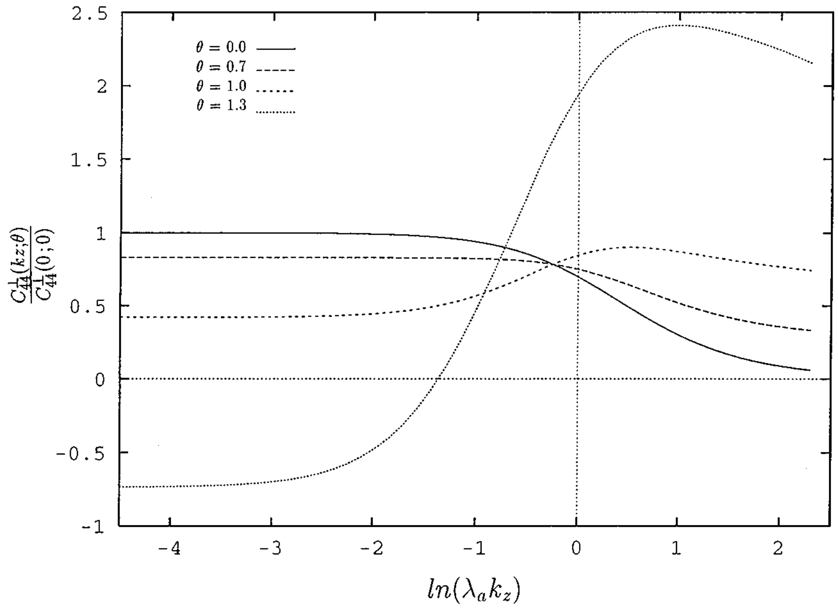

FIG. 6. $c^{\perp}_{44}(k_z;\theta)$ as a function of $\ln(\lambda_a k_z)$, for fixed $\kappa=20$, $\Gamma=20$ for various values of $\theta=[0,\ 0.7,\ 1.0,\ 1.3]$. $c^{\perp}_{44}(k_z;\theta)$ is normalized such that $c^{\perp}_{44}(k_z=0;\theta=0)=1$, and k_z is measured in units of λ_a. A sharp cutoff has been used.

see negative tilt moduli at any value of θ. This is shown in Fig. 8 for $c^{\perp}_{44}(k_z)$ for various values of k_z and using a sharp cutoff. Figure 9 shows the same for $c^{\parallel}_{44}(k_z)$. For $c^{\perp}_{44}(k_z)$, the critical mass anisotropy is increased upon increasing k_z; the trend is the opposite for $c^{\parallel}_{44}(k_z)$. Comparing the $k_z=0$ result with the results of Figs. 2 and 3, it is clear that the critical values of Γ required to see negative tilt moduli are quite different from those required to see double minima in the Gibbs energy. No negativity occurs at $k_z=0$ for the parallel tilt modulus $c^{\parallel}_{44}$, which is the one considered for instance in Ref. 13. On the other hand, we have seen that the perpendicular tilt modulus $c^{\perp}_{44}$ may become negative at $k_z=0$. *This only happens at considerably larger value of* Γ *than those required to see double minima in the Gibbs energy*, as discussed in the previous section. The critical angles (θ_1,θ_2) discussed in Sec. III are shown in Figs. 10 and 11 for $c^{\perp}_{44}$ and $c^{\parallel}_{44}$, respectively.

We conclude that the correct interpretation of the results of the previous section is indeed that a situation has established itself close to the lower critical field where straight flux lines characterized by different orientations coexist in a window of mass anisotropies $\Gamma_{c1}<\Gamma<\Gamma_{c2}$. Γ_{c1} is found from Figs. 2 and 3, whereas Γ_{c2} is found for $k_z=0$ in Fig. 6. When $\Gamma>\Gamma_{c2}$, the ground state of noninteracting flux lines is not one where the flux lines are straight and oriented obliquely with respect to the $\hat{c}$ axis. What the actual ground-state vortex configuration in this case is, in the limit of extremely low inductions, remains for future investigations, and should presumably be investigated using the Lawrence-Doniach theory.

V. EIGENMODES, INTERACTING FLUX-LINE LIQUID

In this section we consider the opposite situation of that in the previous section, namely that of a dense vortex liquid. In the dense limit, it is permissible to consider the ensemble of

