

## Semiclassical analysis of the quantum interference corrections to the conductance of mesoscopic systems

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The Kubo formula for the conductance of a mesoscopic system is analyzed semiclassically, yielding simple expressions for both weak localization and universal conductance fluctuations. In contrast to earlier work that dealt with times shorter than  $O(\ln\hbar^{-1})$ , here longer times are taken to give the dominant contributions. For such long times, many distinct classical orbits may obey essentially the same initial and final conditions on positions and momenta, and the interference between pairs of such orbits is analyzed. Application to a chain of  $k$  classically ergodic scatterers connected in series gives  $-\frac{1}{3}[1-(k+1)^{-2}]$  for the weak localization correction to the zero-temperature dimensionless conductance, and  $\frac{2}{15}[1-(k+1)^{-4}]$  for the variance of its fluctuations. These results interpolate between the well-known ones of random scattering matrices for  $k=1$ , and those of the one-dimensional diffusive wire for  $k\rightarrow\infty$ .

### I. INTRODUCTION

Semiclassical ideas have been central to the understanding of transport effects in mesoscopic systems, specifically weak localization (WL) and universal conductance fluctuations (UCF), from the outset.<sup>1-3</sup> However, they are usually used for handwaving arguments, with the actual calculation being done by resummation of perturbation theory expressions<sup>4</sup> (Feynman diagrams). In recent years, fabrication of ballistic mesoscopic systems has become feasible, and it has been demonstrated that the chaotic or integrable nature of the classical dynamics in such systems is reflected in the quantum interference corrections to their transport properties.<sup>5</sup> The perturbative approach with respect to the impurity potential is inapplicable to such systems, and indeed calculations using other theoretical tools such as random matrix theory<sup>6</sup> (RMT) and the nonlinear  $\sigma$  model<sup>7</sup> have recently appeared. The recent progress in applications of the semiclassical approximation (SCA) to classically chaotic systems<sup>8,9</sup> gives rise to a hope that quantitative results for the mesoscopic transport effects could be obtained from it. This hope is realized below, but it turns out to be necessary to apply the SCA in a somewhat unorthodox manner.

The reason for the difficulty in the semiclassical description of WL and UCF is obvious: both effects involve quantum interference corrections to the classical conductance, which are smaller in powers of  $\hbar$ . Various authors have dealt with this difficulty in different ways. Some have followed the diagrammatic derivation quite closely, and used it for calibration of the magnitude of the effect.<sup>2</sup> Others have limited their attention to effects such as coherent backscattering, where the quantum interference corrections appear in the leading order, and are as large as the classical result.<sup>10</sup> Still others have concentrated on the magnetic field dependence of UCF, and used RMT to calibrate the magnitude.<sup>11-13</sup> All of these studies assume either diffusive or ergodic classical

dynamics, and do not give general semiclassical expressions.

Many of the recent analyses have used the Landauer formula or scattering approach, rather than the bulk approach of the Kubo formula. The proof of the equivalence of these two formulas relies on the unitarity of the quantum-mechanical evolution (current conservation). However, semiclassical evolution in classically chaotic systems is only approximately unitary, and therefore application of the SCA to the Kubo or Landauer formulas may lead to different results. The experience gained from the diagrammatic calculations<sup>14</sup> shows that higher-order corrections to the propagators (diagrams with Hikami boxes<sup>15</sup>) are necessary for an evaluation of the conductance from the Landauer formula, but not when the Kubo formula is used. For this reason, we use the latter in the present work.

A semiclassical analysis has already been developed for the Kubo formula in classically chaotic systems by Wilkinson,<sup>16</sup> with recent applications to transport in antidot arrays.<sup>17</sup> It was assumed there<sup>16</sup> that the relevant propagation times are shorter than the Ehrenfest time,  $t_E \sim O(\ln\hbar^{-1})$ , so that a coherent state or wave packet maintains its correspondence to a point in the classical phase space throughout its evolution. However, it has since become known that semiclassical expressions are applicable also to later times.<sup>18</sup> At such later times the region in classical phase space which evolves from an initially minimal-uncertainty wave packet is stretched and folded by the chaotic dynamics into a very long and curved shape, which may intersect the region defined by some final wave packet several times. The Ehrenfest time thus marks the onset of interference between different classical orbits which correspond to these different intersections (called the mixing regime), and not the breakdown of the SCA. In the present work the focus is on times much longer than the Ehrenfest time, and compact expressions for the contribution of these interference terms to the conductance are developed. Interestingly, some of the terms (includ-

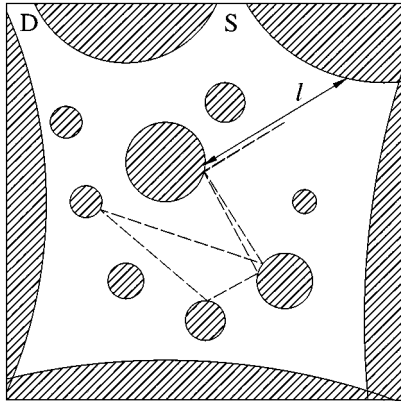


FIG. 1. Sketch of a path which together with its time reverse contributes to weak localization. The initial and final points may be varied along the segment  $l$ . The source ( $S$ ) and drain ( $D$ ) regions are connected through ideal leads to particle reservoirs.

ing the periodic orbit contributions studied by Wilkinson) retain the same form whether the orbits under consideration are short or long on the scale of  $t_E$ , whereas other terms are peculiar to times longer than  $t_E$  and cannot be found by studying the strict semiclassical limit  $\hbar \rightarrow 0$ .

The semiclassical expressions derived below exhibit three rather novel features: first, it will be assumed that the actions of the classical orbits involved are large, and that they thus contribute with essentially random phases and a statistical description in appropriate; second, interference effects between continuous families of classical paths appear explicitly; third, the distribution of classical paths is described as having a continuous density in phase space, even when the initial position and momentum are given, i.e., it will be necessary to introduce a small amount of averaging over the initial (and final) conditions. Such a description is relevant not only for the average over an ensemble of similar mesoscopic systems which differ in their microscopic details (the disorder ensemble), but also for a single system at times longer than the Ehrenfest time. For example, in the expression for weak localization, orbits which start at some given position  $\mathbf{r}$  and momentum  $\mathbf{p}$ , and end after some time  $t$  at the same position  $\mathbf{r}$  but at momentum  $-\mathbf{p}$  will be needed. Rather than retaining only self-retracing paths, which would be implied by strict classical mechanics, the results involve the density of orbits in phase space around the point  $(\mathbf{r}, -\mathbf{p})$  (note that a self-retracing path is not related to a distinct path by time reversal, and is thus irrelevant for interference between *pairs* of time-reversed paths). In incorporating these features we are following the approach of our previous work,<sup>19</sup> which pertains to density-of-states correlations in mesoscopic systems; however, the need to account for contributions which do not strictly obey the initial and final classical conditions did not arise there.

As in other applications of the semiclassical method, interactions are ignored (apart from a possibly self-consistent potential), and the electron fluid is considered as a degenerate thermal distribution of noninteracting particles moving in a mesoscopic sample. We have in mind a situation where the electrons' motion is mostly free, but occasionally they hit an impurity or the boundary of the system and are scattered. This corresponds (in the language of quantum chaos<sup>9</sup>) to the

motion of a “billiard ball” which is scattered by some obstacles (see Fig. 1). In order to apply the SCA, it is assumed that all the classical dimensions, such as the mean free path or the radius of curvature of the obstacles, are much larger than the Fermi wavelength (i.e.,  $\hbar$  is small). The classical dynamics is assumed to be completely chaotic, with all orbits hyperbolically unstable (systems with integrable or intermediate dynamics require separate consideration). The number of dimensions or degrees of freedom,  $N$ , must thus be larger than 1. The exponentially large number of orbits in such systems justifies our statistical approach. Possible complications associated with caustics, etc., are ignored in the present work, and only the generic contributions are considered. In addition we assume that the spectrum is essentially continuous, i.e., that the single-particle level spacing  $\Delta$  is much smaller than the other energy scales in the system, such as those determined by the broadening of the levels  $\Gamma$  due to the external leads, the temperature  $T$ , or the frequency  $\omega$  (even when those tend to zero). This also implies that there is no exponential Anderson localization in the system even in two or quasi-one-dimensions, i.e., the system is smaller than the localization length, because  $\Delta \ll E_c$ , where  $E_c$  is the Thouless energy ( $E_c = \hbar/t_D$  with  $t_D$  the time for an electron to traverse the system).

The various contributions to the conductivity and its fluctuations will be expressed as integrals over the distribution function of classical orbits in the system. Given a Hamiltonian,  $H(\mathbf{r}, \mathbf{p})$ , an initial point in phase space,  $(\mathbf{r}, \mathbf{p})$ , and a propagation time  $t$ , classical dynamics generates a final point which we denote by  $(\mathbf{r}_t, \mathbf{p}_t)$ . The unaveraged distribution of classical orbits is defined as a  $\delta$  function around this point:

$$f(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p}) = \delta(\mathbf{r}' - \mathbf{r}_t) \delta(\mathbf{p}' - \mathbf{p}_t). \quad (1)$$

It is convenient to use a distribution function  $f_E$  limited to the energy hypersurface, by factoring out the energy conservation condition:

$$f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p}) \delta[H(\mathbf{r}', \mathbf{p}') - H(\mathbf{r}, \mathbf{p})] = f(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p}). \quad (2)$$

Whereas these distribution functions describe the chaotic classical dynamics in intricate detail, they can be averaged over small ranges in initial and final conditions to give a smooth distribution which describes the evolution in a statistical sense. This averaging will be denoted by an overline:  $\overline{f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})}$ , with the range of averaging determined by  $\hbar$ . For times  $t$  significantly longer than the Ehrenfest time  $t_E$ , the distribution  $f_E$  becomes independent of the specific details of the averaging procedure. For example, in an ergodic system  $f_E$  becomes independent of initial and final conditions at long times. Apart from the distribution function  $\overline{f_E}$ , additional properties of the classical paths occasionally appear in the expressions (e.g., the distribution of areas enclosed by the classical paths is relevant for the case of weak magnetic fields).

Whereas from the phase-space averaging is denoted by the overline, angular brackets will be used in order to denote averaging over the semiclassical phases, which are assumed to be uncorrelated except for possible symmetries. For example,  $\langle e^{iS_\alpha/\hbar} \rangle = 0$  and  $\langle e^{i(S_\alpha - S_\beta)/\hbar} \rangle = \delta_{\alpha, \beta} + \delta_{\alpha^T, \beta}$ , where  $S_\alpha$  and  $S_\beta$  are the actions of classical orbits  $\alpha$  and  $\beta$ , and  $\alpha^T$  denotes the orbit time reversed to  $\alpha$ . This assumes time re-

versal is the only symmetry in the system, and that the number of orbits is exponentially large so that the possibility of self-symmetric orbits,  $\alpha = \alpha^T$ , may be ignored. This second averaging should be understood either as an averaging over the specific positions of the obstacles (the disorder ensemble average), or, for a single system, as an averaging over a range of possible values of the Fermi energy  $\mu$  (or a range of values of Planck's constant  $\hbar$ ). The average quantum correction to the conductance will be denoted by  $\langle \Delta G \rangle$ , and its variance by  $\langle (\Delta G)^2 \rangle_{\text{var}}$ . The fact that these quantities can be written semiclassically as integrals over the distribution function  $f_E$ , without reference to individual classical orbits and the plethora of their actions and amplitudes, is very useful in applications to classically chaotic systems.<sup>19,20</sup>

The outline of the paper is as follows. In Sec. II the general SCA expression for the Kubo conductance is derived, and shown to reduce to the classical conductance when all interference effects are ignored. The appropriate choice of the electric-field distribution is discussed. Weak localization is analyzed in Sec. III, and universal conductance fluctuations in Sec. IV. In these sections the general expressions in terms of  $f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$  will be applied to diffusive and ergodic systems, for which simple expressions for  $f_E$  are readily available. We concentrate on calculating the magnitude of the interference effects at zero temperature and magnetic field, in order not to repeat the considerations given in the previous semiclassical analyses.<sup>1,2,10-12</sup> Extensions of the analysis, e.g., to finite temperatures, and applications to more complicated systems which consist of several cavities connected in series through ideal leads, will be considered in Sec. V, followed by a discussion in Sec. VI. A short description of this work has been published separately.<sup>21</sup>

## II. THE KUBO FORMULA

According to linear-response theory, the real part of the conductivity tensor is given by (see, e.g., Appendix A of Ref. 2)

$$\sigma_{jk}(\omega) = \frac{e^2}{\omega m^2} \frac{1}{\text{vol}} \text{Re} \sum_{mn} \langle m | \hat{p}_j | n \rangle \times \langle n | \hat{p}_k | m \rangle i \frac{f_{\text{FD}}(\xi_m) - f_{\text{FD}}(\xi_n)}{\hbar \omega + \xi_m - \xi_n + i0}, \quad (3)$$

where  $\xi_m$  and  $|m\rangle$  are the single-particle eigenenergies and eigenstates, respectively,  $\hat{\mathbf{p}}$  is the momentum operator, and  $f_{\text{FD}}(\xi) = 1/\{\exp[\beta(\xi - \mu)] + 1\}$  is the Fermi-Dirac distribution, with  $\mu$  the chemical potential and  $\beta$  the inverse temperature ( $e$  and  $m$  are the electron charge and mass, respectively,  $\omega$  is the frequency, and  $\text{vol}$  is the volume of the system; we are assuming a simple effective-mass description of the electrons, but the derivation applies with minor modifications to systems with a nonspherical Fermi surface). This can be rewritten in terms of the single-particle propagator [e.g.,  $\delta(\hat{H} - \epsilon_1) = \sum_m |m\rangle \delta(\xi_m - \epsilon_1) \langle m|$ ] as

$$\sigma_{jk}(\omega) = \frac{e^2}{m^2} \frac{\hbar}{\text{vol}} \text{Re} \int d\epsilon_1 d\epsilon_2 \frac{i}{\hbar \omega + \epsilon_1 - \epsilon_2 + i0} \times \frac{f_{\text{FD}}(\epsilon_1) - f_{\text{FD}}(\epsilon_2)}{\hbar \omega} \int d\mathbf{r} d\mathbf{r}' \times \left( \frac{\hbar}{i} \frac{\partial}{\partial r'_j} \langle \mathbf{r}' | \delta(\hat{H} - \epsilon_2) | \mathbf{r} \rangle \right) \times \left( \frac{\hbar}{i} \frac{\partial}{\partial r_k} \langle \mathbf{r} | \delta(\hat{H} - \epsilon_1) | \mathbf{r}' \rangle \right), \quad (4)$$

where we have used  $\hat{p}_k = \int d\mathbf{r} | \mathbf{r} \rangle (\hbar/i) \partial / \partial r_k \langle \mathbf{r} |$ , and ignored spin (a spin index summation should be understood with each spatial integration; we avoid the extra indices below by assuming spin degeneracy).

