Stability of multicharged vortices in a model of superflow

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(Received 20 June 1995)

In the framework of the nonlinear Schrödinger equation (NLSE) as a model of superflow we found that multicharged vortices are very long-living objects, contrary to the accepted opinion. The lifetime of these entities is inversely proportional to the dissipation rate which can be incorporated phenomenologically into NLSE. We calculated unstable eigenvalues and corresponding eigenfunctions. The nonlinear stage of the instability is studied numerically. We discuss the implications of our observation in the context of spin-up, flow past a body, and turbulence experiments in a superfluid helium.

Quantum vortices play a crucial role in the dynamics of a strongly correlated system such as superflow or superconductor.[1–3] The vortices in superconductors carry a quant of magnetic flux and the vortices in superfluid possess quantized vorticity. The spinning up of the superfluid and the magnetization of the type-II superconductor is caused by the development of an array of quantum vortices (Tkachenko lattice and Abrikosov lattice, correspondingly[1,4]).

The common belief is that only the simplest, or single-charged vortices with the topological charge ± 1 are stable. That is the case for the well-known Ginzburg-Landau model.[2] More complex multicharged vortices with the topological charge $\pm n$ are known have higher energy and, therefore, decay into n single-charged or elementary vortices. In this paper we show that, for the model of superflow near zero temperature, the decay time of the multicharged vortex can be arbitrarily large. As a result, if this model describes adequately the dynamics of superflow, one can expect to detect multicharged vortices in various experiments. Some indications of the existence of multicharged vortices can be found in several experiments on vortex nucleation.[5,6] We will discuss here the possibility of creating multicharged vortices in spin-up and flow past a body experiments.

For zero temperature, the well-established model of superflow is the nonlinear Schrödinger equation (NLSE) in the form

$$\partial_t \Psi = i(\nabla^2 \Psi + \Psi - |\Psi|^2 \Psi), \tag{1}$$

where we rescaled time as $t \to t/t_0$, space as $r \to r/a$, where $t_0 = g/\hbar$ is the characteristic time on the order of 10^{-11} s, and $a = \hbar/\sqrt{2Mg}$ is the so-called healing length which for ^{4}He superfluid in the low-temperature limit is of the order of a few Å (g is the strength of the short-range interparticle potential, M is the atomic mass of ^{4}He). This equation in the context of superfluid was obtained by Gross and Pitaevskii,[7,8] and is also widely used in nonlinear optics and other applications.[9] The equation can be represented in the Hamiltonian form

$$\partial_t \Psi = -i\, \delta H / \delta \Psi^* \tag{2}$$

with the Hamiltonian $H = \int d\mathbf{r} [|\nabla \Psi|^2 + |\Psi|^4/2 - |\Psi|^2]$.

The n-charged vortex solution is of the form[7,8]

$$\Psi = F(r) \exp[in\theta], \tag{3}$$

where r, θ are polar coordinates. The function $F(r)$ has the following asymptotic behaviors: $F \approx \alpha_n r^n$ for $r \to 0$ and $F^2 \approx 1 - n^2/r^2$ for $r \to \infty$. For the entire interval $0 < r < \infty$ the solution is accessible only numerically. The constants α_n are known for some values of n, e.g., $\alpha_1 = 0.583$, $\alpha_2 = 0.153\,101$, $\alpha_3 = 0.026\,184\,1$, etc. (see, e.g., Ref. 10).

The energy of the n-charged vortex behaves as $n^2 \ln R + CR^2$, where $R \gg 1$ is the outer cutoff radius of the integration and the additive constant C does not depend on the topological charge n. Obviously the state of n single-charged vortices is more energetically favorable than one n-charged vortex. However, in such a dynamical system as NLSE it is not obvious how the n-charged vortex will decay in the presence of the energy conservation and other integrals of motion. One can expect the excess energy to be radiated away due to transformation of the bounding energy coupling of n single charges into the energy of the acoustic excitations. By virtue of the fact that the transformation of the bounding energy into the acoustic field is a very slow process, we can expect the multiple vortex to be a metastable state of the NLSE.