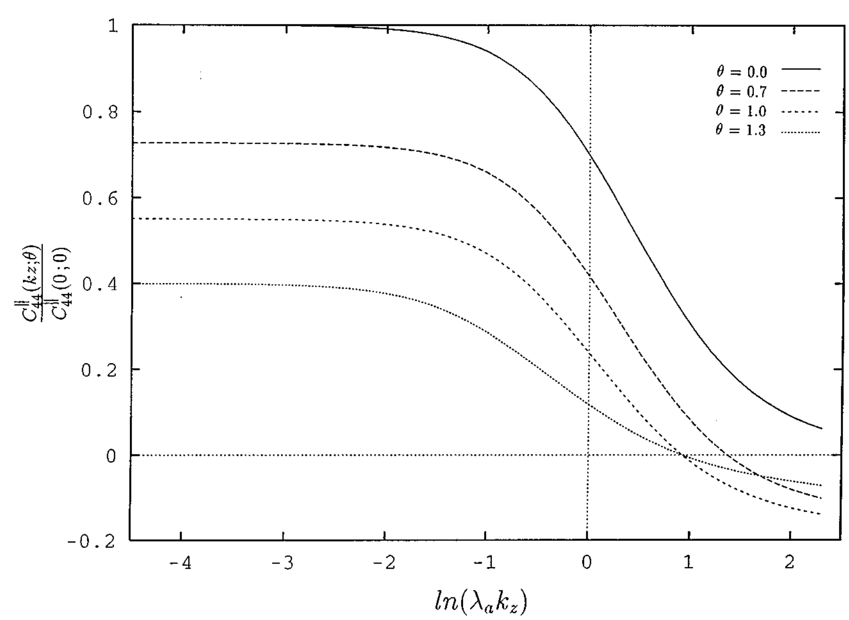

FIG. 7. Same as Fig. 8, for $c^{\parallel}_{44}(k_z;\theta)$, using a sharp cutoff.

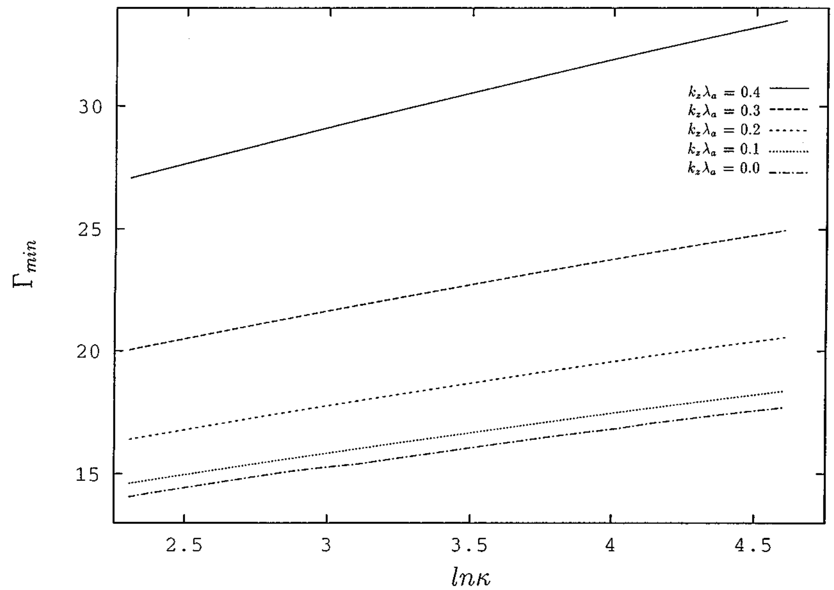

FIG. 8. Critical value of Γ as a function of $\ln\kappa$ for observing a change of sign in $c_{44}^{\perp}(\theta,\Gamma,\kappa;k_z)$ for isolated flux lines at any value of θ, for various values of k_z, $k_z\lambda_a$=[0, 0.1, 0.2, 0.3, 0.4]. A sharp cutoff has been used.

flux lines as a continuum where one averages spatially over the distribution of flux lines, to arrive at a continuum description where all information about the structure of the individual vortices are ignored. Thus, we retain only the $\mathbf{Q}=0$ term in the lattice sums for the matrix elements $\Phi_{\alpha\beta}(\mathbf{k})$. This gives

$$\Phi_{xx}(\mathbf{k})=c_{11}(\mathbf{k})k_x^2+c_{44}^{\perp}(\mathbf{k})k_z^2+2c_{41}(\mathbf{k})k_xk_z\,,$$

$$\Phi_{yy}(\mathbf{k})=c_{11}(\mathbf{k})k_y^2+c_{44}^{\parallel}(\mathbf{k})k_z^2, \tag{18}$$

$$\Phi_{xy}(\mathbf{k})=c_{11}(\mathbf{k})k_xk_y+c_{41}(\mathbf{k})k_yk_z\,,$$

where the elastic moduli are read off directly from the general expressions for $\Phi_{\alpha\beta}(\mathbf{k})$ (Ref. 13) (see Appendix B)

$$c_{11}(\mathbf{k})=\frac{B^2}{4\pi}\left[\frac{1+\lambda_a^2k^2+(\lambda_c^2-\lambda_a^2)\cos^2\theta k^2}{[1+\lambda_a^2k^2][1+\lambda_a^2k^2+(\lambda_c^2-\lambda_a^2)q^2]}\right],$$

$$c_{44}^{\perp}(\mathbf{k})=\frac{B^2}{4\pi}\left[\frac{1+\lambda_a^2k^2+(\lambda_c^2-\lambda_a^2)\sin^2\theta k^2}{[1+\lambda_a^2k^2][1+\lambda_a^2k^2+(\lambda_c^2-\lambda_a^2)q^2]}\right], \tag{19}$$