For the purpose of a semiclassical analysis, it is convenient to define quantities which are bilinear in the quantum-mechanical propagators, and to transform them into a form depending on one energy and one time variable, rather than two energy variables.<sup>19</sup> We are thus led to define the following "form factor":

$$\vec{\mathbf{K}}(E, t; \mathbf{r}, \mathbf{r}') \equiv \int d\epsilon e^{-i\epsilon t/\hbar} \text{Tr}[\hat{\mathbf{j}}(\mathbf{r}') \delta(E + \frac{1}{2} \epsilon - \hat{H}) \times \hat{\mathbf{j}}(\mathbf{r}) \delta(E - \frac{1}{2} \epsilon - \hat{H})], \quad (5)$$

where  $\hat{\mathbf{j}}(\mathbf{r})$  is the current density operator (for a given spin projection). Some of the basic properties of this form factor are (a) it is real; (b) it is symmetric under the interchange of all indices and the sign of the time variable,  $K_{jk}(E, t; \mathbf{r}, \mathbf{r}') = K_{kj}(E, -t; \mathbf{r}', \mathbf{r})$ ; and (c) if an external magnetic field is the only source of time-reversal symmetry breaking, then reversing the sign of this field (together with the sign of the spin indices) has the same effect on  $\vec{\mathbf{K}}$  as reversing the sign of  $t$ . As we will discuss only weak magnetic fields and use the SCA, we can rewrite the form factor as<sup>22</sup>

$$K_{jk}(E, t; \mathbf{r}, \mathbf{r}') \simeq \frac{e^2}{m^2} \int d\epsilon e^{-i\epsilon t/\hbar} \times \left[ \frac{\hbar}{i} \frac{\partial}{\partial r'_j} \langle \mathbf{r}' | \delta(E + \frac{1}{2} \epsilon - \hat{H}) | \mathbf{r} \rangle \right] \times \left[ \frac{\hbar}{i} \frac{\partial}{\partial r_k} \langle \mathbf{r} | \delta(E - \frac{1}{2} \epsilon - \hat{H}) | \mathbf{r}' \rangle \right]. \quad (6)$$

The conductivity is given in terms of this form factor as follows:

$$\vec{\sigma}(\omega) = \int dE dt \text{Re} F(E, t) \int \frac{d\mathbf{r} d\mathbf{r}'}{\text{vol}} \vec{\mathbf{K}}(E, t; \mathbf{r}, \mathbf{r}'), \quad (7)$$

where

$$F(E,t) = \int \frac{d\epsilon}{2\pi} e^{i\epsilon t/\hbar} \times \frac{i}{\hbar\omega - \epsilon + i0} \frac{f_{\text{FD}}(E - \frac{1}{2}\epsilon) - f_{\text{FD}}(E + \frac{1}{2}\epsilon)}{\hbar\omega}. \quad (8)$$

Causality is reflected by the fact that replacing  $\omega$  by  $-\omega$  is tantamount to taking the complex conjugate of  $F(E,t)$ . In order to elucidate the structure of  $F(E,t)$ , note that  $[f_{\text{FD}}(E - \frac{1}{2}\epsilon) - f_{\text{FD}}(E + \frac{1}{2}\epsilon)]/\epsilon$  plays a role similar to  $\delta(E - \mu)$  (it is a box function of width  $\epsilon$  and height  $1/\epsilon$ , smeared by the temperature which will be assumed small; as we will be interested in long times  $t$ , only small values of  $\epsilon$  are relevant). In fact, it is straightforward to show that  $\int dE F(E,t) = e^{i\omega t} \theta(t) - (i/\omega) \delta(t)$ , where  $\theta(t)$  is the step function, and the last term cancels with the diamagnetic term. The detailed form of  $F(E,t)$  becomes considerably simpler if only the time symmetric combination of its real part is needed,  $\frac{1}{2}\text{Re}[F(E,t) + F(E,-t)]$ . Due to the properties of  $\tilde{\mathbf{K}}$  mentioned above, indeed only this combination appears if one is interested in a symmetric integral over  $\tilde{\mathbf{K}}$ , such as the longitudinal conductance, or if the magnetic field vanishes (in the cases to be considered below both of these conditions are satisfied). The integration over  $\epsilon$  then becomes trivial due to a  $\delta(\epsilon - \hbar\omega)$  factor, and in the limit of small frequencies we may write

$$\sigma(\omega \rightarrow 0) = \int_0^\infty dt \int dE (-f'_{\text{FD}}(E)) \int \frac{d\mathbf{r}d\mathbf{r}'}{\text{vol}} \tilde{\mathbf{K}}(E,t;\mathbf{r},\mathbf{r}'). \quad (9)$$

In the zero-temperature limit to be considered below, the integration over  $E$  may be omitted, and its value is identified with the Fermi energy  $\mu$ . Notice that the part of  $F(E,t)$  which is asymmetric in  $t$  and responsible for the Hall effect has been simplified in this analysis, which *assumes* that  $\tilde{\sigma}$  is a Fermi surface property.<sup>23</sup>

In the case of restricted geometries or nondiffusive systems, the conductance rather than the volume-averaged conductivity is the appropriate quantity to study. One defines the space-dependent conductivity  $\tilde{\sigma}(\mathbf{r},\mathbf{r}')$  exactly as in Eqs. (7) and (9), except that the integrations over  $\mathbf{r}$  and  $\mathbf{r}'$  and the division by the volume are not performed. The current is given by  $\mathbf{j}(\mathbf{r}') = \int d\mathbf{r} \tilde{\sigma}(\mathbf{r},\mathbf{r}') \mathbf{E}(\mathbf{r})$ , where  $\mathbf{E}(\mathbf{r})$  is the electric field. This dissipative conductance of a sample of general geometry with two leads, at zero frequency, can be written as the dissipated power  $\int d\mathbf{r}' \mathbf{E}(\mathbf{r}') \cdot \mathbf{j}(\mathbf{r}')$ , divided by the voltage  $V$  squared:

$$G = \frac{1}{V^2} \int d\mathbf{r}d\mathbf{r}' \mathbf{E}(\mathbf{r}') \tilde{\sigma}(\mathbf{r},\mathbf{r}') \mathbf{E}(\mathbf{r}). \quad (10)$$

Current conservation is expressed as  $\nabla_{\mathbf{r}} \cdot \tilde{\sigma}(\mathbf{r},\mathbf{r}') = \nabla_{\mathbf{r}'} \cdot \tilde{\sigma}(\mathbf{r},\mathbf{r}') = 0$ , in the absence of a magnetic field [ $\nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}'} \cdot \tilde{\sigma}(\mathbf{r},\mathbf{r}') = 0$  still holds even if a magnetic field is present]. This can be shown by integration by parts in Eq. (10) (see e.g., Ref. 14) to imply that the electric field  $\mathbf{E}(\mathbf{r})$  need not be calculated self-consistently, and instead one can use any electric-field distribution which gives the voltage  $V$  when integrated along any path connecting the two leads [the boundary conditions

require that the components of  $\mathbf{E}(\mathbf{r})$  and of  $\tilde{\sigma}(\mathbf{r},\mathbf{r}')$  perpendicular to an insulating boundary vanish]. In fact, one may take different electric-field distributions for the two  $\mathbf{E}(\mathbf{r})$  factors in Eq. (10)—in the scattering approach of the Landauer formula the electric field is in effect concentrated in the source lead for one factor, and in the drain for the other (another example is the case of multilead devices, for which the conductance  $G_{ij}$  is a matrix, and the boundary conditions for the two electric-field factors may be different). As already mentioned, the semiclassical expressions derived below for  $\tilde{\sigma}(\mathbf{r},\mathbf{r}')$  do not necessarily obey current conservation, because higher-order corrections in  $\hbar$  are not included. Thus, the SCA expressions for the conductance do depend on the use of the actual electric-field distribution in the sample, as discussed below.

The next step is to write down the semiclassical expression for  $\tilde{\mathbf{K}}(E,t;\mathbf{r},\mathbf{r}')$  of Eq. (6). As a starting point we use the van Vleck formula:

$$\langle \mathbf{r}' | \exp(-i\hat{H}t) | \mathbf{r} \rangle \approx \sum_{\alpha \in \{\mathbf{r},\mathbf{r}';t\}} A_\alpha e^{iS_\alpha/\hbar}, \quad (11)$$

where  $\alpha$  is a discrete index which runs over all the classical paths that start at the point  $\mathbf{r}$  and end after a time  $t$  at the point  $\mathbf{r}'$ . The amplitude and action of the classical path  $\alpha$  are given by

$$A_\alpha = i^{\nu_\alpha} \left| \det \left( \frac{1}{\hbar} \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{r}'} \right) \right|^{1/2}, \quad S_\alpha = \int_{\mathbf{r}}^{\mathbf{r}'} (\mathbf{p}d\mathbf{r} - Hdt), \quad (12)$$

where  $\mathbf{p}_\alpha$  is the initial momentum of the path  $\alpha$  and  $\nu_\alpha$  is the integer Maslov index (this and other factors of  $i$  may be ignored for the purposes of the present work). After Fourier transforming from time  $t$  to energy  $E$ , we have (see, e.g., Ref. 9)

$$\langle \mathbf{r}' | \delta(E - \hat{H}) | \mathbf{r} \rangle \approx \sum_{\alpha \in \{\mathbf{r},\mathbf{r}';E\}} \tilde{A}_\alpha e^{i\tilde{S}_\alpha/\hbar}, \quad (13)$$

where the index  $\alpha$  counts classical paths of energy  $E$ , and the modified amplitude and action are given by

$$\tilde{A}_\alpha = i^{\tilde{\nu}_\alpha} \left| \det \left( \frac{1}{\hbar} \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{r}'} \right) \right|^{1/2} \left| \frac{1}{\hbar} \frac{dT_\alpha}{dE} \right|^{1/2}, \quad \tilde{S}_\alpha = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{p}d\mathbf{r}. \quad (14)$$

The derivative  $\partial \mathbf{p}_\alpha / \partial \mathbf{r}'$  appearing in this amplitude is taken at a constant duration of the orbit  $t = T_\alpha$  [it is also possible to reexpress this amplitude in terms of an  $(N+1) \times (N+1)$  matrix of derivatives taken at constant energy  $E$ , but this will not be helpful here;  $N$  denotes the number of dimensions]. Again, the Maslov index  $\tilde{\nu}_\alpha$  may be ignored, since only the magnitude of  $\tilde{A}_\alpha$  will be needed below. It is necessary to note the derivatives of the action:

$$\frac{\partial \tilde{S}_\alpha}{\partial E} = T_\alpha, \quad \frac{\partial \tilde{S}_\alpha}{\partial \mathbf{r}} = -\mathbf{p}_\alpha, \quad \frac{\partial \tilde{S}_\alpha}{\partial \mathbf{r}'} = \mathbf{p}'_\alpha, \quad (15)$$

where  $T_\alpha$  is the duration of the classical path  $\alpha$ ,  $\mathbf{p}_\alpha$  is the initial momentum mentioned above, and  $\mathbf{p}'_\alpha$  is the final momentum.

Substituting Eq. (13) in the expression for the form factor, Eq. (6), and using the fact that only contributions from small values of  $\epsilon$  will be important (because  $t$  is integrated over a large range) to develop the action around the mean energy  $E$ , gives

$$\begin{aligned} \tilde{K}(E, t; \mathbf{r}, \mathbf{r}') &\approx \frac{e^2}{m^2} \int d\epsilon e^{-i\epsilon t/\hbar} \\ &\times \sum_{\alpha, \beta \in \{\mathbf{r}, \mathbf{r}'; E\}} \tilde{A}_\alpha \tilde{A}_\beta^* \mathbf{p}'_\alpha \mathbf{p}_\beta e^{i(\tilde{S}_\alpha + \frac{1}{2}\epsilon T_\alpha - \tilde{S}_\beta + \frac{1}{2}\epsilon T_\beta)/\hbar} \\ &= \frac{e^2}{m^2} h \sum_{\alpha, \beta \in \{\mathbf{r}, \mathbf{r}'; E\}} \tilde{A}_\alpha \tilde{A}_\beta^* \mathbf{p}'_\alpha \mathbf{p}_\beta e^{i(\tilde{S}_\alpha - \tilde{S}_\beta)/\hbar} \\ &\times \delta\left(t - \frac{T_\alpha + T_\beta}{2}\right). \end{aligned} \quad (16)$$

The  $\delta$  function over time should be understood to have a width determined by the higher-order corrections in  $\epsilon$ . Inserting this result in Eqs. (9) and (10) will give the semiclassical approximation for the conductivity and the conductance.

Before discussing quantum corrections to the conductivity, we observe how the classical results can be regained from this expression. To this end, all interference terms in Eq. (16) are ignored, and only the ‘‘diagonal’’ part of the double sum,  $\alpha = \beta$ , is retained:

$$\tilde{K}^D(E, t; \mathbf{r}, \mathbf{r}') = \frac{e^2}{m^2} h \sum_{\alpha \in \{\mathbf{r}, \mathbf{r}'; E\}} |\tilde{A}_\alpha|^2 \mathbf{p}'_\alpha \mathbf{p}_\alpha \delta(t - T_\alpha). \quad (17)$$

In order to proceed, the amplitudes of Eq. (14) should be substituted here. As similar expressions will be used below, we note here the general form of a sum of this type:

$$\begin{aligned} h \sum_{\alpha \in \{\mathbf{r}, \mathbf{r}'; E\}} |\tilde{A}_\alpha|^2 \delta(t - T_\alpha) (\cdots)_\alpha \\ &= \sum_{\alpha \in \{\mathbf{r}, \mathbf{r}'; t\}} |\tilde{A}_\alpha|^2 \delta(E_\alpha - E) (\cdots)_\alpha \\ &= \int \frac{d\mathbf{r}_0 d\mathbf{p}_0}{h^N} \delta[H(\mathbf{r}_0, \mathbf{p}_0) - E] \\ &\times \delta(\mathbf{r}_0 - \mathbf{r}) \delta(\mathbf{r}_t - \mathbf{r}') (\cdots)_{(\mathbf{r}_0, \mathbf{p}_0)} \\ &= \frac{1}{h^N} \int d\mathbf{p}_E d\mathbf{p}'_E f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p}) (\cdots)_{(\mathbf{r}, \mathbf{p})}. \end{aligned} \quad (18)$$

Again, the phase-space point  $(\mathbf{r}_t, \mathbf{p}_t)$  is that which evolves from the initial point  $(\mathbf{r}_0, \mathbf{p}_0)$  by following the classical dynamics for a time  $t$ . In the first equality the factor  $|(1/h)dT_\alpha/dE|$  in  $|\tilde{A}_\alpha|^2$  was used to turn from a fixed energy representation to fixed time. In the second equality the sum over  $\alpha$  was rewritten as an integral over the initial coordinate and momentum, using the factor  $|A_\alpha|^2 = |\det(1/h \partial \mathbf{p}_\alpha / \partial \mathbf{r}')|$  (the integration over the initial position is trivial, and in fact superfluous at this stage). The result is an integral over the phase-space energy hypersurface, where the properties of the individual paths which were denoted by the dots are now identified by the initial coordinate and momentum of each path. Notice that all the determinants of derivatives which

appeared in the amplitudes have been replaced by integrations over  $\delta$  functions,<sup>20</sup> in such a way that allows for the introduction of the classical distribution function  $f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$  in the last line (integration over the energy surface is denoted by  $\int d\mathbf{p}_E \cdots = \int d\mathbf{p} \delta[H(\mathbf{r}, \mathbf{p}) - E] \cdots$ ).