Although NLSE describes some important properties of superflow (see, e.g., Ref. 11), it cannot be an appropriate model to describe the dissipation processes due to its Hamiltonian nature. In order to allow the dissipation we consider the simplest possible generalization of the model

$$\partial_t \Psi = (\epsilon + i)(\nabla^2 \Psi + \Psi - |\Psi|^2 \Psi), \tag{4}$$

where the small phenomenological parameter $\epsilon \ll 1$ describes the bulk dissipation of superflow towards the condensate. We assume that the dissipation rate ϵ is temperature dependent and is the function of normal component density. The role of the normal component may be played by normal ^{3}He atoms. We assume that the only channel for the bulk dissipation for this condition (very low temperature) is the absorption of acoustic excitations of superflow by the normal component. We would like to point out that this particular form of the dissipation term does not change the stationary solution of

NLSE. This equation formally recalls the Pitaevskii-Ginzburg equation for a superfluid near Λ point[12,13] with the dissipation rate ϵ of the order one. We expect, therefore, that the decay time of the n-charged vortex will be strongly affected by the bulk dissipation rate ϵ.

In order to investigate the stability of the n-charged vortex with respect to small perturbations, we consider the perturbative solution in the form

$$\Psi=[F(r)+\eta(x,y,t)]\exp[in\theta], \tag{5}$$

where η is a complex function. Performing linearization with respect to η we obtain the perturbative equation

$$\partial_t\eta=(\epsilon+i)\left(\nabla^2\eta-\frac{n^2}{r^2}\eta+\frac{2in}{r^2}\partial_\theta\eta+(1-2F^2)\eta-F^2\eta^*\right). \tag{6}$$

Separating the real and imaginary part of $\eta=a+ib$ and representing the solution in the form

$$\begin{pmatrix}a\\b\end{pmatrix}=\sum_{m=-\infty}^{\infty}\exp[\lambda t+im\theta]\begin{pmatrix}a_m(r)\\b_m(r)\end{pmatrix}, \tag{7}$$

where λ stays for the growth rate of linear perturbations, m denotes the azimuthal number of the perturbations, and $(a_m(r),b_m(r))$ are real functions of r. After simple algebra we obtain

$$\lambda\frac{\epsilon a_m+b_m}{1+\epsilon^2}=\partial_r^2a_m+\frac{1}{r}\partial_r a_m-\frac{m^2+n^2}{r^2}a_m-\frac{2imn}{r^2}b_m+(1-3F^2)a_m,$$

$$\lambda\frac{\epsilon b_m-a_m}{1+\epsilon^2}=\partial_r^2b_m+\frac{1}{r}\partial_r b_m-\frac{m^2+n^2}{r^2}b_m+\frac{2imn}{r^2}a_m+(1-F^2)b_m. \tag{8}$$

The spectrum of the problem (8) consists of the following parts: (1) A continuous band, describing the extended perturbation of the vortex far away from the core. This part of the spectrum is linearly stable for $\epsilon\geqslant 0$. (2) Three neutral modes $(\lambda=0)$ corresponding to the translation of the core in x and y directions $(m=\pm 1)$ and the rotation of the phase $(m=0)$. (3) A discrete spectrum of exponentially localized modes of Eqs. (8) which exists for $|m|\geqslant 2$. These modes are responsible for the split of the multicharged core.

In order to find localized modes of Eq. (8) we solved Eqs. (8) numerically using a matching-shooting method.[14] The procedure requires very high numerical precision in order to reproduce the solution in the interval of localization of the eigenmode. It transpires that even double precision (16 digits after the decimal point) is not sufficient in the general case. The problem is significantly simplified in the limit of small ϵ. For $\epsilon=0$ Eqs. (8) are an anti-Hermitian problem. Therefore, all eigenvalues λ are purely complex, i.e., $\lambda=i\omega$, ω is a real number. Replacing for simplicity $b_m\to ib_m$, we obtain the system of two real ordinary differential equations with the phase-space dimension equal to 4:

TABLE I. The values of ω and λ_1 for $n=m=2,3,4,5$.