$$c_{44}^{\parallel}(\mathbf{k})=\frac{B^2}{4\pi}\left[\frac{1}{[1+\lambda_a^2k^2+(\lambda_c^2-\lambda_a^2)q^2]}\right],$$

$$c_{41}(\mathbf{k})=\frac{B^2}{4\pi}\left[\frac{(\lambda_c^2-\lambda_a^2)\cos\theta\,\sin\theta k^2}{[1+\lambda_a^2k^2][1+\lambda_a^2k^2+(\lambda_c^2-\lambda_a^2q^2]}\right],$$

where $\mathbf{q}=\mathbf{k}\times\hat{c}$. The modulus $c_{41}(\mathbf{k})$ is a mixed shear and bulk modulus discussed by Sardella, and it has no counterpart in the isotropic case.

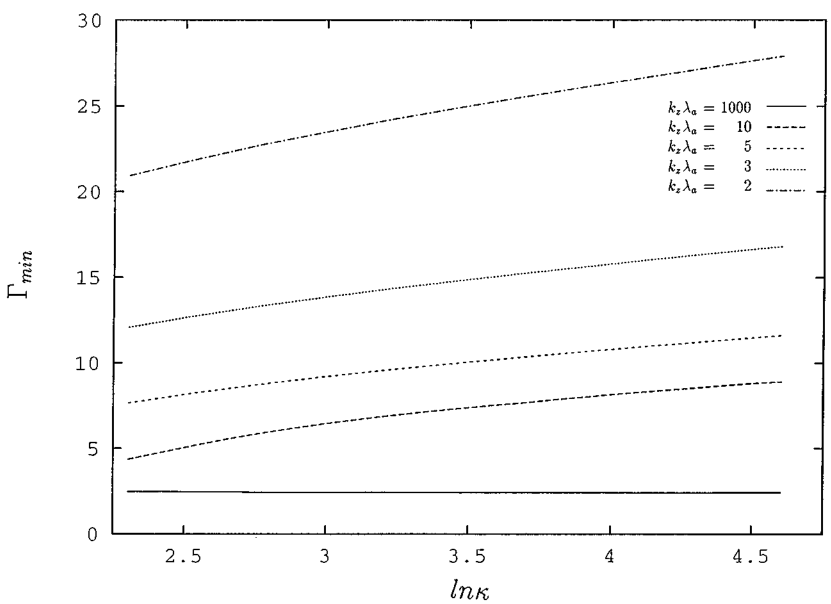

FIG. 9. Same as Fig. 12, for $c_{44}^{\parallel}(\theta)$, using a sharp cutoff.

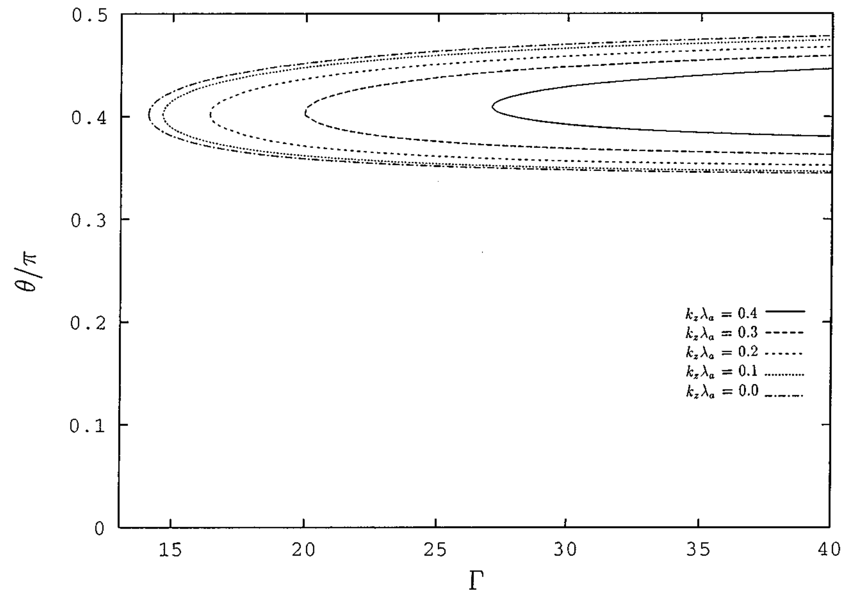

FIG. 10. The critical orientations (θ_1,θ_2) where $c_{44}^{\perp}(\theta)$ for isolated flux lines changes sign, as a function of Γ, for fixed $\kappa=10$ for various values of $k_z, k_z\lambda_{ab}=[0,\ 0.1,\ 0.2,\ 0.3,\ 0.4]$. A sharp cutoff has been used.