Applying this trick to the diagonal approximation of the form factor, Eq. (17), gives the classical contribution to the conductivity of Eq. (9):

$$\tilde{\sigma}^{\text{cl}} = \frac{e^2}{m^2} \frac{1}{h^N \text{vol}} \int_0^\infty dt \int d\mathbf{r} d\mathbf{p}_E d\mathbf{r}' d\mathbf{p}'_E \mathbf{p}'_E \mathbf{p}_E f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p}) \quad (19)$$

(at zero frequency and temperature). This may be rewritten as

$$\tilde{\sigma}^{\text{cl}} = \frac{e^2}{m^2} \nu \int_0^\infty dt \langle \mathbf{p}' \mathbf{p}' \rangle_t, \quad (20)$$

where  $\nu = (1/h^N \text{vol}) \int d\mathbf{r} d\mathbf{p} \delta[\mu - H(\mathbf{r}, \mathbf{p})]$  is the density of states (implicitly including the spin summation), and the last factor is a momentum correlator:

$$\langle \cdots \rangle_t = \frac{\int d\mathbf{r} d\mathbf{p}_E d\mathbf{r}' d\mathbf{p}'_E \cdots f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})}{\int d\mathbf{r} d\mathbf{p}_E} \quad (21)$$

[it is the classical counterpart of the Fermi-surface correlator  $\langle p(0)p(t) \rangle$  defined in Ref. 2]. Equation (2) is the classical contribution to the Kubo conductivity. Notice that the density of states contains a factor of  $h^{-N}$ , so that the quantum corrections to the conductance, of the order of  $e^2/h$ , are small corrections to it (higher power of  $\hbar$ ). In the present work, the semiclassical limit is considered with the Fermi momentum and the mobility (or other characterization of the scattering potential) taken as classical parameters, so that the density of electrons and the conductance become trivially  $\hbar$  dependent.

For diffusive motion, given some initial value of the momentum  $\mathbf{p}$ , the average of  $\mathbf{p}'$  at a time  $t$  shortly thereafter is equal to  $\mathbf{p}$  multiplied by  $\exp(-t/\tau)$ , where  $\tau$  is the momentum relaxation (or transport) mean free time. Integrating over the directions of the initial momentum, the classical conductivity is found to be diagonal,  $\sigma_{jk}^{\text{cl}} = \delta_{j,k} \sigma^D$ , and

$$\sigma^D(\omega) = \frac{e^2}{m^2} \text{Re} \int_0^\infty dt \cos(\omega t) \exp(-t/\tau) \nu \frac{p_F^2}{N}, \quad (22)$$

where the frequency dependence has been restored ( $p_F$  denotes the Fermi momentum). Since the density of electrons  $n$  is equal to  $\nu(p_F^2/2m)2/N$ , this evaluates to

$$\sigma^D(\omega) = \frac{ne^2}{m} \text{Re} \frac{\tau}{1 - i\tau\omega}, \quad (23)$$

which is just the Drude conductivity.

The simple description of the momentum correlations used above is inadequate for the calculation of the space-dependent conductivity. In fact, it is known from diagrammatic theory that when all interference terms are ignored, it may be written (for  $\omega = 0$ ) as<sup>14</sup>

$$\sigma_{jk}^D(\mathbf{r}, \mathbf{r}') \approx \sigma^D(0) [\delta_{j,k} \bar{\delta}(\mathbf{r} - \mathbf{r}') - \nabla_j \nabla'_k d(\mathbf{r}, \mathbf{r}')]. \quad (24)$$

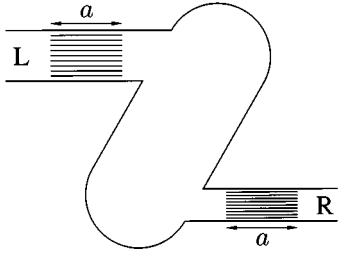


FIG. 2. Schematic sketch of conductance through a ballistic cavity, which is considered to be completely chaotic and ergodic. The electric field is taken to be concentrated in regions of size  $a$  in the left and right leads (the results are independent of  $a$ ).

Here  $\bar{\delta}(\mathbf{r}-\mathbf{r}')$  is a smeared  $\delta$  function of range equal to the mean free path  $l$ , which represents the short-range part of the conductivity (analogous to the Chambers formula), and is due to paths which have not scattered at all. The scaled “diffuson”  $d(\mathbf{r},\mathbf{r}')$  represents the long-range contributions of paths which have scattered at least once, and obeys the equation  $-\nabla^2 d(\mathbf{r},\mathbf{r}') = \bar{\delta}(\mathbf{r}-\mathbf{r}')$  with vanishing boundary conditions at the conducting leads and vanishing normal derivative at insulating boundaries. This ensures that  $\vec{\sigma}^D(\mathbf{r},\mathbf{r}')$  conserves current.

It is possible to render the long-range part unimportant by using the classical electric-field distribution in Eq. (10) (for the standard rectangular geometry this is just a constant field). Indeed, if the condition  $\nabla \cdot \mathbf{E} = 0$  is valid, then it follows from the boundary conditions on  $\mathbf{E}$  and  $d(\mathbf{r},\mathbf{r}')$  that  $\int d\mathbf{r} d\mathbf{r}' \nabla_j \nabla'_k d(\mathbf{r},\mathbf{r}') \mathbf{E}(\mathbf{r}) \mathbf{E}(\mathbf{r}')$  vanishes by integration by parts. In this case one may use a simple approximation to the space-dependent conductivity, keeping only the short-range part [the first term in Eq. (24)], without compromising the accuracy of the classical conductance  $G^{\text{cl}}$ . It is emphasized here that this choice of the electric field represents the actual electric field in the sample, if it is interpreted as the gradient of the electrochemical potential, rather than the externally applied perturbation which can be arbitrary. In other words, if charge neutrality is assumed (the chemical potential cannot vary, and the self-consistent electric field is just that which will not cause any charge perturbations), then the long-range part of the conductivity cannot contribute, because it represents the currents due to the gradient of the induced charge perturbations.

This idea can be generalized to nondiffusive systems, such as a chaotic cavity<sup>24</sup> (see Fig. 2). We will assume that the classical dynamics in such a cavity is not only ballistic, but also ergodic, so that the probability for an electron at any point  $\mathbf{r}$  inside the cavity to leave through (or to have come from) the left (right) lead is proportional to its width  $W_L$  ( $W_R$ ). The self-consistent electrostatic potential is thus a constant within the cavity, and is equal to  $(W_L V_L + W_R V_R)/(W_L + W_R)$ , where  $V_L$  and  $V_R$  denote the potentials in the corresponding reservoirs. All of the potential drop occurs in the leads (or at the boundaries between the leads and the cavity or the reservoirs). Note that the velocity correlator of Eq. (20) is multiplied by the electric field in Eq. (10) and integrated over time, in such a way that the contribution of a certain path [determined by its initial conditions  $(\mathbf{r},\mathbf{p})$ ] to the conductivity has a simple interpretation; it is

proportional to the potential difference between the initial point  $\mathbf{r}$  and the reservoir which an electron with the given initial conditions will eventually reach. If the electron passes through an ergodic cavity on its way, and if the electrostatic potential is chosen self-consistently, then it is no longer necessary to integrate along the remainder of the path—the potential in that cavity is already the averaged potential of the reservoirs, weighted by the probability that the electron would leave through the corresponding lead. The long-range part of the electron paths, i.e., following the electron all the way to the reservoirs, thus becomes unimportant (in the diffusive case any elastic scattering event which randomizes the electron’s direction of propagation plays the role of an ergodic cavity, in that it ends the short-range part of the propagation).

According to this discussion, the precise form of the electric field in the leads does not matter. For specificity, we take the classical electric field  $\mathbf{E}(\mathbf{r})$  to be constant over regions of size  $a$  in each lead. Combining Eq. (10) with Eq. (19), the classical conductance may be written as

$$G^{\text{cl}} = \frac{e^2}{m^2} \frac{1}{h^N V^2} \int_0^\infty dt \int d\mathbf{r} d\mathbf{p}_E d\mathbf{r}' d\mathbf{p}'_E [\mathbf{p} \cdot \mathbf{E}(\mathbf{r})] \times [\mathbf{p}' \cdot \mathbf{E}(\mathbf{r}')] f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p}). \quad (25)$$

The contribution of the short-range part to the conductance thus becomes [the integration over  $f_E(\mathbf{r}', \mathbf{p}', t)$  is trivial]

$$G_C^{\text{cl}} = \frac{e^2}{h} \sum_{i=R,L} \frac{\Delta V_i^2}{V^2 a^2} \int_i d\mathbf{r} d\mathbf{p}_E \tau(\mathbf{r}, \mathbf{p}) v_F^2 \cos^2(\theta), \quad (26)$$

where  $v_F$  is the Fermi velocity,  $\Delta V_i$  denotes the voltage drop over the corresponding lead,  $\theta$  is the angle between  $\mathbf{p}$  and the direction of the lead,  $\tau(\mathbf{r}, \mathbf{p})$  is the time that an electron starting at  $(\mathbf{r}, \mathbf{p})$  spends in the region  $a$  of the electric field, and  $\int_i d\mathbf{r}$  denotes integration over the electric-field region in the lead  $i$ . The integration over the directions in  $\int d\mathbf{p}_E$ , together with a factor of  $v_F |\cos(\theta)|$ , may be replaced by an integration over the transverse momentum  $\int d\mathbf{p}_\perp$  and a summation over the two possible directions along the lead. When summed over these two directions, the free time  $\tau(\mathbf{r}, \mathbf{p})$  gives  $a/v_F |\cos(\theta)|$ . The integration over the position along the lead gives a further factor of  $a$ , which thus cancels out as it should. The remaining integral gives the number of transverse channels in the lead  $g_i = h^{-(N-1)} \int_i d\mathbf{r}_\perp d\mathbf{p}_\perp$ , which is proportional to the width of the lead (in two dimensions  $g_i = 2W_i p_F/h$ , in three dimensions  $W_i$  is an area and  $g_i = \pi W_i p_F^2/h^2$ ). The final result is (ignoring spin)

$$G_C^{\text{cl}} = \frac{e^2}{h} \left[ \left( \frac{g_R}{g_R + g_L} \right)^2 g_L + \left( \frac{g_L}{g_R + g_L} \right)^2 g_R \right] = \frac{e^2}{h} \frac{g_R g_L}{g_R + g_L}. \quad (27)$$

This result corresponds to adding the resistances of the two ideal leads classically in series. Obviously, in order to reach this result with any other choice of the electric-field factors, one would have to evaluate also the contributions of the long-range parts of  $f_E$  (for example, the transmission which enters the Landauer formula is due to paths that start in one lead, scatter inside the cavity, and then leave through the other lead).

The issue of an appropriate choice of the electric field becomes much more important for the quantum corrections to the conductivity, to be considered below. The reason is that it is quite hard to find a current conserving approximation for  $\vec{\sigma}(\mathbf{r}, \mathbf{r}')$  which includes these quantum corrections, even in the diffusive case which is treated well by perturbation theory (see Ref. 13). When the self-consistent electric-field configuration is used, current conservation and the long-range part of the conductivity become less important. For example, if quantum interference gives an enhanced probability to find an electron at some  $(\mathbf{r}, \mathbf{p})$  at some time  $t$ , it becomes unnecessary to follow the propagation of that electron to possibly correlated momenta  $\mathbf{p}'$  at later times.<sup>25</sup> Loosely speaking, the missing electrons represented by the non-strictly-vanishing values of  $\nabla \cdot \mathbf{j}(\mathbf{r})$  may be thought of as being reinjected into the system with a random direction, so that the self-consistent potential at  $\mathbf{r}$  automatically takes care of their contribution. Unfortunately, the validity of this approach can only be strictly proven if one can write down an expression for the higher-order corrections to  $\vec{\sigma}(\mathbf{r}, \mathbf{r}')$ , and show that they do not contribute when integrated with the electric-field factors. While this is readily done in the diagrammatic analysis, it is not easily generalized to other, non-diffusive systems. In the following sections we proceed by analogy with the diffusive case, and calculate the quantum corrections to the conductivity of a general classically chaotic system, using the self-consistent electric-field configuration and the leading order quantum corrections to  $\vec{\sigma}(\mathbf{r}, \mathbf{r}')$ . We return to this issue in the final section and show that this approach is indeed justifiable, at least for an ideally ergodic cavity.

### III. WEAK LOCALIZATION

In this section the SCA is used to calculate the average of the quantum correction to the conductivity, i.e., the weak localization correction.<sup>2,26</sup> We concentrate on the long-range part of the conductivity, i.e., on times  $t$  larger than  $t_E \sim \tau$ —the short-range part does not have a weak localization correction. As advertised, the actions of the classical orbits in Eq. (16) for the form factor  $\tilde{K}(E, t; \mathbf{r}, \mathbf{r}')$  will be assumed random and uncorrelated, except if the two orbits are related by a symmetry. After averaging, only two types of contributions remain: the classical contribution  $\beta = \alpha$ , and that of interference between time-reversed orbits  $\beta = \alpha^T$  (it is assumed that time-reversal symmetry is the only symmetry in the system). For any orbit  $\alpha \in \{\mathbf{r}, \mathbf{r}'; E\}$ , we have  $\alpha^T \in \{\mathbf{r}', \mathbf{r}; E\}$ , with  $\alpha$  and  $\alpha^T$  sharing the same values of action, amplitude, and duration, but  $\mathbf{p}_{\alpha^T} = -\mathbf{p}'_{\alpha}$  and  $\mathbf{p}'_{\alpha^T} = -\mathbf{p}_{\alpha}$ . The possibility of having a strict equality  $\beta = \alpha^T$  arises only in the case that  $\mathbf{r}' = \mathbf{r}$ , giving rise to a factor of 2 enhancement of  $\tilde{K}(E, t; \mathbf{r}, \mathbf{r})$  relative to its classical value. As the coordinates  $\mathbf{r}$  and  $\mathbf{r}'$  are integrated over, it is necessary to find how this enhancement is reduced when  $\mathbf{r}'$  deviates from  $\mathbf{r}$ . Therefore, we include in the weak localization term all pairs of orbits for which  $\beta = \alpha^T$ , in the sense that  $\beta$  and  $\alpha$  smoothly deform into a pair of time-reversed orbits when  $\mathbf{r}'$  approaches  $\mathbf{r}$ . Orbits which are self-symmetric are excluded, because their contribution is already accounted for in the classical term.

The expression for the weak localization correction to the form factor thus reads

$$\langle \Delta \tilde{K}(E, t; \mathbf{r}, \mathbf{r}') \rangle \approx \frac{e^2}{m^2} h \sum_{\substack{\alpha, \beta \in \{\mathbf{r}, \mathbf{r}'; E\} \\ \beta = \alpha^T \neq \alpha}} |\tilde{A}_{\alpha}|^2 \delta(t - T_{\alpha}) \mathbf{p}'_{\alpha} \\ \times (-\mathbf{p}'_{\alpha}) e^{i(\tilde{S}_{\alpha} - \tilde{S}_{\beta})/\hbar}, \quad (28)$$

where it is assumed that  $\mathbf{r}'$  is near  $\mathbf{r}$ , so that the only important  $\mathbf{r}'$  dependence is in the rapidly varying phase factor. A direct evaluation of the  $\mathbf{r}$  and  $\mathbf{r}'$  integrations over the form factor in the  $\hbar \rightarrow 0$  limit gives vanishing results. In fact, the stationary phase conditions would imply  $\mathbf{p}_{\alpha} = \mathbf{p}_{\beta}$  and  $\mathbf{p}'_{\alpha} = \mathbf{p}'_{\beta}$ , which in classical mechanics can only hold for self-symmetric orbits,  $\alpha = \beta$ . In principle, one could try to evaluate higher order corrections to the nonstationary phase integrals which arise, in the  $\hbar \rightarrow 0$  limit. However, in practice (cf. Ref. 5)  $\hbar$  is not extremely small, and the number of possible orbits  $\alpha$  in a chaotic system can be exponentially large. In the mixing regime ( $t > t_E$ ) many of these orbits have such small momentum differences  $\mathbf{p}'_{\alpha} - \mathbf{p}'_{\alpha^T}$  so as to make the phase practically stationary *throughout* the integration region. The spatial integration region is limited by the size of the system (in practice it may be smaller because the paths  $\alpha$  and  $\beta$  may cease to exist due to caustics or shadowing). Thus, the contribution of orbits with momentum differences smaller than  $\hbar/l_{\perp}$  is just proportional to the size of the integration region,  $l_{\perp}$ . Rather than following the exact distribution of possible values of  $l_{\perp}$  for different orbits, in the following we describe this result effectively by a  $\delta$  function over the momentum difference. Note that this represents a nonstandard application of the SCA, because it is assumed that many orbits can fit into the width of the  $\delta$  function, which is proportional to  $\hbar$ . We return to this point in the discussion of Sec. VI A.