$n=m$	2	3	4	5
ω	-0.4376	-0.6634	-0.778	-0.842
λ_1	0.4241	0.658	0.777	0.842

$$\omega b_m=\partial_r^2a_m+\frac{1}{r}\partial_r a_m-\frac{m^2+n^2}{r^2}a_m-\frac{2mn}{r^2}b_m+(1-3F^2)a_m,$$

$$-\omega a_m=\partial_r^2b_m+\frac{1}{r}\partial_r b_m-\frac{m^2+n^2}{r^2}b_m-\frac{2mn}{r^2}a_m+(1-F^2)b_m. \tag{9}$$

The solution of Eqs. (9) consists of two steps. In the first step we obtain numerically unperturbed function F. Than we applied rational polynomial approximation of this numerically obtained solution. Expressions for several n are given in the Appendix. On the second step we solved Eqs. (9) using shooting-matching method. The conditions at $r\to\infty$ were satisfied demanding exponential decay of a,b, i.e., $(a_m,b_m)\sim(1,g)\exp[-\beta r]$, where $\beta^2=1+\sqrt{1-\omega^2}$ and $g=\omega/(\beta^2-2)$. For $r\to 0$ we applied leading-order expansions of a,b; $(a_m,b_m)=s_1r^{|n+m|}+s_2r^{|n-m|}$ where $s_{1,2}$ are unknown two-component vectors which are used in the matching procedure.

In order to obtain the numerical solution of Eqs. (9) one requires an initial assumption for ω and $s_{1,2}$. Because there is no apparent idea how to choose them, we set our initial values for $\omega=-0.5$ and, e.g., $s_1=1$, $s_2=0$. For this guess, we started our matching-shooting procedure on a relatively short interval $r_0<r<r_e$, with $r_0\approx 0.05-0.1$ and $r_e\approx 2-3$. Then we repeated the procedure increasing slowly the interval of integration using as an improved initial value the results from the previous step.

The ϵ correction can be obtained perturbatively as far as a_m,b_m and ω are known. Representing the solution in the form $(a_m,b_m)=(a_m^0,b_m^0)+\epsilon(a_m^1,b_m^1)+\cdots$, $\lambda=i\omega+\epsilon\lambda_1+\cdots$ and imposing the solvability conditions for the functions a_m^1,b_m^1 we obtain

$$\lambda_1=-2\omega\langle a_m^0b_m^0\rangle/\langle(a_m^0)^2+(b_m^0)^2\rangle, \tag{10}$$

where the scalar product is defined as $\langle ab\rangle=\int_0^\infty r\,dr\,a(r)b(r)$.

The values of ω,λ_1 for several m,n are presented in Tables I and II. The localized eigenfunction for the solution of $n=m=2$ and $n=m=3$ is shown in Fig. 1. The most unstable mode occurs for $m=n$ (it is energetically favorable to split the multiple core to m single cores rather than to tear off only one core) (see also Ref. 10). A single-charged vortex does not possess an exponentially localized eigenfunction for $m=\pm 1$, and, therefore, is stable according our analysis. Detailed proof of the stability for a single-charged vortex is given in Ref. 10. The meaning of the core mode for $\epsilon=0$ is the rotation with the frequency ω of n single zeros of Ψ around the center of symmetry fixed at an infinitely small distance.

TABLE II. The values of ω and λ_1 for $n=4$ and $m=0-4$.

m	0	1	2	3	4
ω	0	0	-0.1847	-0.4963	-0.778
λ_1	0	0	0.1712	0.4844	0.777

Let us now discuss the result. We determined that the lifetime $T\sim 1/\mathrm{Re}\lambda = 1/(\epsilon\lambda_1)$ of the multicharged vortex diverges as ϵ decreases and is formally infinity for NLSE. For the NLSE we have no exponential instability of the multicharged vortex. However we can expect a type of slower instability mode in higher orders. In particular, generic perturbations may grow linearly with time. In this sense the multicharged vortex is metastable and may exist for a significant period depending on initial perturbations.

The instability has a nonlinear character and originates from the interaction between localized eigenmodes and the continuous spectrum emitted by the vortex radiation. The instability does not contradict the energy conservation because the radiated waves and the localized modes contribute to the energy with opposite signs. Indeed, simple analysis shows that in the linear approximation the extended excitations of the form $\eta = w(q)\exp[iqx]$ have the energy density $H\sim\int dq|w(q)|^2 q^2 > 0$, where $w(q)$ is the spectral density of the initial conditions. On the other hand, the localized eigenmodes of Eq. (9) contribute to the Hamiltonian with the sign opposite to extended modes. We observe that conservation of the Hamiltonian does not contradict simultaneous growth of localized and radiated modes. This process is similar to the growth of waves with negative energy.[15]

Our simulations with Eq. (4) confirm the results. The simulations were performed in a rectangular domain by a quasispectral split-step method based on fast-Fourier transform (FFT). We observed very long-living vortices (e.g., for $\epsilon = 0.001$ the vortex remains unsplit until 1500 units of time). A rotating vortex pair excites the acoustic waves radiated away. In Fig. 2 one clearly sees the acoustic waves produced by the decaying double-charged and triple-charged vortices.