When $\theta=0$, the mixed bulk-tilt modulus vanishes, while the two tilt moduli become identically equal. In this case, the eigenvalues corresponding to the longitudinal and transverse eigenmodes of the elastic matrix, λ_L and λ_T, respectively, are given by the well-known expressions

$$\lambda_L = c_{11}(\mathbf{k})k_{\perp}^2 + c_{44}(\mathbf{k})k_z^2,$$
$$\lambda_T = c_{44}(\mathbf{k})k_z^2. \tag{20}$$

Note that for $\theta=0$, positive elastic constants guarantee stable nonsoft modes, i.e., no instabilities occur in the flux-line liquid if elastic moduli are positive definite. Note also that in the interacting flux-line-liquid limit, $c_{44}^{\perp} > c_{44}^{\parallel}$, contrary to the isolated vortex liquid case. Their dispersions are Lorentzians directly reflecting the screened Coulomb-type interaction between flux-line elements (which has a tensorial nature); moreover they never change sign.

When $\theta>0$, the longitudinal and transverse eigenvalues of a tilted flux-line liquid, λ_L and λ_T, are given by

$$\lambda_L = \frac{1}{2}\left[\nu + \sqrt{\nu_+\nu_-}\right],$$

$$\lambda_T = \frac{1}{2}\left[\nu - \sqrt{\nu_+\nu_-}\right],$$

$$\nu = c_{11}k_{\perp}^2 + (c_{44}^{\perp} + c_{44}^{\parallel})k_z^2 + 2c_{41}k_yk_z, \tag{21}$$

$$\nu_{\pm} = c_{11}k_{\perp}^2 + (c_{44}^{\perp} - c_{44}^{\parallel})k_z^2 + 2\tilde{c}_{41}^{\pm}k_yk_z,$$

$$\tilde{c}_{41}^{\pm} = c_{41} \pm \sqrt{c_{11}(c_{44}^{\perp} - c_{44}^{\parallel})}.$$

The stability of a tilted flux-line liquid is somewhat less of a trivial matter than what one naively would have thought, based simply on the positivity of all the elastic constants in

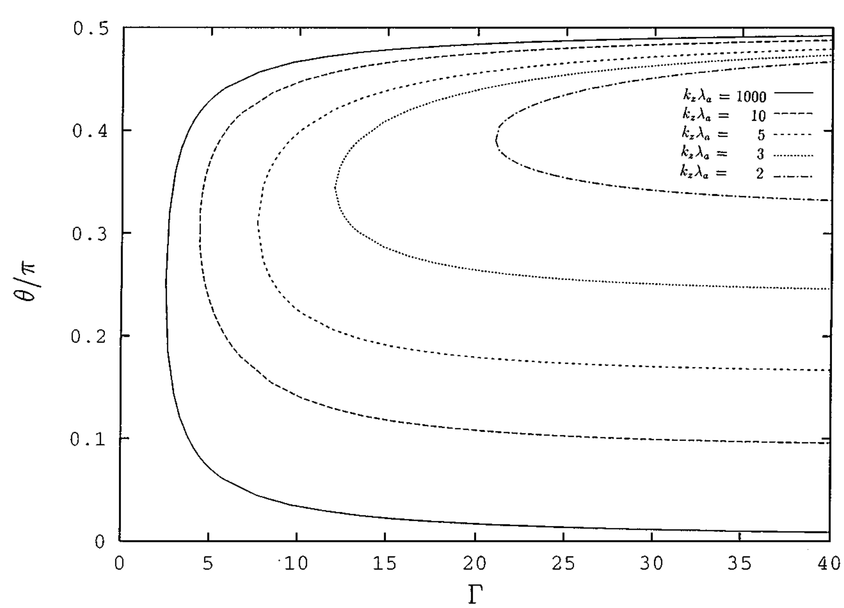

FIG. 11. The critical orientations (θ_1,θ_2) where $c_{44}^{\parallel}(\theta)$ for isolated flux lines changes sign, as a function of Γ for fixed $\kappa=10$ for various values of $k_z, k_z\lambda_{ab}=2,3,5,10,1000$. A sharp cutoff has been used.

the dense flux-line liquid limit Eq. (19). The reason is the following: due to the appearance of a term $c_{41}k_y k_z$ *even in the eigenvalues*, it is clear that even if $c_{41}>0$, this term does not have a definite sign *and can be negative*. We now investigate whether or not the lowest eigenvalue λ_T could give rise to a soft mode $\lambda_T=0$ in this interacting case. Setting $\lambda_T=0$ leads to the condition

$$c_{44}^{\parallel}\Delta + c_{11}(c_{44}^{\perp}-c_{44}^{\parallel})k_y^2=0,$$
$$c_{11}k_{\perp}^2+c_{44}^{\perp}k_z^2+2c_{41}k_y k_z\equiv\Delta. \tag{22}$$