The weak localization correction to the form factor, Eq. (28), is again in the form described by Eq. (18), which allows us to reexpress it in terms of the classical distribution of orbits. Furthermore, Eq. (15) may be used to develop the actions around the point  $\mathbf{r}' = \mathbf{r}$ :

$$\tilde{S}_{\alpha}(\mathbf{r}, \mathbf{r}'; E) \approx \tilde{S}_{\alpha}(\mathbf{r}, \mathbf{r}; E) + (\mathbf{r}' - \mathbf{r}) \mathbf{p}'_{\alpha}, \\ \tilde{S}_{\beta}(\mathbf{r}, \mathbf{r}'; E) = \tilde{S}_{\alpha}(\mathbf{r}', \mathbf{r}; E) \\ \approx \tilde{S}_{\alpha}(\mathbf{r}, \mathbf{r}; E) - (\mathbf{r}' - \mathbf{r}) \mathbf{p}_{\alpha} \quad (29)$$

(inclusion of higher-order terms in this expansion turns out to be unnecessary), giving

$$\langle \Delta \tilde{K}(E, t; \mathbf{r}, \mathbf{r}') \rangle \approx -\frac{e^2}{m^2} \frac{1}{h^N} \int d\mathbf{p}_E d\mathbf{p}'_E f_E(\mathbf{r}, \mathbf{p}', t; \mathbf{r}, \mathbf{p}) \mathbf{p}'_{\alpha} \\ \times e^{i(\mathbf{r}' - \mathbf{r})(\mathbf{p} + \mathbf{p}')/\hbar}. \quad (30)$$

In order to perform the integrations using the stationary phase approximation, it is necessary to treat the preexponential factor in the integrand as slowly varying. To this end, we replace  $f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$  with  $f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$ , where the overline denotes averaging of the initial and final positions and momenta over small ranges. As long as the range of averaging is much smaller than a Fermi wavelength in position, and much smaller than a typical value of  $\hbar/l_{\perp}$  in momentum,

there is no way that this replacement can affect the result of the integrals in Eq. (30) and Eq. (9). After the replacement, the preexponential factor is in fact smoothly varying, if the time  $t$  is indeed significantly longer than the Ehrenfest time  $t_E$  [it is not necessary to explicitly remove the contributions of self-symmetric orbits from  $f_E(\mathbf{r}, \mathbf{p}', t; \mathbf{r}, \mathbf{p})$ —such orbits are exponentially rare in the mixing regime].

The next step uses the stationary phase approximation,  $\int dx dp f(x, p) e^{ixp/\hbar} \approx hf(0, 0)$  for small  $\hbar$  and a smooth  $f(x, p)$ , in order to perform the integrals over the angular variables of  $\mathbf{p}'$  and the transverse components of  $\mathbf{r}'$ . The stationary phase conditions identify  $\mathbf{p}'$  with  $-\mathbf{p}$ , and the components of  $\mathbf{r}'$  perpendicular to  $\mathbf{p}$  with those of  $\mathbf{r}$ . The longitudinal component of  $\mathbf{p}'$  is not integrated over because of the limitation to the Fermi surface, which gives rise to a factor of  $1/v_F$ . The integration over the longitudinal component of  $\mathbf{r}'$ , parallel to  $\mathbf{p}$ , cannot be done in the stationary phase approximation. However, as occurs also in the case of the spectral form factor,<sup>19</sup> this integration is trivial—the integrand is constant—and the result is just equal to the effective length of the integration region, which we denote by  $l(\mathbf{r}, \mathbf{p})$ . This integration region is not limited by a small  $\hbar$ , and may extend over relatively long distances along the direction of the classical path: it represents constructive interference between a continuous family of classical orbits, labeled by the longitudinal component of  $\mathbf{r}'$ . As a result, one has

$$\langle \Delta \vec{\sigma} \rangle = -\frac{e^2}{m^2} \frac{1}{h \text{vol}} \int_0^\infty dt \int d\mathbf{r} d\mathbf{p} \mathbf{p} \mathbf{p} f_E(\mathbf{r}, -\mathbf{p}, t; \mathbf{r}, \mathbf{p}) \frac{l(\mathbf{r}, \mathbf{p})}{v_F}, \quad (31)$$

which is a general semiclassical expression for the weak localization correction to the conductivity [the precise meaning of  $l(\mathbf{r}, \mathbf{p})$  will be discussed further below]. Note that the integrations over  $\mathbf{r}$  and  $\mathbf{r}'$  always lead to an identification of  $\mathbf{p}_\alpha$  with  $\mathbf{p}_\beta$ , and of  $\mathbf{p}'_\alpha$  with  $\mathbf{p}'_\beta$  (cf. Ref. 22). In the present case of the weak localization contribution, we also have an identification of  $\mathbf{p}$  with  $-\mathbf{p}'$  (see Fig. 1), leading to the negative sign of the complete expression.

For diffusive behavior with isotropic scattering, the meaning of  $l(\mathbf{r}, \mathbf{p})$  is identified with the free path (see Fig. 1)—when  $\mathbf{r}'$  deviates from  $\mathbf{r}$  further than the next or the previous scattering event, the momentum factors in Eq. (16) become essentially random [as opposed to the approximation used in Eq. (28)], leading on the average to a vanishing contribution. In  $N=3$  dimensions, and for times  $t > \tau$ , one may describe the distribution of classical orbits by

$$\overline{f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})} = W(\mathbf{r}, \mathbf{r}'; t) \frac{v_F}{4\pi p^2}, \quad (32)$$

where  $W(\mathbf{r}, \mathbf{r}'; t) d\mathbf{r}'$  is the probability for a diffusing particle which started at  $\mathbf{r}$  to be within  $d\mathbf{r}'$  of  $\mathbf{r}'$  at time  $t$  (in  $N=2$  dimensions  $2\pi p$  replaces  $4\pi p^2$  in the last factor). The factorization of the distribution into separate spatial and momentum space dependencies, with  $f_E$  independent of the momentum direction, implies that the factors of  $\mathbf{p}\mathbf{p}$  may be replaced by  $\delta_{j,k} p^2/N$ . The averaging over the integration segment  $l(\mathbf{r}, \mathbf{p})$  gives  $2v_F\tau$ —the factor of 2 is due to  $l(\mathbf{r}, \mathbf{p})$  being defined as the sum of the free paths in the backward and the forward directions [another way to justify this factor is to recall that a classical path with a long integration seg-

ment  $l(\mathbf{r}, \mathbf{p})$  for the integration over the longitudinal component of  $\mathbf{r}'$ , will also have the same integration region for that component of  $\mathbf{r}$ , which means that in the averaging  $l(\mathbf{r}, \mathbf{p})$  is weighted by its own length]. The resulting correction to the conductivity is again a diagonal tensor:

$$\langle \Delta \sigma_{jk} \rangle = -e^2 \delta_{j,k} \frac{2sD}{h} \int_\tau^{\tau_\phi} dt \int \frac{d\mathbf{r}}{\text{vol}} W(\mathbf{r}, \mathbf{r}; t), \quad (33)$$

where  $D = v_F^2 \tau / N$  is the diffusion constant, and an  $s=2$  spin degeneracy factor has been explicitly restored. This result coincides with Eq. (3.8) of Chakravarty and Schmid,<sup>2</sup> and thus their quantum-mechanical derivation of the numerical prefactor in this equation (their Appendix D, which assumes a white-noise disorder potential) may be replaced by a semiclassical one.

The time integration appearing here is limited from above by the dephasing time  $\tau_\phi$ , which is due to interactions of the electron with other particles not included in the single-particle Hamiltonian (the assumption of a continuous spectrum implies that the times involved are shorter than  $\hbar/\Delta$ ). It is also limited from below, by the mean free time  $\tau$ . The probability density  $W$  satisfies the diffusion equation:

$$\frac{\partial W(\mathbf{r}, \mathbf{r}'; t)}{\partial t} - D \nabla_{\mathbf{r}'}^2 W(\mathbf{r}, \mathbf{r}'; t) = \delta(t) \delta(\mathbf{r} - \mathbf{r}'). \quad (34)$$

For short times,  $W(\mathbf{r}, \mathbf{r}'; t) = (4\pi Dt)^{-N/2} \exp(-|\mathbf{r} - \mathbf{r}'|^2/4Dt)$ . For times of the order of the diffusion time through the sample  $t_D = L^2/D$ , it is necessary to expand  $W$  in the eigenfunctions  $\Phi_n$  of the diffusion operator,  $W(\mathbf{r}, \mathbf{r}'; t) = \sum_n \Phi_n(\mathbf{r}) \Phi_n^*(\mathbf{r}') \exp(-t/T_n)$ . In this representation the integrals over  $W(\mathbf{r}, \mathbf{r}; t)$  in Eq. (33) become trivial, and for  $\tau_\phi \rightarrow \infty$ ,

$$\langle \Delta \sigma \rangle = -\frac{e^2}{h} \frac{2sD}{\text{vol}} \sum_n T_n. \quad (35)$$

For example, for a quasi-one-dimensional wire extending from  $x=0$  to  $x=L$  (with finite cross-sectional area, i.e., many transverse modes) the longest lasting eigenmodes are  $\Phi_n(x) \propto \sin(\pi n x/L)$  (where  $n=1, 2, \dots$ ) with decay times of  $T_n^{-1} = \pi^2 n^2 D/L^2$ . The boundary conditions are essential in determining this—they are closed in the transverse directions but open in the direction along the wire. This reflects the fact that trajectories hitting the latter boundaries will continue through the hypothesized ideal leads into the reservoirs, and will not return to any point  $\mathbf{r}'$  in the sample ( $\int W dt$  is essentially the “diffuson” of the diagrammatic technique, which was mentioned earlier). Defining the dimensionless conductance per spin direction  $g$ , and using the fact that  $\sum_n (\pi n)^{-2} = 1/6$ , gives (for the  $j=k=1$  component)

$$\sigma_{11} = s \left( \frac{e^2}{h} \frac{L^2}{\text{vol}} \right) g, \quad \langle \Delta g \rangle = -\frac{1}{3}, \quad (36)$$

which is a well-known result for the diffusive wire geometry.

In order to apply the semiclassical expression, Eq. (31), to a more general system, one must specify the meaning of the free path factor  $l(\mathbf{r}, \mathbf{p})$ , or in other words one must perform the integration over the longitudinal component of  $\mathbf{r}'$  with care. Note that in a two-dimensional system the longitudinal direction may be defined as the locus of points for which the



action difference  $S_\alpha(\mathbf{r}, \mathbf{r}'; E) - S_\beta(\mathbf{r}, \mathbf{r}'; E)$  vanishes—there is then no reason to stop the integration at the next scattering event (which in itself may be ill-defined for a smoothly varying potential). Following the discussion of the preceding section for the classical conductance of such systems, the value of the integral is identified as the potential difference  $\Delta V(\mathbf{r}, \mathbf{p})$ , which is defined as the difference between the potential at the point eventually reached by an electron at  $(\mathbf{r}, \mathbf{p})$  and the point from which it emerged. Strictly speaking,  $\Delta V(\mathbf{r}, \mathbf{p})$  is equal to either  $\pm V$  or 0, because both the original and the eventual points are in one of the reservoirs. However, the quantities in Eq. (31) are averaged, and so we are led to define  $l(\mathbf{r}, \mathbf{p}) = \Delta V(\mathbf{r}, \mathbf{p}) p_F / [\mathbf{E}(\mathbf{r}) \cdot \mathbf{p}]$  [the averaging in  $\Delta V(\mathbf{r}, \mathbf{p})$  is over the same range as in  $f_E$ ]. The integration over the longitudinal component of  $\mathbf{r}'$  is thus effectively limited by the “Ehrenfest length”  $v_F t_E$ .

With this definition, the application of Eq. (31) [cf. also Eq. (10)] to the chaotic cavity of Fig. 2 involves  $\Delta V(\mathbf{r}, \mathbf{p}) = \pm \Delta V_i$ , due to the fact that the escape time from the cavity is assumed to be much longer than  $t_E$ . With the specific choice of the electric field within the leads as before,  $|\mathbf{E}(\mathbf{r})| = \Delta V_i / a$ , this gives

$$\langle \Delta G_C \rangle = -\frac{e^2}{h} \sum_{i=L,R} \frac{\Delta V_i^2}{V^2 a^2} \int_0^\infty dt \int_i d\mathbf{r} d\mathbf{p} v_F \cos^2(\theta) \times \overline{f_E(\mathbf{r}, -\mathbf{p}, t; \mathbf{r}, \mathbf{p})} l(\mathbf{r}, \mathbf{p}), \quad (37)$$

where the factors of  $a$  cancel due to the longitudinal component of the  $d\mathbf{r}$  integration, and the relation  $l(\mathbf{r}, \mathbf{p}) |\cos(\theta)| = a$ . The time integral of the distribution function  $f_E$  is found by requiring that the total number of electrons escaping from the cavity, which can be written as an integral over any cross section in the leads  $\sum_{i=R,L} \int_i d\mathbf{r}'_i d\mathbf{p}'_i \int_{t_E}^\infty dt f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$ , is equal to unity [the escape velocity  $v_F |\cos(\theta)|$  is used as before to transform from  $\int_i d\mathbf{p}_E \dots$  to  $\int_i d\mathbf{p}_\perp \dots$ ]. The time integral  $\int f_E dt$  is independent of the detailed initial and final positions and momenta due to ergodicity in the cavity, and is thus equal to  $h^{1-N} / (g_L + g_R)$  (as long as  $\mathbf{r}$  and  $\mathbf{r}'$  are in the leads,  $\mathbf{p}$  is in the inward direction, and  $\mathbf{p}'$  is in the outward direction). The remaining integral gives just  $h^{N-1} g_i$ , so that

$$\langle \Delta G_C \rangle = -\frac{e^2}{h} \left[ \left( \frac{g_R}{g_R + g_L} \right)^2 \frac{g_L}{g_R + g_L} + \left( \frac{g_L}{g_R + g_L} \right)^2 \frac{g_R}{g_R + g_L} \right] = -\frac{e^2}{h} \frac{g_L g_R}{(g_R + g_L)^2}. \quad (38)$$

In the case of equal leads,  $g_L = g_R$ , one obtains  $\langle \Delta G_C \rangle = -\frac{1}{4}$  [with  $G = g(e^2/h)$ ], in agreement with RMT results.<sup>6</sup> Applications to additional systems will be considered in Sec. V.