Let us discuss some important implications of our results. The double-charged vortex is a limiting configuration of two single vortices for intervortex distance $d\rightarrow 0$. It is well known that for large separations $d\gg 1$ such vortices can be modeled by point vortices in an ideal incompressible fluid.

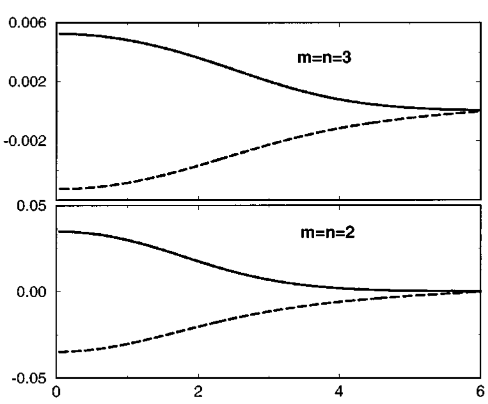

FIG. 1. Real $a_m(r)$ (solid line) and imaginary $b_m(r)$ (dashed line) parts localized eigenmodes of Eq. (9) for $m=n=2,3$.

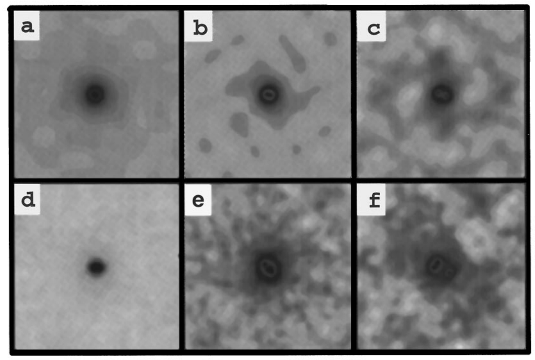

FIG. 2. Gray-coded snapshots of $|\Psi(x,y)|^2$ (zero is shown in black, $|\Psi|=1$ in white) for double-charged vortex (a)–(c); and triple-charged (d)–(f) at the moments of time: (a) $t=1700$; (b) $t=2000$; (c) $t=2500$; (d) $t=1700$; (e) $t=2000$; and (f) $t=2500$. The parameters of the simulations are: The domain size 100×100 units, number of FFT harmonics 128×128; $\epsilon=0.001$, boundary conditions no-flux; initial conditions were slightly perturbed double-charged vortex. Single vortices are presented by black spots, the acoustic field is seen in gray shade.

These vortices rotate around the center of symmetry with the frequency $\Omega = 4/d^2$.[16] The frequency formally diverges at small d. However our analysis shows that indeed the frequency remains finite and approaches the asymptotical value ω for $d\rightarrow 0$, although its value remains rather high on the order of t_0^{-1}.

A more striking phenomenon occurs if we take into account the dissipation in Eq. (4), i.e., set $\epsilon\neq 0$. Dissipation causes the vortices to repel. Again, for large separation the velocity of radial motion of the vortices v behaves as $v\sim\epsilon/d$. Formally, repulsion becomes infinite at small distances. However, according to our analysis the repulsion *vanishes* at small distances and the radial velocity behaves as $v\sim\epsilon\lambda_1 d$, which is in the order of $10^3\epsilon$ (in our scaling of time), and can be low enough at small ϵ. One can speculate that an external force could bring the vortices together, and then they will remain as a double-charged object for a long time. Moreover, in the experimental measurements of the quantized vorticity,[19] such multicharged vortices may result in multiple and even fractional vorticity (due to transient effects).