The appearance of a soft mode λ_T requires at least $\Delta<0$, since $c_{44}^{\perp}>c_{44}^{\parallel}>0$. Such a lack of positive definiteness in an eigenmode can only come from the third term in Δ. Hence, an instability of the flux-line liquid, were it to exist, would thus be due to a mixed bulk and tilt deformation of the flux lines. Inserting the expression for the various elastic moduli into the expression for Δ, we find the condition for $\Delta=0$

$$1+\lambda_{1\theta}^2 k_{\perp}^2+\lambda_{2\theta}^2 k_z^2+(\lambda_c^2-\lambda_a^2)\sin(2\theta)k_y k_z=0, \tag{23}$$

where $\lambda_{1\theta}^2=\lambda_a^2\sin^2\theta+\lambda_c^2\cos^2\theta$, $\lambda_{2\theta}^2=\lambda_a^2\cos^2\theta+\lambda_c^2\sin^2\theta$, and where the last term on the left-hand side originates in c_{41}. We may use this condition to find a "dispersion relation" for a soft mode $k_z=k_z(k_x,k_y)$, where however the condition for real solutions is given by

$$(\lambda_c^2-\lambda_a^2)^2 k_y^2\sin^2\theta\cos^2\theta\geqslant\lambda_{2\theta}^2(1+\lambda_{1\theta}^2 k_{\perp}^2). \tag{24}$$

This inequality is most easily satisfied, if at all, for $k_x=0$. After some algebra, it is found that this implies that $\lambda_a^2\lambda_c^2<0$, which is not possible. We conclude that any tendency toward a bulk-tilt instability in an interacting tilted vortex liquid in uniaxially anisotropic superconductors, is suppressed by interactions between flux lines reflected by the presence of the bulk modulus c_{11}. However, the mixed bulk-tilt modulus discovered by Sardella[13] leads to a softening of the transverse elastic response of the flux-line liquid, *in certain regions of* **k** *space*. Whether or not this result is of any importance for such phenomena as flux-line-lattice melting,[18,19] where the dominant Fourier modes contributing to destabilizing the flux-line lattice are located at the Brillouin-zone boundary,[19,20] remains to be investigated. This requires incorporation, into the linear elastic energy, of the easy and hard shear moduli discussed by Kogan and Campbell,[14] as well as the mixed shear and tilt modulus discovered by a scaling approach by Blatter and co-workers.[12,21]

VI. CONCLUSION

A flux-line liquid with average induction tilted an angle θ away from the symmetry axis of a uniaxial superconductor exhibits two different tilt modes associated with tilt components perpendicular and parallel to the CuO_2 planes. At $k_z=0$ the perpendicular tilt mode may go soft for large enough mass anisotropy, whereas the parallel tilt mode does not. At large values of k_z, the situation is reversed. The situation where k_z is large $k_z\xi_a\sim 1$ is presumably unphysical since the London theory is a continuum theory of superconductors. A more useful starting point for discussing short-wavelength tilt instabilities is the Lawrence-Doniach theory.

We have shown that the softness of Φ_{yy} in the noninteracting case is not connected with the possibility of coexisting species of flux lines characterized by different orientations relative to the uniaxial symmetry axis in an anisotropic superconductor. The relevant tilt instability to compare the coexistence with is the one associated with the softness of Φ_{xx}. The $k_z=0$ tilt instability associated with Φ_{xx} sets in at much larger values of the mass anisotropy, and thus does not preempt the coexistence of straight flux lines of different orientations with respect to the $\hat{c}$ axis.

The above picture could conceivably be altered by the correct physical interpretation of the large-k_z instability found in $\Phi_{yy}(k_z)$. This must await further treatment outside the scope of anisotropic London theory, or any theory which describes layered superconductors as an anisotropic continuum. *This is not an issue of how to introduce core cutoffs in London theory*. Anisotropic Ginzburg-Landau theory would suffer from the same short-comings as the London theory in correctly describing the short-distance physics (in the $\hat{c}$ direction) of layered superconductors. The correct framework for such an investigation appears to be the Lawrence-Doniach theory. If the large-k_z instability of $\Phi_{yy}(k_z)$ survives a correct treatment of the short-distance physics in the $\hat{c}$ direction of layered superconductors, and the $k_z\rightarrow\infty$ tilt instability is found to be the most unstable mode, then it is not ruled out that the phase diagram (Γ,κ) for this mode may be such as to entirely preempt coexistence of straight flux lines.

An interesting issue is to what extent thermally excited Fourier tilt modes will affect the melting of the flux-line lattice in a tilted field configuration at low inductions. When interactions between flux lines are taken into account in a flux-line liquid, it is found that in the dense case, all pure tilt moduli are positive. The tilt instabilities discovered in the noninteracting case thus are suppressed by interactions between flux lines. A modulus which mixes tilt and bulk modes gives rise to a term in the eigenmode spectrum which is not positive definite, and, in principle, may go soft. It is found that this tilt-bulk instability is also suppressed by interactions between flux lines. The lack of positive definiteness of the mixed bulk-tilt term does however soften the lowest eigenvalue. Thus, the tilt-bulk contribution to the transverse eigenmode may also have important consequences for such phenomena as flux-line-lattice melting in tilted magnetic-field configurations.