#### IV. UNIVERSAL CONDUCTANCE FLUCTUATIONS

We now turn to the off-diagonal terms  $\beta \neq \alpha, \alpha^T$ , which do not contribute to  $\bar{\sigma}$  on the average, and calculate the typical magnitude or variance of the fluctuations. A close inspection of the perturbative derivation (Refs. 3, 27, and references therein) shows that there are in fact three different types of contributions. A sketch of these three types of dia-

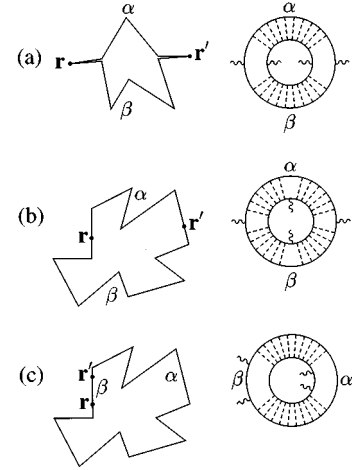


FIG. 3. Sketch of the three different types of interference contributions to the conductivity, together with some of the corresponding perturbative diagrams. The classical paths  $\alpha$  and  $\beta$  start at  $\mathbf{r}$  and end at  $\mathbf{r}'$ . (a)  $\alpha$  different from  $\beta$ , (b)  $\alpha$  and  $\beta$  lie on a periodic orbit  $\gamma$ , and one of them is of negative duration, (c)  $\alpha$  and  $\beta$  lie on  $\gamma$ , and  $\beta$  is short (in this case  $\alpha$  traverses the whole of  $\gamma$  and then repeats the segment  $\beta$  a second time). The sketch assumes diffusive motion with  $t_E \sim \tau$ .

grams and the corresponding classical paths is given in Fig. 3, and the semiclassical analysis will be detailed in this section.

The contribution of all  $\beta \neq \alpha, \alpha^T$  paths to the conductivity of Eqs. (9) and (16) is denoted here by  $\Delta \bar{\sigma}^{ND}$ . At first sight it would seem that each term, defined by a specific choice of  $\alpha$  and  $\beta$ , has an uncorrelated phase  $(S_\alpha - S_\beta)/\hbar$ , and thus its absolute magnitude squared gives an independent contribution to  $\langle |\Delta \bar{\sigma}^{ND}|^2 \rangle_{\text{var}}$ . This is indeed true for terms with  $T_\alpha, T_\beta \geq t_E$ , and forms the first type of contribution [Fig. 3(a) and Sec. IV A].

If one of  $T_\alpha, T_\beta$  is negative [note that according to Eq. (16) their sum must be positive], then the integrations over  $\mathbf{r}$  and  $\mathbf{r}'$  in Eq. (9) effectively “join” the paths  $\alpha$  and  $\beta$  into a single periodic orbit of duration  $|T_\alpha| + |T_\beta|$  (the path with negative duration  $-|T|$  may be considered as starting at  $\mathbf{r}'$  and ending at  $\mathbf{r}$  after a time  $|T|$ ). Since a single periodic orbit can be bisected into two segments in many different ways, all contributing with the same phase, this type of contribution can show significant interference between different  $\alpha, \beta$  pairs. It is useful to first add up all the contributions for each periodic orbit, which will be labeled  $\gamma$ , and then consider the contribution to the variance from each such term. This is done below, and forms the second type of contribution [Fig. 3(b) and Sec. IV B]. As a result of the momentum factors, and the fact that the integral of the momentum along a periodic orbit must vanish, these terms do not contribute in the simple cases considered here.

The third type of contribution arises when  $T_\alpha$  or  $T_\beta$  is positive, but smaller than the Ehrenfest time. For definiteness, take  $0 < T_\beta < t_E$ ; in this case the path  $\alpha$  forms a periodic orbit, which returns to its starting coordinate and momentum after a time  $T_\alpha - T_\beta$  and then continues along the same direction as  $\beta$  for a time  $T_\beta$ . Thus, the action along this last segment cancels in the expression  $(S_\alpha - S_\beta)/\hbar$ , and just as in the second case described above, this contribution is

close to a periodic orbit. These periodic orbit contributions form the third and last type of contribution to  $\langle |\Delta \tilde{\sigma}|^2 \rangle_{\text{var}}$  [see Fig. 3(c) and Sec. IV C].

In the following subsections, the semiclassical expressions for each of these three types of contributions are derived. For simplicity, we consider only the case for which the temperature and the frequency approach zero, and for which time-reversal symmetry holds.

#### A. Contribution with $T_\alpha, T_\beta \gtrsim t_E$

The different contributions to  $\langle \Delta \sigma_{j_1 k_1}^{ND} \Delta \sigma_{j_2 k_2}^{ND*} \rangle$  will be denoted by  $F^1, F^2$ , and  $F^3$ , with the  $j_1 k_1, j_2, k_2$  indices suppressed in most of the equations (the complex conjugate of the real quantity  $\Delta \sigma_{j_2 k_2}^{ND}$  is taken for convenience in notation; it amounts only to exchanging the  $\alpha$  and  $\beta$  indices). According to Eqs. (9) and (16), the expression for  $|\Delta \tilde{\sigma}^{ND}|^2$  involves an integration over four different spatial coordinates. The first type of contribution,  $F^1$  (or at least that part of it that will not vanish after averaging) comes from regions where both  $\mathbf{r}$  coordinates and both  $\mathbf{r}'$  coordinates are close to each other, and both copies of  $\alpha$  and of  $\beta$  coincide. Due to time-reversal symmetry, there is also a similar contribution from the case in which these coordinates are interchanged. As will become evident, these two cases contribute terms with indices  $\delta_{j_1, j_2} \delta_{k_1, k_2}$  and  $\delta_{j_1, k_2} \delta_{k_1, j_2}$ , respectively, which are otherwise identical. The first of these, denoted by  $F^{1a}$ , gives

$$F^{1a} = \left( \frac{e^2}{m^2} \frac{1}{\text{vol}} \right)^2 \int_{t_E}^{\infty} dt_a \int_{t_E}^{\infty} dt_b \int d\mathbf{r}_+ d\mathbf{r}'_+ \int d\mathbf{r}_- d\mathbf{r}'_- \frac{1}{h^{2N}} \int d\mathbf{p}_{aE} d\mathbf{p}'_{aE} \int d\mathbf{p}_{bE} d\mathbf{p}'_{bE} \times f_E(\mathbf{r}'_+, \mathbf{p}'_+, t_a; \mathbf{r}_+, \mathbf{p}_a) f_E(\mathbf{r}'_-, \mathbf{p}'_-, t_b; \mathbf{r}_-, \mathbf{p}_b) \mathbf{p}'_a \mathbf{p}_b \mathbf{p}'_b \mathbf{p}_a e^{i\Delta S/\hbar}, \quad (41)$$

where the quantities relating to the possible  $\alpha$  orbits are denoted by a subscript  $a$ , and those of the  $\beta$  orbits by  $b$ . It is assumed here that the relevant contributions come from orbits longer than the Ehrenfest time, so that  $t_a$  and  $t_b$  are bigger than  $t_E$ . This allows us to replace the two factors of  $f_E$  by their smooth averages  $\overline{f_E}$ , and to neglect the contributions of  $\alpha = \beta$  to these averaged distributions. The next step is to use the phase factor in order to perform the integration over the relative coordinates and momenta, giving

$$F^{1a} \approx \left( \frac{e^2}{m^2} \frac{1}{h \text{vol}} \right)^2 \int_0^{\infty} dt_a \int_0^{\infty} dt_b \int d\mathbf{r} d\mathbf{p}_E \int d\mathbf{r}' d\mathbf{p}'_E \overline{f_E(\mathbf{r}', \mathbf{p}', t_a; \mathbf{r}, \mathbf{p})} \overline{f_E(\mathbf{r}', \mathbf{p}', t_b; \mathbf{r}, \mathbf{p})} \frac{l(\mathbf{r}, \mathbf{p}) l(\mathbf{r}', \mathbf{p}')}{v_F^2} \mathbf{p}' \mathbf{p} \mathbf{p}' \mathbf{p}, \quad (42)$$

where again the integration over the longitudinal directions gives factors of the “free path”  $l(\mathbf{r}, \mathbf{p})$ , just as in the evaluation of the weak localization term.

For diffusive behavior, Eq. (32) can again be used, and the momentum direction integrations performed. The free paths  $l(\mathbf{r}_0, \mathbf{p}_0)$  are replaced as before by factors of  $2v_F \tau$ , giving

$$F^{1a} = s^2 \left( \frac{2e^2}{h} \frac{L^2}{\text{vol}} \right)^2 \delta_{j_1, j_2} \delta_{k_1, k_2} \int_0^{\infty} \frac{D dt_a}{L^2} \int_0^{\infty} \frac{D dt_b}{L^2} \int d\mathbf{r} d\mathbf{r}' W(\mathbf{r}, \mathbf{r}', t_a) W(\mathbf{r}, \mathbf{r}', t_b) \quad (43)$$

(again the spin degeneracy factor  $s=2$  has been restored). Using the decomposition of  $W(\mathbf{r}, \mathbf{r}'; t)$  in terms of orthonormal eigenfunctions, the spatial integrals give simply  $\sum_n \exp(-t_a/T_n) \exp(-t_b/T_n)$ . Together with the term  $F^{1b}$ , this yields

$$F^1 = s^2 \left( \frac{2e^2}{h} \frac{L^2}{\text{vol}} \right)^2 (\delta_{j_1, j_2} \delta_{k_1, k_2} + \delta_{j_1, k_2} \delta_{k_1, j_2}) \sum_n \left( \frac{T_n D}{L^2} \right)^2. \quad (44)$$

This expression can be seen to coincide with the first part of Eq. (46) of Altshuler and Shklovskii.<sup>27</sup> For the example of the quasi-1D wire, with  $T_n D/L^2 = 1/\pi^2 n^2$ , this contribution to the conductance fluctuations gives  $\langle |\Delta g|^2 \rangle_{\text{var}} = (8/\pi^4) \sum_{n=1}^{\infty} n^{-4} = \frac{8}{90}$ , in terms of the dimensionless conductance  $g$ .

$$F^{1a} = \left( \frac{e^2}{m^2} \frac{1}{\text{vol}} \right)^2 \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \int d\mathbf{r}_+ d\mathbf{r}'_+ \int d\mathbf{r}_- d\mathbf{r}'_- \times h^2 \sum_{\alpha, \beta \in \{\mathbf{r}_+, \mathbf{r}'_+, i\mu\}} |\tilde{A}_\alpha|^2 |\tilde{A}_\beta|^2 \mathbf{p}'_\alpha \mathbf{p}_\beta \mathbf{p}'_\beta \mathbf{p}_\alpha \delta\left(t_1 - \frac{T_\alpha + T_\beta}{2}\right) \times \delta\left(t_2 - \frac{T_\alpha + T_\beta}{2}\right) e^{i\Delta S/\hbar}. \quad (39)$$

The summation here includes only orbits with positive times  $T_\alpha$  and  $T_\beta$ , which scatter at least once, and are not related to each other by symmetry (other contributions are included in the other terms). The notation  $\mathbf{r}_+ = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ ,  $\mathbf{r}_- = \mathbf{r}_1 - \mathbf{r}_2$ , etc., is used, and only the contribution for which  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2$  is retained. The deviation of the orbits from  $\{\mathbf{r}_+, \mathbf{r}'_+, \mu\}$  leads to corrections to the actions, which to first order are given by

$$\Delta S \approx -\mathbf{r}_- (\mathbf{p}_\alpha - \mathbf{p}_\beta) + \mathbf{r}'_- (\mathbf{p}'_\alpha - \mathbf{p}'_\beta). \quad (40)$$

Consider first the integrations over the time variables. Necessarily  $t_1 = t_2$ , and this time variable will be noted by  $t_+$ . It will be convenient to add a fictitious integration over  $\int dt_- \delta[t_- - (T_\alpha - T_\beta)]$ . The  $\delta$  functions over time may then be rewritten in the form  $\delta(t_a - T_\alpha) \delta(t_b - T_\beta)$ , where  $t_a = t_+ + t_-/2$  and  $t_b = t_+ - t_-/2$ . After this is done, it is possible to transform the sums over  $\alpha, \beta \in \{\mathbf{r}_+, \mathbf{r}'_+, \mu\}$  into phase-space integrations using again Eq. (18). This gives

A further application of Eq. (42) is to the chaotic cavity of Fig. 2. This gives (including the factor of 2 due to time-reversal symmetry, but ignoring spin)

$$\begin{aligned} \langle (\Delta G_C)^2 \rangle_{\text{var}} &= 2 \left( \frac{e^2}{h} \right)^2 \sum_{i,j=L,R} \frac{\Delta V_i^2 \Delta V_j^2}{V^4 a^4} \int_{t_E}^{\infty} dt_a \int_{t_E}^{\infty} dt_b \int_i d\mathbf{r} d\mathbf{p}_E \int_j d\mathbf{r}' d\mathbf{p}'_E \\ &\quad \times \overline{f_E(\mathbf{r}', \mathbf{p}', t_a; \mathbf{r}, \mathbf{p}) f_E(\mathbf{r}', \mathbf{p}', t_b; \mathbf{r}, \mathbf{p})} l(\mathbf{r}, \mathbf{p}) l(\mathbf{r}', \mathbf{p}') \cos^2(\theta) \cos^2(\theta') v_F^2 \\ &= 2 \left( \frac{e^2}{h} \right)^2 \sum_{i,j=L,R} \frac{g_i^2 g_j^2}{(g_R + g_L)^4} \frac{g_i g_j}{(g_R + g_L)^2} = 2 \left( \frac{e^2}{h} \right)^2 \frac{g_L^2 g_R^2}{(g_R + g_L)^4}. \end{aligned} \quad (45)$$

Here we have used our previous result for the time integral of  $\overline{f_E}$ , and all the factors of  $a$  have canceled as before. The notation  $\bar{i}$  denotes the lead opposite to the lead  $i$ . This is just twice the square of the weak localization result, and in the case of symmetric leads  $g_R = g_L$  reduces to the well-known result  $\langle (\Delta g_C)^2 \rangle_{\text{var}} = \frac{2}{16}$  (see Ref. 6). Notice that in this case all of the conductance fluctuations originate from the first type of contribution, Fig. 3(a) because there are no periodic orbits which traverse the region in which the classical electric field does not vanish.

### B. Contributions with $T_\alpha T_{\beta'} < 0$

Consider next the contributions which are concentrated around periodic orbits. For definiteness, assume that  $T_\beta$  is negative. The description  $\tilde{A}_\beta^* e^{-iS_\beta/\hbar}$  of the path  $\beta \in \{\mathbf{r}, \mathbf{r}'; E\}$  may then be replaced by the identical term  $\tilde{A}_{\beta'} e^{i\tilde{S}_{\beta'}/\hbar}$ , associated with the path  $\beta' \in \{\mathbf{r}', \mathbf{r}; E\}$  where  $\beta$  corresponds to retracing  $\beta'$  backward in time ( $T_{\beta'} = -T_\beta > 0$ ). One may rewrite Eq. (16) as

$$\begin{aligned} \tilde{\mathbf{K}}(E, t; \mathbf{r}, \mathbf{r}') &\simeq \frac{e^2}{m^2 h} \sum_{\substack{\alpha \in \{\mathbf{r}, \mathbf{r}'; E\} \\ \beta' \in \{\mathbf{r}', \mathbf{r}; E\}}} \tilde{A}_\alpha \tilde{A}_{\beta'} \mathbf{p}'_\alpha \mathbf{p}'_{\beta'} e^{i(\tilde{S}_\alpha + \tilde{S}_{\beta'})/\hbar} \\ &\quad \times \delta\left(t - \frac{T_\alpha - T_{\beta'}}{2}\right), \end{aligned} \quad (46)$$

which is completely equivalent, but more convenient if the time  $T_\alpha$  and  $T_{\beta'}$  are positive. When the spatial integrations over the form factor are performed, it is seen that indeed the stationary phase points occur when  $\mathbf{p}_{\beta'} = \mathbf{p}'_\alpha$  and  $\mathbf{p}'_{\beta'} = \mathbf{p}_\alpha$ . This means that the path  $\beta'$  must continue the path  $\alpha$ , and vice versa—together they form a periodic orbit.