Another important implication is that multicharged vortices are effective sources of acoustic waves. In the limit $\epsilon\rightarrow 0$ a large part of the energy of the n-charged vortex is transferred to the acoustic excitations. One can speculate that the acoustic field resulting from decaying of multicharged vortices may create necessary conditions for initialization of superfluid turbulence.[17,18]

Let us consider applications of our theory for a system of superflow in a rotating bucket. We performed numerical simulations of Eq. (4) coupled with a solid-body rotation of the normal fluid written by analogy with Ref. 12:

$$\partial_t\Psi = (\epsilon + i)[(\nabla - i\mathbf{v_n})^2\Psi + \Psi - |\Psi|^2\Psi] \qquad (11)$$

with the velocity of normal flow $v_n = \zeta r$, and $\zeta=$const describes the vorticity of the normal flow.[12] The simulations were performed in a circular vessel of radius R. As initial conditions we took the state without vortices. We applied a quasispectral split-step method based on FFT for the rectangular domain. The circular geometry was enforced by ramping the coefficient in front of the linear term Ψ in Eq. (11) to a large negative value away from the radius of the vessel R. In such a way we modeled the condition $\Psi=0$ at the circular boundary. We indeed determined that beyond the threshold of nucleation the vortices appear at the edges of the vessel as multicharged aggregates. Our numerical simulation shows the following possible mechanism for the creation of multicharged vortices. Initially, a single-charged vortex appears at the edge of the vessel, and moves away. The moving vortex suppresses the condensate amplitude in the region behind it, which looks like a long tail (see Fig. 3). The tail initiates the creation of the next vortex mainly in tail's location. The newly created vortex moves along the tail with velocity higher than that of the initial vortex. Thus, the vortices bound each other and create a multicharged vortex. In relation to Fig. 3, let us discuss the spin-up problem of superfluid He in the framework of NLSE. At zero ϵ there is no mechanism to bring up the superflow into rotation due to momentum conservation. Even if the vortices are generated at the boundary, they do not propagate into the vessel. The situation changes at any $\epsilon \neq 0$. Then, due to modulational instability of the condensate[20,21] the vortices are created, and only one-sign vortices will propagate into the interior. The natural question here is whether it is the only system where the multicharged vortices can be created. We conjecture that far enough from the threshold one should find multicharged vortex pairs in the flow wake of the body.

The authors are grateful to L. Kramer, S. Popp, and O. Stiller for fruitful discussions. The work of I.A. was supported in part by the Raschi Foundation. The support of the Israel Science Foundation and Israeli-French Foundation is kindly acknowledged. The work of V.S. was partially supported by the Minevra Foundation and Minevra Center for Nonlinear Physics of Complex Systems.

APPENDIX

Here we present a rational polynomial approximation for unperturbed vortex solution of NLSE for $n=1,2,3,4$

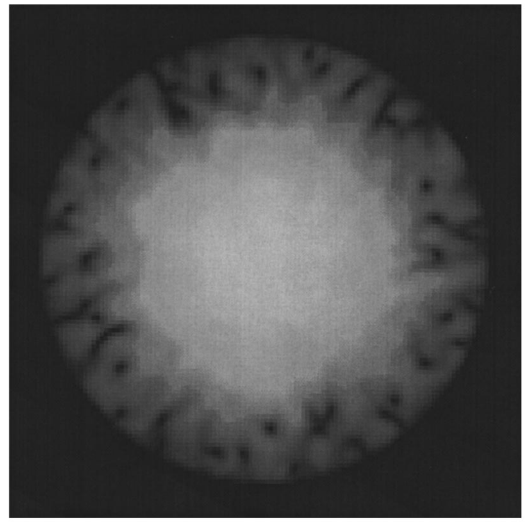

FIG. 3. The snapshot of $|\Psi(x,y)|^2$ for Eq. (11). The parameters of the simulations are radius of the vessel $R=65$; angular velocity of the normal component $\zeta=0.016$ and $\epsilon=0.01$. Initial conditions are $\Psi=1$ plus small amplitude noise. Isolated vortices are seen as small black dots; multiple vortices are bold dots or black areas.

$$n=1, \quad F^2=r^2\frac{0.34+0.07r^2}{1+0.41r^2+0.07r^4};$$

$$n=2, \quad F^2=r^4\frac{0.023\,44+5053r^2}{1+252\,537r^2+20\,212.023\,44r^4+5053r^6};$$

$$n=3,$$

$$F^2=r^6\frac{0.000\,685\,608\,38+r^2}{1+122.57r^4+2233.6r^2+8.999\,314\,39r^6+r^8};$$

$$n=4,$$

$$F^2=r^8(0.000\,011\,072\,650\,76+r^2)/(1+31\,735r^2 + 17\,232.2r^4+192.85r^6+15.999\,988\,927\,34r^8+r^{10}) \; .$$

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