ACKNOWLEDGMENTS

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APPENDIX A: REAL-SPACE ANISOTROPIC LONDON POTENTIAL

We start from the expression in momentum space for the anisotropic London potential, when the average induction is oriented at an angle θ with respect to the $\hat{c}$ axis

$$\tilde{V}_{\alpha\beta}(\mathbf{k})=\frac{1}{1+\Lambda_1 k^2}\left[\delta_{\alpha\beta}-\frac{\Lambda_2 q_\alpha q_\beta}{1+\lambda_1 k^2+\Lambda_2 q^2}\right],$$
$$\Lambda_1=\lambda_a^2;\quad \Lambda_2=\lambda_c^2-\lambda_a^2;\quad \mathbf{q}=\mathbf{k}\times\hat{c},\tag{A1}$$

where the real-space potential is given by

$$\tilde{V}_{\alpha\beta}(\mathbf{r})=\int\frac{d^3k}{(2\pi)^3}\tilde{V}_{\alpha\beta}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$=\frac{\delta_{\alpha\beta}}{4\pi\lambda_a^2}\frac{e^{-r/\lambda}}{r}+V_2^{\alpha\beta}(\mathbf{r}),$$
$$V_2^{\alpha\beta}(\mathbf{r})=-\Lambda_2[[\nabla[(\nabla I(\mathbf{r}))\times\hat{c}]_\alpha]\times\hat{c}]_\beta,\tag{A2}$$
$$I(\mathbf{r})=\Lambda_2\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(1+\Lambda_1k^2)(1+\Lambda_1k^2+\Lambda_2q^2)}.$$

In the above, $(\alpha,\beta)\in(x,y,z)$. The procedure for Fourier transforming $\tilde{V}_{\alpha\beta}(\mathbf{k})$ is described in Ref. 11, for the case $\theta=0$, and here we generalize it to the case $\theta>0$. This result is useful for considering for instance the stability of twisted multiplets of vortices in vortex lattices and vortex liquids, when the vortices are oriented at an average oblique angle with respect to the $\hat{c}$ axis.

The first step for $\theta>0$ is to introduce new variables

$$k_x=\cos\theta k_x'-\sin\theta k_z',$$
$$k_y=k_y',\tag{A3}$$
$$k_z=\sin\theta k_x'+\cos\theta k_z',$$

and similarly in real space

$$x'=\cos\theta x+\sin\theta z,$$
$$y'=y,\tag{A4}$$
$$z'=-\sin\theta x+\cos\theta z.$$

Now we can express the auxiliary function $I(\mathbf{r})$, after an angular integration, and redefining $\mathbf{k}'\to\mathbf{k}$

$$I(\mathbf{r})=\int_{-\infty}^{\infty}\frac{dk_z}{(2\pi)^2}\frac{\cos(k_z z')}{1+\Lambda_1k_z^2}F(k_z,\rho'),$$
$$F(k_z,\rho')=\int_0^\infty dk_\perp k_\perp\left[\frac{(\Lambda_1+\Lambda_2)J_0(k_\perp\rho')}{1+(\Lambda_1+\Lambda_2)k_\perp^2+\Lambda_1k_z^2}-\frac{\Lambda_1J_0(k_\perp\rho')}{1+\Lambda_1k_\perp^2+\Lambda_1k_z^2}\right],\tag{A5}$$
$$\rho'=\sqrt{(\mathbf{r}\times\hat{c})^2}.$$

The integrations in F are of a variety considered previously,[11] for the case $\theta=0$; all θ dependence is now subsumed in the variables ρ',z', which are not involved in the integrations. The $k_\perp$ integrations produce modified zeroth-order Bessel functions. We get

$$I(\mathbf{r})=\int_{-\infty}^{\infty}\frac{dk_z}{(2\pi)^2}\frac{\cos(k_z)}{1+\Lambda_1k_z^2}[K_0(z_1)-K_0(z_2)]$$
$$\equiv I_1-I_2,$$
$$z_i=\alpha_i\sqrt{1+\Lambda_1k_z^2},\tag{A6}$$
$$\alpha_i=a_i\rho';\quad a_1=1/\sqrt{\Lambda_1+\Lambda_2};\quad a2=1/\sqrt{\Lambda_1},$$

where K_0 are modified zeroth-order Bessel functions. The remaining integrations in $I(\mathbf{r})$ appear difficult to perform, but are not required: All we need are various *derivatives* of $I(\mathbf{r})$ which are easily obtained. We list two results that are useful when working out the derivatives in Eq. (A1)

$$\nabla I_i(\mathbf{r})=\nabla\alpha_i\frac{\partial I_i}{\partial\alpha_i}+\nabla z'\frac{\partial I_i}{\partial\alpha_i},$$
$$\frac{\partial I_i}{\partial\alpha_i}=-\frac{1}{2\pi^2}\frac{1}{\sqrt{\Lambda_1}}\sqrt{\frac{\pi}{2}}\frac{1}{\alpha_i}\left[\alpha_i^2+\frac{(z')^2}{\Lambda_a}\right]^{1/4}\times K_{1/2}\left(\sqrt{\alpha_i^2+(z')^2/\Lambda_1}\right),\tag{A7}$$

where $K_{1/2}(x)=\sqrt{\pi/2x}\,\exp(-x)$. Performing all necessary differentiations, and adding together, we get the results listed in Sec. II.