The periodic orbits of energy  $E$  may be enumerated by the discrete index  $\gamma \in \{\mathbf{r} = \mathbf{r}', \mathbf{p} = \mathbf{p}'; E\}$ . Each periodic orbit is in fact a continuous family of periodic classical trajectories, which differ from each other by the choice of the initial position along the orbit. The contributions of all the different pairs of paths  $\alpha$  and  $\beta$  which fall along the periodic orbit  $\gamma$  must now be found. Note that if the integral over all positive times  $t$  is taken as indicated in Eq. (9), the last  $\delta$  function may be replaced by a restriction to  $T_\alpha > T_{\beta'}$ . The next step is to undo the Fourier transforms which led to the energy representation in terms of  $\tilde{A}_\alpha$  and  $\tilde{A}_{\beta'}$ , and to return to a representation in terms of  $A_\alpha$  and  $A_{\beta'}$ :

$$\begin{aligned} &\int_0^\infty dt \int d\mathbf{r} d\mathbf{r}' \tilde{\mathbf{K}}(E, t; \mathbf{r}, \mathbf{r}') \\ &\sim \frac{e^2}{m^2 h} \int d\mathbf{r} d\mathbf{r}' \int_0^\infty dt_1 \int_0^{t_1} dt_2 e^{iEt_1/\hbar} e^{iEt_2/\hbar} \\ &\quad \times \sum_{\substack{\alpha \in \{\mathbf{r}, \mathbf{r}'; t_1\} \\ \beta' \in \{\mathbf{r}', \mathbf{r}; t_2\}}} A_\alpha A_{\beta'} \mathbf{p}'_\alpha \mathbf{p}'_{\beta'} e^{i(S_\alpha + S_{\beta'})/\hbar}. \end{aligned} \quad (47)$$

The relation  $\sim$  (instead of  $\simeq$ ) is used to indicate that only the contribution with  $T_\beta < 0$  is included here. The contribution of a periodic orbit  $\gamma$  may now be calculated. The spatial integrations here resemble the convolution formula for the propagator at time  $t_1 + t_2$  in terms of the propagators at times  $t_1$  and  $t_2$ , of which the trace is then taken. Thus, apart from the integration over the time difference  $t_1 - t_2$  and the appearance of the momentum factors, the stationary phase integrations can be performed just as is normally done for the periodic orbits involved in the Gutzwiller trace formula.<sup>8,9,20</sup> This gives

$$\begin{aligned} &\int_0^\infty dt \int d\mathbf{r} d\mathbf{r}' \tilde{\mathbf{K}}(E, t; \mathbf{r}, \mathbf{r}') \\ &\sim \frac{e^2}{m^2} \sum_{\gamma \in \{\mathbf{r} = \mathbf{r}', \mathbf{p} = \mathbf{p}'; E\}} A_\gamma^p e^{i\tilde{S}_\gamma/\hbar} \\ &\quad \times \int_0^{T_\gamma} dt_0 \int_0^{T_\gamma/2} dt_2 \mathbf{p}_\gamma(t_0 + t_1) \mathbf{p}_\gamma(t_0), \end{aligned} \quad (48)$$

where  $\mathbf{p}_\gamma(t)$  denotes the momentum along the periodic orbit  $\gamma$ , at a point parametrized by a time variable  $t$ , and  $t_1 + t_2 = T_\gamma$ . The periodic orbit amplitude,

$$|A_\gamma^p| = \frac{1}{h} |\det[M_\gamma - I]|^{-1/2}, \quad (49)$$

where  $M_\gamma$  is the monodromy matrix describing the stability of the orbit  $\gamma$ , is the same as the Gutzwiller amplitude apart from a time factor. As in the case of the nonperiodic orbits of the propagator, the squares of these amplitudes may be written as integrals over the distribution of classical orbits,<sup>19</sup>

$$|A_\gamma^p|^2 \delta(t - T_\gamma) = \frac{1}{h^2 T_\gamma} \int dr dp f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p}), \quad (50)$$

where the integration is restricted to the region in phase space surrounding the periodic orbit  $\gamma$  (the time variable too is restricted—the right-hand side contains additional contributions at times  $t = mT_\gamma$  with any integer  $m$ ).

Equation (48) implies in fact that these periodic orbit contributions vanish in all the cases considered in the present work. One may extend the time integral of Eq. (9) to negative times, because one is interested only in the contribution which is symmetric with respect to reversal of the magnetic field.<sup>23</sup> Another way to make this point is to recall that the time-reversed orbit  $\gamma^T$  will also contribute, with  $\mathbf{p}_\gamma^T(t) = -\mathbf{p}_\gamma(-t)$ . Thus both time arguments appearing in the momenta  $\mathbf{p}_\gamma(t)$  can be taken to vary over the whole periodic orbit. However, the factors of  $\int_0^{T_\gamma} dt \mathbf{p}_\gamma(t)$  must vanish for a periodic orbit, in order for it to return to the starting point. Such a cancellation has also been observed on the diagrammatic level,<sup>28</sup> where the condition  $T_\alpha, T_\beta > 0$  corresponds to the condition that a pair of propagators has either both propagators advanced or both retarded. This kind of contribution will not vanish only if the frequency  $\omega$  is nonzero, or if the integral  $\int_\gamma \mathbf{E} \cdot d\mathbf{r}$  along the periodic orbit is nonzero (this can happen if the electric field is driven by a time-dependent flux, and  $\gamma$  surrounds the flux). These exceptions will not be discussed further here.

### C. Contributions with $T_\alpha$ or $T_\beta \leq t_E$

We now turn to the last type of contribution, which involves cases when either  $|T_\alpha|$  or  $|T_\beta|$  is smaller than the Ehrenfest time  $t_E$ , so that the stationary phase conditions force the two orbits to overlap over this period. In this case too the integrations over the end points  $\mathbf{r}$  and  $\mathbf{r}'$  lead to the appearance of periodic orbits. In fact, most of the analysis of the preceding subsection still holds, but with  $T_{\beta'} < 0$ : the particle is performing periodic motion along a path of period  $T_\gamma = T_\alpha - T_\beta$  (assuming for the moment that  $T_\beta < t_E$ ), and after revolving around for a time  $T_\alpha$  it retraces its path for a short period  $T_\beta$ , and reaches its starting point. The corresponding contribution is

$$\begin{aligned} & \int_0^\infty dt \int d\mathbf{r} d\mathbf{r}' \vec{\mathbf{K}}(E, t; \mathbf{r}, \mathbf{r}') \\ & \sim \frac{e^2}{m^2} \sum_{\gamma \in \{\mathbf{r}=\mathbf{r}', \mathbf{p}=\mathbf{p}'; E\}} A_\gamma^p e^{i\tilde{S}_\gamma/\hbar} \\ & \quad \times \int_0^{T_\gamma} dt_0 \int_{-t_E}^0 dt_2 \mathbf{p}_\gamma(t_0 - t_2) \cdot \mathbf{p}_\gamma(t_0). \end{aligned} \quad (51)$$

Note that the only difference between this and Eq. (48) is in the limits of the  $t_2$  integration. The absolute square of each such term is an independent contribution to the variance of the conductance, which thus includes a sum over all periodic orbits  $\gamma$ .

Assuming that the relevant periodic orbits are longer than the Ehrenfest time,  $T_\gamma > t_E$ , we identify the  $dt_2$  integration in Eq. (51) as an integration over the momentum correlator along the orbit  $\gamma$ . A periodic-orbit-dependent diffusion con-

stant  $\vec{\mathbf{D}}_\gamma$  may be defined, such that the time integrations in Eq. (51) give simply  $m^2 T_\gamma \vec{\mathbf{D}}_\gamma$ . In order to express the results in terms of the phase-space distribution function  $f_E$ , this diffusion coefficient is relabeled as  $\vec{\mathbf{D}}(\mathbf{r}, \mathbf{p})$ , which is identical to  $\vec{\mathbf{D}}_\gamma$  for all points  $(\mathbf{r}, \mathbf{p})$  on the periodic orbit  $\gamma$ . Using this notation and the expression for the amplitudes, Eq. (50), gives

$$F^3 = 4 \left( \frac{e^2}{h \text{vol}} \right)^2 \int_0^\infty dt \int d\mathbf{r} d\mathbf{p} f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p}) \vec{\mathbf{D}}(\mathbf{r}, \mathbf{p}) \vec{\mathbf{D}}(\mathbf{r}, \mathbf{p}). \quad (52)$$

A factor of 2 arises due to the contributions with  $T_\alpha < t_E$  rather than  $T_\beta < t_E$  [these give the complex conjugate of Eq. (51)], and a further factor of 2 allows for time-reversal symmetry, i.e., contributions of pairs of orbits with  $\tilde{S}_\gamma = \tilde{S}_{\gamma^T}$ .

In the case of diffusive motion (with  $s=2$ ), and assuming  $t \gg \tau \sim t_E$ , the diffusion constants  $\vec{\mathbf{D}}_\gamma$  can be approximated by their average  $D \delta_{j,k}$ , giving

$$F^3 = \left( \frac{e^2 D}{\text{vol}} \right)^2 \delta_{j_1, k_1} \delta_{j_2, k_2} |d_{\text{osc}}(\mu)|^2, \quad (53)$$

where (see Ref. 19)

$$|d_{\text{osc}}(\mu)|^2 = s^2 \frac{4}{h^2} \int_\tau^{\tau+\phi} dt \int d\mathbf{r} W(\mathbf{r}, \mathbf{r}; t). \quad (54)$$

In these expressions one sees most explicitly the observation made by Altshuler and Shklovskii<sup>27</sup> that the corresponding contribution to the UCF is associated with the fluctuations in the density of states ( $d_{\text{osc}}$ ). The integrations may be done explicitly, giving again a simple sum over extinction times of the modes:

$$F^3 = 4s^2 \left( \frac{e^2 L^2}{h \text{vol}} \right)^2 \delta_{j_1, k_1} \delta_{j_2, k_2} \sum_n \left( \frac{T_n D}{L^2} \right)^2. \quad (55)$$

This is identical with Eq. (44) apart from the spatial indices, again in agreement with Eq. (46) of Altshuler and Shklovskii. It enhances the fluctuations of the conductance by 50%, which thus totals for our example of the quasi-1D wire  $\langle (\Delta g)^2 \rangle_{\text{var}} = \frac{12}{90} = \frac{2}{15}$ , once again a well-known result.

For the calculation of the conductance of chaotic cavities, Eq. (10), one must take into account the fact that the electric-field factors are position dependent, and may vary over the region covered by a periodic orbit  $\gamma$ . The periodic-orbit-dependent diffusion constants should be taken to reflect this:

$$\begin{aligned} D_\gamma &= \frac{1}{m^2 T_\gamma} \int_0^{T_\gamma} dt_0 \int_{-t_E}^0 dt_2 [\mathbf{p}_\gamma(t_0 - t_2) \cdot \mathbf{E}(\mathbf{r}_\gamma(t_0 - t_2))] \\ & \quad \times [\mathbf{p}_\gamma(t_0) \cdot \mathbf{E}(\mathbf{r}_\gamma(t_0))]. \end{aligned} \quad (56)$$

The limit  $-t_E$  of the  $dt_2$  integration is to be understood in the same manner as the integration over the longitudinal components of the  $\mathbf{r}'$  integrations above—it is useful to define a free time  $\tau(\mathbf{r}, \mathbf{p}) = \Delta_+ V(\mathbf{r}, \mathbf{p}) m / [\mathbf{E}(\mathbf{r}) \cdot \mathbf{p}]$ , in analogy with the free path  $l(\mathbf{r}, \mathbf{p})$  defined above. The potential difference  $\Delta_+ V(\mathbf{r}, \mathbf{p})$  is the difference between the reservoir eventually reached by electrons starting at  $(\mathbf{r}, \mathbf{p})$  and the potential at  $\mathbf{r}$  (the integration is only over the forward direction, not the backward one<sup>29</sup>). Again, although the propagation times

to the reservoirs may be long, the averaging effectively limits the length of the path contributing to  $\tau(\mathbf{r}, \mathbf{p})$  by the Ehrenfest length. Strictly speaking, the averaging in  $\Delta_+ V(\mathbf{r}, \mathbf{p})$  should be taken over the region in phase space corresponding to a minimal-uncertainty wave packet defined by the “width” of the periodic orbit  $\gamma$ , i.e., by the Monodromy matrix  $M_\gamma$ . Thus, the fact that the classical path for electrons starting precisely on a periodic orbit never reaches the reservoirs is irrelevant. Our assumption that the Ehrenfest time is much smaller than the typical propagation times in the system (such as  $T_\gamma$  for the relevant orbits) means that  $\Delta_+ V(\mathbf{r}, \mathbf{p})$  becomes a smooth function already with a significantly smaller averaging region, and therefore the precise form of the averaging is unimportant.

With this definition, Eq. (56) becomes

$$D_\gamma = \frac{1}{m^2 T_\gamma} \int_0^{T_\gamma} dt_0 [\mathbf{p}_\gamma(t_0) \cdot \mathbf{E}(\mathbf{r}_\gamma(t_0))]^2 \tau(\mathbf{r}, \mathbf{p}). \quad (57)$$

Notice that despite the averaging involved in defining  $\tau(\mathbf{r}, \mathbf{p})$ , the resulting  $D(\mathbf{r}, \mathbf{p})$  function can in principle have large fluctuations, so that it becomes important to use the actual fluctuating  $f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p})$  in Eq. (52), rather than the smooth averaged one. A possible way to avoid this is to use a weighted average for  $D(\mathbf{r}, \mathbf{p})$ , i.e., define an averaging such that  $\overline{D^2(\mathbf{r}, \mathbf{p}, t)} = \overline{f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p}) D^2(\mathbf{r}, \mathbf{p}) / f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p})}$ .

As already mentioned, the periodic orbits do not contribute to conductance fluctuations in the case of the chaotic cavity in Fig. 2 because the electric field vanishes in the region in which periodic motion can occur, so that  $D(\mathbf{r}, \mathbf{p}) = 0$ . A simple example of a class of chaotic systems for which the periodic orbits of Eq. (52) do contribute will be considered in the next section. Note that the relationship between the contribution of periodic orbits to conductance fluctuations and their contribution to the fluctuations in the density of states is essentially modified by the presence of the electric-field factors, and no longer follows Eq. (53).