APPENDIX B: ELASTIC MATRIX ELEMENTS

For numerical calculations at arbitrary angle θ, it is convenient to express the elastic matrix in the following form:

$$\Phi_{\alpha\beta}(\mathbf{k})=\frac{B^2}{4\pi}\sum_{\mathbf{Q}}\left[\frac{N_{\alpha\beta}(\mathbf{K})}{D(\mathbf{K})}S(\mathbf{K}_\perp)-\frac{N_{\alpha\beta}(\mathbf{Q})}{D(\mathbf{Q})}S(\mathbf{Q})\right],$$
$$D(\mathbf{K})=(1+\Lambda_1K^2)(1+\Lambda_1K^2+\Lambda_2q^2),$$
$$N_{xx}(\mathbf{K})=K_x^2[1+(\Lambda_1+\Lambda_2\cos^2\theta)K^2]+k_z^2[1+(\Lambda_1+\Lambda_2\sin^2\theta)K^2]+k_zK_x\sin2\theta K^2,$$
$$N_{yy}(\mathbf{K})=K_y^2[1+(\Lambda_1+\Lambda_2\cos^2\theta)K^2]+k_z^2[1+\Lambda_1K^2],$$
$$N_{xy}(\mathbf{K})=K_xK_y[1+(\Lambda_1+\Lambda_2\cos^2\theta)K^2]+\Lambda_2k_zK_y\sin\theta\cos\theta K^2,\tag{B1}$$

where $\mathbf{K}=\mathbf{k}+\mathbf{Q}$, and $\mathbf{q}=\mathbf{K}\times\hat{c}$. $S(\mathbf{k}_\perp)$ formally denotes a cutoff function introduced *ad hoc* into the London theory as an ''ultraviolet regularization,'' due to the lack of an explicit description of the vortex core. It may be chosen quite freely, either as a sharp or a smooth cutoff. In this paper we consider both sharp and smooth cutoff procedures, i.e., we have chosen two model cutoff functions, S_1 and S_2, defined by

$$S_1(\mathbf{k}_\perp)=\Theta[1-g(\mathbf{k}_\perp)],$$
$$S_2(\mathbf{k}_\perp)=e^{-g(\mathbf{k}_\perp)},\tag{B2}$$
$$g(\mathbf{k}_\perp)=\xi_a^2k_y^2+\xi_\theta^2k_x^2,$$

where $\Theta(x)$ is the Heavyside step function. Note that we have little freedom in choosing the function $g(\mathbf{k}_\perp)$, since the correct elliptical shape of the vortex core is essential at large mass anisotropies, and must be accounted for to get correct results; the "logarithmically weak" dependence of the core cutoff on quantities such as self-energy and elastic constants does become significant when M_z/M is very large.

In principle, the above expression for the elastic matrix contains all information on the elasticity of the flux-line lattice, since even structure-dependent terms emerge as a result of the discrete summation over lattice vectors $\mathbf{Q}$. These vectors should only belong to an equilibrium lattice configuration, otherwise the expansion around the vortex state with energy E_0 is meaningless. There are two, in some sense opposite, limits of Eq. (B1). The simplest one to treat analytically is the one involving only the $\mathbf{Q}=0$ term in $\Phi_{\alpha\beta}(\mathbf{k})$. The assumption that the $\mathbf{Q}=0$ term is dominant amounts to a *high-density* limit, since the remaining terms will be of order $1/(1+4\pi\kappa^2 b/\sqrt{3})$ compared to the leading term, where $b=B/H_{c2}(T)$. The quality of this approximation increases with increasing κ, since vortex fields overlap to a stronger degree when the penetration depth increases. Physically, this approximation amounts to considering a dense, interacting flux-line *liquid*, since all information stemming from any structure dependence of the flux-line ensemble, is lost. When the induction becomes low, such that $\kappa^2 b \ll 1$, i.e., the dilute limit, the terms in $\Phi_{\alpha\beta}$ fall off sufficiently slowly that the first term no longer adequately represents the entire sum. In this case the summation over $\mathbf{Q}$ must be performed, in general numerically. There is however a limit which leads to drastic simplifications, namely the extreme low-induction limit where the lattice vectors may be considered to span a continuum. Then the sum may be converted to an integral, which in turn implies that all structure-dependent information again is lost. Physically, this corresponds to the limit of a gas of flux lines with no interaction *between different* flux lines.

We first consider the dense flux-line-liquid limit. Retaining only the $\mathbf{Q}=0$ term in $\Phi_{\alpha\beta}(\mathbf{k})$, we find

$$\Phi_{xx}(\mathbf{k})=\frac{B^2}{4\pi}\,\frac{k_x^2[1+(\Lambda_1+\Lambda_2\cos^2\theta)k^2]+k_z^2[1+(\Lambda_1+\Lambda_2\sin^2\theta)k^2]+\Lambda_2 k_z k_x\sin 2\theta k^2}{(1+\Lambda_1 k^2)(1+\Lambda_1 k^2+\Lambda_2 q^2)},$$

$$\Phi_{yy}(\mathbf{k})=\frac{B^2}{4\pi}\,\frac{k_y^2[1+(\Lambda_1+\Lambda_2\cos^2\theta)k^2]+k_z^2(1+\Lambda_1 k^2)}{(1+\Lambda_1 k^2)(1+\Lambda_1 k^2+\Lambda_2 q^2)},$$

$$\Phi_{xy}(\mathbf{k})=\frac{B^2}{4\pi}\,\frac{k_x k_y[1+(\Lambda_1+\Lambda_2\cos^2\theta)k^2]+\Lambda_2 k_z k_y\sin\theta\cos\theta k^2}{(1+\Lambda_1 k^2)(1+\Lambda_1 k^2+\Lambda_2 q^2)}. \tag{B3}$$

Comparing this with Eq. (18), we immediately identify the elastic constants in Eq. (19).

We next consider the elastic matrix elements in the limit of a noninterating vortex liquid, i.e., $k_x=k_y=0$, and where the discrete summation over $\mathbf{Q}$ is replaced by an integration. In this case we have

$$\Phi_{\alpha\beta}(k_z)=\frac{B\Phi_0}{4\pi}\int\frac{d\mathbf{Q}}{(2\pi)^2}\left[\frac{N_{\alpha\beta}(\mathbf{Q},k_z)}{D(\mathbf{Q},k_z)}-\frac{N_{\alpha\beta}(\mathbf{Q})}{D(\mathbf{Q})}\right]S(\mathbf{Q}). \tag{B4}$$

Using Eq. (B1), it is seen that $\Phi_{xy}(k_z)=0$, moreover it can be shown that linear terms in k_z vanish by symmetry. The above integral expressions define the two tilt moduli $c_{44}^{\perp}(k_z)$ and $c_{44}^{\parallel}(k_z)$ for a noninteracting flux-line liquid. The necessary integration can be carried out analytically at $\theta=0$ and $\theta=\pi/2$. For arbitrary values of θ the integrations have been done numerically.

APPENDIX C: SELF-ENERGY OF RIGID FLUX LINES

The self-energy for a straight, rigid flux line is given by

$$J(\theta)=\frac{\Phi_0^2}{8\pi}\int\frac{d^2\mathbf{k}}{4\pi^2}\,\frac{1+\lambda_\theta^2 k_\perp^2}{[1+\lambda_a^2 k_\perp^2][1+\lambda_\theta^2 k_x^2+\lambda_c^2 k_y^2]},$$
$$\lambda_\theta^2\equiv\lambda_a^2\sin^2\theta+\lambda_c^2\cos^2\theta, \tag{C1}$$

where the symbols have been defined in the text. The $\mathbf{k}$-space summation is cutoff on an ellipse defined by Eq. (7), which we circularize by scaling the $\mathbf{k}$ variable

$$k_x\rightarrow k_x'=\frac{\xi_\theta}{\xi_{ab}}\,k_x\,;k_y\rightarrow k_y\,. \tag{C2}$$

This enables us to use simple two-dimensional polar coordinates (ϕ,k') for the angular and radial integrations, at the expense of introducing additional angular dependence into the integrand

$$1+\lambda_a^2 k_\perp^2\rightarrow 1+\lambda_a^2 k'^2+[\lambda_c^2-\lambda_\theta^2]\,\frac{\lambda_a^2}{\lambda_\theta^2}\,k'^2\cos^2\phi,$$

$$1+\lambda_\theta^2 k_\perp\rightarrow 1+\lambda_\theta^2 k'^2+[\lambda_c^2-\lambda_\theta^2]k'^2\cos^2\phi, \tag{C3}$$

$$1+\lambda_\theta^2 k_x^2+\lambda_c^2 k_y^2\rightarrow 1+\lambda_c^2 k'^2.$$

Inserting these transformed quantities into the expression for the self-energy, and renaming the integration variable $\mathbf{k}'\rightarrow$k, we get after performing the angular integration over ϕ

$$J(\theta)=\frac{\Phi_0^2}{16\pi^2}\frac{\xi_a}{\xi_\theta}\int_0^{\xi_a^{-1}}\frac{kdk}{1+\lambda_c^2k^2}\left[\frac{2c}{\sqrt{(b+2a)^2-b^2}}+\frac{d}{b}\left(1-\frac{2a}{\sqrt{(b+2a)^2-b^2}}\right)\right],$$

$$a\equiv 1+\lambda_a^2k^2,$$

$$b\equiv\frac{k^2\lambda_a^2}{\lambda_\theta^2}(\lambda_c^2-\lambda_\theta^2), \quad \text{(C4)}$$

$$c\equiv 1+\lambda_\theta^2k^2,$$

$$d\equiv(\lambda_c^2-\lambda_\theta^2)k^2.$$

Previous analytical approximations to $J(\theta)$ were obtained by approximating the square roots $2/\sqrt{(a+2b)^2-b^2}\approx 1/(b+a/2)$, which is a good approximation in hard type-II superconductors.

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