## V. APPLICATIONS TO SPECIFIC SYSTEMS

In the preceding sections, the semiclassical formulas for weak localization and universal conductance fluctuations were applied to two simple systems, the diffusive wire and the ergodic chaotic cavity. In the case that the leads are of a constant width, both of the quantum interference effects are stronger for the diffusive system than for the chaotic scatterer ( $-\frac{1}{3}$  vs  $-\frac{1}{4}$  for weak localization,  $\frac{2}{15}$  vs  $\frac{2}{16}$  for the variance of the conductance). Furthermore introducing asymmetry in the chaotic cavity by taking  $g_L \neq g_R$  decreases the interference. In the present section systems which are intermediate between these two cases are discussed: in the first subsection a system of two cavities connected in series is considered, allowing for any combination of widths of the different leads; in the second subsection a string of  $k$  chaotic cavities in series is considered, with all leads equal in width. In the course of the treatment of these systems, results which are valid more generally, for networks of ergodic cavities connected by ideal leads, will be given.

Before embarking on the detailed treatment of these special systems, several possible extensions will be mentioned. Consider first the effects of symmetry breaking by weak

magnetic fields or by spin-flip or spin-orbit scattering. This requires knowledge of the area distribution associated with the paths contributing to  $f_E(\mathbf{r}'\mathbf{p}', t; \mathbf{r}, \mathbf{p})$ , or the distribution of  $2 \times 2$  spin scattering matrices along them, and has been considered in Ref. 2 for weak localization. Because of the close parallelism between the present approach and the diagrammatic theory, it is not surprising that the results for diffusive systems are reproduced. Specifically, in the case of complete symmetry breaking the weak localization correction either vanishes, if time reversal invariance is broken, or is multiplied by a factor of  $-\frac{1}{2}$  for the case of symplectic symmetry (strong spin-orbit scattering). The variance of the conductance is reduced by a factor of 2 or 4, respectively. These results hold whether the system is diffusive, ergodic, or simply chaotic.

Temperature and frequency dependences may likewise be treated, and again the diagrammatic results will be reproduced for the diffusive case. Observation of Eq. (16) shows that the frequency couples to the sum of the periods of the two interfering orbits  $T_\alpha + T_\beta$ , whereas temperature (averaging of  $E$  over a range around  $\mu$ ) limits the contributions from orbits with large period differences  $|T_\alpha - T_\beta|$ . This gives different behaviors for the four types of interference effects considered (weak localization, and the three different contributions to conductance fluctuations). The temperature and magnetic-field dependences of UCF, for ergodic cavities without time-reversal symmetry, were very recently studied by Efetov<sup>7</sup> using the nonlinear  $\sigma$  model. As noted there, the results are in agreement with previous semiclassical analyses,<sup>10-12</sup> except for the amplitude of the effect which has been reproduced semiclassically only in the present work. The most striking result of this reference is that temperature smearing, as opposed to dephasing, does not change the typical area which enters into the magnetic-field dependence, although it does change the form of that dependence somewhat. This is of direct relevance to the experimental work,<sup>30</sup> as it may allow the measurement of the dephasing rate from the magnetic-field dependence, by inferring this typical area. It may be intuitively explained by noting that as the finite temperature limits only the difference between  $T^\alpha$  and  $T^\beta$ , the areas they encircle may be arbitrary, and the area difference will be of the same order as the typical area which enters in the zero-temperature case. Dephasing, on the other hand, limits the sum of  $T^\alpha$  and  $T^\beta$ , leaving only the contributions with typically smaller areas and area differences.

### A. Two ergodic cavities in series

The system considered here consists of two ergodic cavities connected in series through a lead with  $g_M$  conducting modes, with  $g_L$  and  $g_R$  denoting the number of modes in the right and left leads as before. In this case ergodicity is not achieved, and there are two different regions of initial conditions which must be considered in order to evaluate  $f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$ . For electrons originating in the left cavity, the probability of leaving the system through the left lead is the sum of a geometric series:

$$\begin{aligned} P_{L,L} &= [g_L / (g_L + g_M)] \{ 1 - [g_M / (g_L + g_M)] [g_M / (g_R + g_M)] \}^{-1} \\ &= g_L (g_R + g_M) / (g_M g_R + g_R g_L + g_L g_M), \end{aligned}$$

whereas the probability of leaving the system through the right lead is

$$P_{L,R} = [g_M/(g_L + g_M)]g_R/g_R + g_M \{1 - [g_M/(g_L + g_M)]g_M/(g_R + g_M)\}^{-1} \\ = g_M g_R / (g_M g_R + g_R g_L + g_L g_M).$$

The probabilities for an electron in the right cavity (or situated in one of the leads but headed towards the right cavity) are given by similar expressions for  $P_{R,L}$  and  $P_{R,R}$ . The classical electrostatic potential in the left cavity is therefore  $[g_L(g_R + g_M)V_L + g_M g_R V_R] / (g_M g_R + g_R g_L + g_L g_M)$ , and that in the right cavity  $[g_R(g_L + g_M)V_R + g_M g_L V_L] / (g_M g_R + g_R g_L + g_L g_M)$ . The potential drops in the left, middle, and right leads are

$$\Delta V_L = V \frac{g_M g_R}{g_M g_R + g_R g_L + g_L g_M}, \\ \Delta V_M = V \frac{g_R g_L}{g_M g_R + g_R g_L + g_L g_M}, \quad (58) \\ \Delta V_R = V \frac{g_L g_M}{g_M g_R + g_R g_L + g_L g_M}.$$

The classical conductance for this system may be written as

$$G_{2C} = \frac{e^2}{h} \sum_{i=L,M,R} g_i \left( \frac{\Delta V_i}{V} \right)^2 = \frac{e^2}{h} \frac{g_L g_M g_R}{g_M g_R + g_R g_L + g_L g_M}. \quad (59)$$

This is the result of adding classically in series the resistances  $h/(e^2 g_i)$  of the three leads [cf. Eq. (27) for the case of a single cavity].

The weak localization correction for this system is given as a sum of four terms, corresponding to different  $(\mathbf{r}, \mathbf{p})$  integration regions in Eq. (31): electrons in the left lead moving towards the left cavity, in the middle lead moving towards either the left or the right cavity, and in the right lead moving into the right cavity. It is convenient to generalize the probabilities  $P_{i,j}$  mentioned above, so as to allow indices which describe electrons in the middle lead, moving either to the right  $i = M_+$ , or to the left  $i = M_-$ :  $P_{i,j} = \int dt \int_j d\mathbf{r}'_{\perp} d\mathbf{p}'_{\perp} f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})$ , where  $(\mathbf{r}, \mathbf{p})$  can be any phase-space point in the “directed lead”  $i$ , and the time integration excludes very short times for which no chaotic scattering has occurred. The sixteen “probabilities”  $P_{i,j}$  can be obtained from the four probabilities  $P_{i,j}$  with  $i, j = L, R$  by noting that  $i = M_-$  is equivalent to  $i = L$ ,  $i = M_+$  is equivalent to  $i = R$ , and similar equivalences can be obtained for the final condition,  $j$ , up to factors of  $g_M/g_L$  and  $g_M/g_R$ , respectively (the directions of the arrows for  $i = L, R$  is obvious and omitted in the notation). In analogy with Eq. (38), one obtains

$$\langle \Delta G_{2C} \rangle = - \frac{e^2}{h} \sum_{i=L,M_-,M_+,R} \left( \frac{\Delta V_i}{V} \right)^2 P_{i,i^T} \\ = - \frac{e^2}{h} \frac{g_L g_M g_R (g_L + g_M)(g_M + g_R)(g_R + g_L)}{(g_L g_M + g_M g_R + g_R g_L)^3}, \quad (60)$$

where the index  $i^T$  denotes motion time reversed to that denoted by  $i$ . In the case of leads of equal width  $g_L = g_M = g_R$ , this gives  $\langle \Delta g_{2C} \rangle = -\frac{8}{27}$ , which is intermediate between the  $-\frac{1}{3}$  result for diffusive systems, and the  $-\frac{1}{4}$  result for completely random scattering.

The  $F^1$  and  $F^3$  types of contributions to the fluctuations of the conductance may also be evaluated in a quite straightforward manner. The first gives, as in Eq. (45),

$$\langle (\Delta G_{2C})^2 \rangle_{\text{Var}}^1 = 2 \left( \frac{e^2}{h} \right)^2 \sum_{i,j=L,M_-,M_+,R} \frac{\Delta V_i^2 \Delta V_j^2}{V^4} \frac{g_i}{g_j} P_{i,j}^2 \\ = 2 \left( \frac{e^2}{h} \right)^2 g_L^2 g_M^2 g_R^2 (g_L + g_M)(g_R + g_M) \\ \times \frac{(g_L + g_M)(g_R + g_M)(g_L^2 + g_R^2) + 2g_L g_R g_M^2}{(g_M g_R + g_R g_L + g_L g_M)^6}. \quad (61)$$

Again, in the case  $g_L = g_M = g_R$  we find  $\langle (\Delta g_{2C})^2 \rangle^1 = \frac{80}{729}$ , which is intermediate between the  $\frac{2}{16}$  and  $\frac{4}{45}$  results of chaotic and diffusive systems, respectively.

In order to find the contribution of periodic orbits from Eq. (52), consider a periodic orbit which traverses the middle lead  $n_\gamma$  times, which must be even. The integral of Eq. (57) gives  $D_\gamma = (n_\gamma/T_\gamma)(\Delta V_M^2/2)$  [the one-half comes from  $\tau(\mathbf{r}, \mathbf{p})$ , which is on the average  $a/2|\cos(\theta)|v_F$  for  $\mathbf{r}$  in the electric-field region]. Equation (52) then gives

$$\langle (\Delta G_{2C})^2 \rangle_{\text{Var}}^3 = 4 \left( \frac{e^2}{h} \right)^2 \sum_{i=M_-,M_+} \frac{\Delta V_i^4}{V^4} \int_0^\infty dt \\ \times \int_i d\mathbf{r}_{\perp} d\mathbf{p}_{\perp} f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p}) \frac{n(\mathbf{r}, \mathbf{p}, t)}{4}, \quad (62)$$

where the integration over the whole length of the periodic orbits,  $\int_0^\infty dt \int d\mathbf{r} d\mathbf{p} f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p}) (n(\mathbf{r}, \mathbf{p}, t)^2/4t)$ , has been replaced by an equivalent integral over the cross section of the lead,  $\int_0^\infty dt \sum_i \int_i d\mathbf{r}_{\perp} d\mathbf{p}_{\perp} f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p}) [n(\mathbf{r}, \mathbf{p}, t)/4]$ . The notation  $n(\mathbf{r}, \mathbf{p}, t)$  identifies the value of  $n_\gamma$  corresponding to the different periodic orbits, as was done with  $D(\mathbf{r}, \mathbf{p})$  above.<sup>31</sup>

Due to the appearance of the  $n(\mathbf{r}, \mathbf{p}, t)$  factors, this result cannot be written directly in terms of the “probabilities”  $P_{i,j}$ , and requires instead the following modification: the geometric sum in  $P_{M_-,M_+}$  and  $P_{M_+,M_-}$  is weighted by a factor of  $m/2$  for the  $m$ th term in the sum ( $n_\gamma = 2m$ ), and thus gives  $\frac{1}{2}z/(1-z)^2$ , where  $z = g_M^2/(g_L + g_M)(g_M + g_R)$ . Taking into account the contributions from both  $M_-$  and  $M_+$ , one has

$$\langle (\Delta G_{2C})^2 \rangle_{\text{Var}}^3 = 4 \left( \frac{e^2}{h} \right)^2 \left( \frac{g_L g_R}{g_L g_M + g_M g_R + g_R g_L} \right)^4 \frac{z}{(1-z)^2}, \quad (63)$$

which for  $g_L = g_M = g_R$  (with  $z = \frac{1}{4}$ ) gives  $\langle (\Delta g_{2C})^2 \rangle^3 = \frac{16}{729}$ . This too is intermediate between the zero result for the single chaotic cavity, and the  $\frac{2}{45}$  result of the diffusive wire.

Adding the results of Eqs. (61) and (63) gives for the total conductance fluctuations

$$\begin{aligned} \langle (\Delta G_{2C})^2 \rangle_{\text{var}} &= 2 \left( \frac{e^2}{h} \right)^2 g_L^2 g_M^2 g_R^2 (g_L + g_M)(g_R + g_M)(g_L + g_R) \\ &\times \frac{(g_L + g_M)(g_R + g_M)(g_L + g_R) - 2g_L g_R g_M}{(g_M g_R + g_R g_L + g_L g_M)^6}. \end{aligned} \quad (64)$$

For the case of equal leads this gives  $\langle (\Delta g_{2C})^2 \rangle_{\text{var}} = (\frac{2}{3})^5$ , which is again intermediate between  $\frac{2}{16}$  and  $\frac{2}{15}$  results.

The quantum interference effects obtained in Eq. (60) and (64) can be considered in the two limiting cases of a very wide or very narrow middle lead. In the case  $g_M \gg g_L, g_R$ , the middle lead connects the two cavities very efficiently, so that a particle in one of them will explore the phase space of both cavities ergodically before having a chance to leave through one of the external leads. Thus, the results of a single cavity, Eqs. (38) and (45), are obtained in this limit. This can be understood already from the electric-field distribution, Eq. (58), which vanishes in the middle lead in this case. In the opposite limit  $g_M \ll g_L, g_R$  the middle lead acts as a weak link (the use of semiclassics assumes  $g_i \gg 1$ , so it is still much larger than a quantum point contact), and the quantum interference effects are suppressed, giving  $\langle \Delta G_{2C} \rangle \approx -(e^2/h) g_M (g_L^{-1} + g_R^{-1})$  and  $\langle (\Delta G_{2C})^2 \rangle_{\text{var}} \approx 2(e^2/h)^2 g_M^2 (g_L^{-1} + g_R^{-1})^2$ . As the expressions of Eqs. (60) and (64) are symmetric with respect to any permutation of the indices  $L, M, R$  it does not matter whether the middle lead or any one of the other leads is taken as very wide or very narrow. However, the ratio of the periodic orbit contribution, Eq. (61), to that of interference between pairs of different paths, Eq. (63), is not invariant under such permutations. In fact, this ratio is maximal when  $g_M \ll g_L = g_R$ , and in that case it is equal to 1, which is *larger* than the ratio of 1/2 familiar from diffusive systems (it is 0 for a single cavity). The quantum interference effects turn out to be largest when the three leads are of equal width, which is the case to be studied and generalized in the next subsection.

### B. Chains of ergodic cavities

It is clear that by adding more and more ergodic cavities in series, the situation of the diffusive one-dimensional wire may be approached. Consider the case of  $k$  cavities connected by leads of equal width,  $g_i = \text{const}$ . Even though the leads may be identical to each other, the chaotic cavities are taken to be different in order to avoid a periodic situation in which Bloch states would emerge (i.e., we assume the ab-

sence of translation symmetry). In this case the electric-field configuration is trivial, with  $\Delta V_i/V = 1/(k+1)$  in all leads. In order to make use of the semiclassical formulas of the type appearing in Eqs. (60), (61), and (63), one needs to study and generalize the classical probabilities  $P_{i,j}$ .

It is convenient to define a classical dynamic probability  $p_{l,m}(t)$ , equal to the probability that an electron will be found in the  $m$ th cavity, given that it was initially in the  $l$ th cavity and that it has since traversed through a lead  $t$  times ( $l, m = 1, \dots, k$  and  $t = 0, 1, \dots$ ). Notice that the “time” variable  $t$  is discretized, with no reference to the actual time the electron may spend in the leads and in the cavities on its way. Quite generally,  $P_{i,j} = (g_j/g_m) \sum_{t=0}^{\infty} p_{l,m}(t)$ , where  $l$  is the cavity that the “directed lead”  $i$  is flowing into,  $m$  is the cavity out of which  $j$  is flowing, and  $g_m$  is the sum of the conductances of the two leads connected to the cavity  $m$ . The simple electric-field configuration mentioned above is directly related to these “probabilities.”

A dynamic difference equation, similar to the diffusion equation, may be written for  $p_{l,m}(t)$ :

$$p_{l,m}(t+1) = \frac{1}{2} [p_{l,m-1}(t) + p_{l,m+1}(t)], \quad (65)$$

with the initial condition  $p_{l,m}(0) = \delta_{l,m}$  and the boundary conditions  $p_{l,0}(t) = p_{l,k+1}(t) = 0$ . The classical evolution can be decomposed into  $i = 1, \dots, k$  eigenvalues  $\alpha_i$  and eigenfunctions  $\beta_{i,l}$ , such that  $p_{l,m}(t) = \sum_{i=1}^k \beta_{i,l} \alpha_i^t \beta_{i,m}$ . For the simple system considered here explicit expressions are available:  $\alpha_i = \cos[(i/k+1)\pi]$ , and  $\beta_{i,l} = \sqrt{2/k+1} \sin[(il/k+1)\pi]$ . Just as has happened for the diffusive case, the eigenfunctions turn out to be unimportant, and only the eigenvalues  $-1 < \alpha_i < 1$  will play a role.

Generalizing the results of Eqs. (60), (61), and (63) to the present case, and rewriting them in terms of the dynamical probabilities, gives for the weak localization correction

$$\begin{aligned} \langle \Delta G_{kC} \rangle &= -\frac{e^2}{h} \frac{1}{(k+1)^2} \sum_{l,t} p_{l,l}(t) \\ &= -\frac{e^2}{h} \frac{1}{(k+1)^2} \sum_i \frac{1}{1-\alpha_i} = -\frac{e^2}{h} \frac{1}{3} \left( 1 - \frac{1}{(k+1)^2} \right), \end{aligned} \quad (66)$$

where the explicit values of  $\alpha_i$  were used in the last equality. The result for the conductance fluctuations is

$$\begin{aligned} \langle (\Delta G_{kC})^2 \rangle &= 2 \left( \frac{e^2}{h} \right)^2 \frac{1}{(k+1)^4} \sum_{l,m,t,t'} p_{l,m}(t) p_{l,m}(t') + \left( \frac{e^2}{h} \right)^2 \frac{1}{(k+1)^4} \sum_{l,t} t p_{l,l}(t) \\ &= \left( \frac{e^2}{h} \right)^2 \frac{1}{(k+1)^4} \left( 2 \sum_i \frac{1}{(1-\alpha_i)^2} + \sum_i \frac{\alpha_i}{(1-\alpha_i)^2} \right) \\ &= \left( \frac{e^2}{h} \right)^2 \frac{1}{(k+1)^4} \frac{2}{45} \{ [(k+1)^2 - 1][2(k+1)^2 + 7] + [(k+1)^2 - 1][(k+1)^2 - 4] \} \\ &= \left( \frac{e^2}{h} \right)^2 \frac{2}{15} \left( 1 - \frac{1}{(k+1)^4} \right), \end{aligned} \quad (67)$$

where the two different contributions are kept separate until the last equality. Notice that the factor of  $\frac{1}{2}=g_j/g_m$  in the relationship between  $P_{i,j}$  and  $p_{l,m}$  exactly compensates the fact that the number of terms in the  $i,j$  summations is twice as large as that in the  $l,m$  summations (there are twice as many directed leads as cavities).

Naturally, these results reproduce those described above for  $k=1$  and  $k=2$  cavities ( $k=0$  describes an ideal wire), and approach the diffusive results for  $k\rightarrow\infty$ . A similar crossover behavior has been calculated for a different kind of system which interpolates between the same ergodic and diffusive limits, using supersymmetry techniques.<sup>32</sup> In that case too the idea was to connect  $k$  ideally chaotic cavities in series, but the connections were made directly in the Hamiltonian, by introducing matrix elements which mix between the quantum states of adjacent cavities (each cavity was ascribed a GOE Hamiltonian). The results were as here rational function of  $k$  which reproduce the  $k=1$  and the  $k\rightarrow\infty$  limits, but their form was much more complicated than Eqs. (66) and (67). Even the asymptotic behavior for large  $k$  is not similar, and in fact the crossover for UCF described in Ref. 32 is slightly nonmonotonous. It would be very interesting to compare our results also with a continuous crossover between an ideal lead ( $k=0$  here) and a diffusive wire, obtained when a disordered region of size  $L$  and mean free path  $l$  is introduced in the lead (with the continuous parameter  $L/l$ ). In principle such an analysis can be carried out using the semiclassical methods developed here, but would depend on the details of the disordered wire [ $f_E$  and  $\mathbf{E}(\mathbf{r})$  depend on whether the scattering is isotropic or small-angle scattering].

## VI. DISCUSSION

The main results of the present work are the semiclassical expressions for the mean and variance of the quantum interference corrections to the conductance, i.e., weak localization and universal conductance fluctuations. For classically chaotic systems, these quantities are expressed as integrals over the distribution of classical orbits,  $f_E(\mathbf{r}',\mathbf{p}',t;\mathbf{r},\mathbf{p})$ , involving also additional quantities which can be derived from this distribution: the self-consistent electric field  $\mathbf{E}(\mathbf{r})$ , and the effective free paths  $l(\mathbf{r},\mathbf{p})$ , and diffusion constants  $D(\mathbf{r},\mathbf{p})$ . Knowledge of this distribution of classical orbits is readily available in the applications considered here (see Sec. V), due to the assumption of a *diffusive* system or a system consisting of several *ergodic* cavities connected through ideal leads. The important task of demonstrating the use of these expressions on a generic system lies beyond the scope of the present work. However, it is stressed that finding this classical distribution numerically for a given potential should be relatively easy, because only a statistical knowledge of the classical orbits is necessary, and there is no need to form a full database consisting of exponentially many orbits.

It was originally thought<sup>1</sup> that SCA could help to bridge the gap between the down-to-earth experimentalists and the abstract mathematical analysis of the theorists. Unfortunately, the recent developments in the theory have considerably widened this gap, with the introduction of diffusion equations in  $n$ -dimensional transmission-eigenvalue spaces, and supersymmetric techniques. It is hoped that the present method will contribute to reversing this trend,<sup>33</sup> although it is

acknowledged that the semiclassical approach has its own limitations (in particular, the case of a few, or partially open, channels is outside its scope). The following two subsections give a detailed discussion of the novel features of the present work, and in that context an attempt is made to bridge a different gap—that between the theory of disordered systems and the theory of quantum chaos. Possibilities for cross fertilization between these two fields, in both directions, are pointed out.

### A. Modified semiclassical approximation for the mixing regime

It should be emphasized that the semiclassical analysis was applied here in an unorthodox manner. A strict stationary phase argument would require the initial momenta (and final momenta) of the two classical paths  $\alpha$  and  $\beta$  appearing in the semiclassical expression for the conductivity, Eq. (16), to be strictly equal to each other. As these two paths start at the same position, they would have to be either identical to each other (the classical contribution to the conductivity), or to differ by completing a different number of revolutions around a strictly periodic orbit<sup>16</sup> [as in  $F^3$  of Eq. (52)]. Thus no weak localization corrections, and no contributions of the first kind [ $F^1$  of Eq. (39)] to universal conductance fluctuations, would be obtained. However, the value of  $\hbar$  is never infinitesimally small, and thus the stationary phase argument is never completely strict. In fact, in chaotic systems interesting contributions to physical quantities often arise from the mixing regime, i.e., from orbits with propagation times between the Ehrenfest time  $t_E$  and the typical escape time  $t_{esc}$ . This regime disappears in the extreme  $\hbar\rightarrow 0$  limit, and it is thus not surprising that its detailed treatment deviates from the standard SCA.

Consider for example the classical orbit of Fig. 1, and its behavior when  $\mathbf{r}'$  is not strictly equal to  $\mathbf{r}$ . Labeling the impurities by digits, we refer to the interference term between the path  $\alpha=\mathbf{r}-1-2-3-4-2-1-\mathbf{r}'$  and the path  $\beta=\mathbf{r}-1-2-4-3-2-1-\mathbf{r}'$ , in which the order of scattering has been reversed. Clearly  $\tilde{S}_\alpha$  and  $\tilde{S}_\beta$  are guaranteed to be equal only if  $\mathbf{r}=\mathbf{r}'$ . It is also clear [from Eq. (15)] that when  $\mathbf{r}'$  deviates from  $\mathbf{r}$ , the difference of actions will be proportional to the difference in the final momenta of these two paths, which for the case depicted in the figure is about 1% of  $p_F$ . This momentum difference,  $\mathbf{p}'_\alpha-\mathbf{p}'_\beta$ , is always perpendicular to the difference in the final momenta of these two paths, which for the case depicted in the figure is about 1% of  $p_F$ . This momentum difference,  $\mathbf{p}'_\alpha-\mathbf{p}'_\beta$ , is always perpendicular to the difference in the final momenta of these two paths, which for the case depicted in the figure is about 1% of  $p_F$ . This momentum difference,  $\mathbf{p}'_\alpha-\mathbf{p}'_\beta$ , is always perpendicular to the difference in the final momenta of these two paths, which for the case depicted in the figure is about 1% of  $p_F$ . This momentum difference,  $\mathbf{p}'_\alpha-\mathbf{p}'_\beta$ , is always perpendicular to the difference in the final momenta of these two paths, which for the case depicted in the figure is about 1% of  $p_F$ .



larger than  $10^{-3}p_F l$  the phase will be negligible throughout the whole integration range, and there will be a finite contribution.

Quantitatively, such contributions may be described by a factor of  $h^{N-1} \delta_{\perp}(\mathbf{p}'_{\alpha} - \mathbf{p}'_{\beta})$ , where the  $\delta$  function is over the perpendicular components of the momentum, and has a finite width  $\sim h/l_{\perp}$  and height  $\sim l_{\perp}^{N-1}$ . In the analysis of Secs. III and IV the averaged distribution function  $f_E$  was introduced already at an earlier stage and the integration over  $\mathbf{r}'$  and  $\mathbf{p}'$  was done in one step, so that this  $\delta_{\perp}$  function never appeared explicitly. For long times there are exponentially many classical orbits from  $\mathbf{r}$  to  $\mathbf{r}'$ , which initial and final momenta within  $h/l_{\perp}$  of  $\mathbf{p}$  and  $\mathbf{p}'$ , respectively, and thus the averaging in  $\overline{f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})}$  gives a smooth distribution. The fact that  $f_E$  is smooth makes it invariant to the details of the averaging procedure. Thus there is no need to keep track of the actual distribution of sizes of  $l_{\perp}$  (at least as long as the main contributions come from long orbits).

The crossover between “short” and “long” times for this purpose occurs at an Ehrenfest time  $t_E$ , which depends on  $\hbar$  and the typical Lyapunov exponents. A pair of paths starting very near to each other in phase space will have their momentum difference multiplied by a factor of order  $l/R$  after each collision with an obstacle, where  $l$  is the free path and  $R$  is the radius of curvature of the obstacle. Thus, if the initial momentum difference is  $h/l_{\perp}$  and the momentum difference at  $t_E$  is required to be of order  $p_F$ , the resulting estimate for the Ehrenfest time is

$$t_E \sim \tau \frac{\ln(p_F l_{\perp} / h)}{\ln(l/R)} \quad (68)$$

(suitably averaged value of  $\tau$ ,  $l$ , and  $l_{\perp}$  should be used here). Often this estimate can be taken to imply  $t_E \sim \tau$ , especially in diffusive systems for which it is customary to take the limit of small scatterers  $R \ll l$  (recall that diffusive motion is not a valid description for times  $t \sim \tau$  anyway). However, in the semiclassical limit  $\hbar$  will become so small that we will have  $t_E \gg \tau$ , and in principle this should be taken into account. For instance, the magnetic-field dependence is determined by the distribution of areas enclosed by diffusive paths of length  $t$ , which should be replaced by  $t - 2t_E$  since the paths are so close to each other that they enclose a negligible area for a time  $t_E$  near their beginning and end (this argument does not apply to the contribution of periodic orbits). Since the dominant contributions come from long times, such corrections are unimportant for diffusive systems, except for exceedingly small values of  $\hbar$ . They are also unimportant in the chaotic systems studied here, because of the assumption of ergodicity which was used for the individual cavities (we have essentially assumed that both  $t_E$  and the time taken to traverse each cavity are much smaller than the escape time from the cavity, which typifies the length of the shortest relevant orbits). It would be very interesting to study systems for which  $t_E$  is not negligible compared to the typical propagation times.

The physics associated with the Ehrenfest time can be clarified by considering a different geometry. It is often claimed in the context of disordered systems, on the basis of a perturbative analysis, that any disordered system can be

described by a single parameter—the diffusion constant—on all length scales larger than some physical cutoff such as the transport mean free path or the grain size for a granular material. However, an additional length scale related to the Ehrenfest time occurs naturally in the semiclassical analysis. Furthermore, this length scale depends not only on the microscopic characteristics of the potential, but also on  $\hbar$ , and diverges for  $\hbar \rightarrow 0$ . As an example, consider a thick slab of transparent disordered material. If it is illuminated by a *well-collimated* beam of light, a speckle pattern may be observed in the transmitted light. Based on the perturbative analysis, one may expect that the condition on the thickness of the slab is given by the transport mean free path, but in fact it is clear that the relevant length is related to the Ehrenfest time (provided that the microscopic features of the disordered sample are “soft,” i.e., they must be larger than a wavelength and must not act as beam splitters). For slabs of thickness intermediate between these two lengths, the beam will remain well collimated and no speckle pattern will be observed, even though the direction of propagation of the transmitted (or reflected) light will be random.

It is somewhat surprising that the importance of the averaging in  $\overline{f_E(\mathbf{r}', \mathbf{p}', t; \mathbf{r}, \mathbf{p})}$ , or equivalently the width of the  $\delta_{\perp}(\mathbf{p}'_{\alpha} - \mathbf{p}'_{\beta})$  functions, was not emphasized earlier. This may be due in part to the fact that much of the attention was devoted to physical quantities which involve periodic classical orbits, such as the density of states.<sup>8,9</sup> In fact, this kind of averaging does not arise naturally in the analysis of the contribution of periodic orbits to the conductance either (see Sec. IV C above), and even if it were to occur, it would have no dramatic effect on integrals involving  $\overline{f_E(\mathbf{r}, \mathbf{p}, t; \mathbf{r}, \mathbf{p})}$ . In contrast, in the case of weak localization the averaging is around the point  $\overline{f_E(\mathbf{r}, -\mathbf{p}, t; \mathbf{r}, \mathbf{p})}$ , and it introduces qualitatively new types of orbits (i.e., non-self-retracing orbits) into the calculation for long times. Likewise, the averaging for  $\overline{f_E(\mathbf{r}', \mathbf{p}', t_a; \mathbf{r}, \mathbf{p})} \overline{f_E(\mathbf{r}', \mathbf{p}', t_b; \mathbf{r}, \mathbf{p})}$  has an important effect as it allows distinct classical orbits to overlap [see Fig. 3(a)].

## B. Corrections to the conductance from leading-order propagators

It is remarkable that the use of the Kubo formula, with a self-consistent choice of the electric field, has enabled here the calculation of corrections to the classical conductance which are of higher order in  $\hbar$ , without having to evaluate such high-order corrections to the propagators themselves. As explained already in Sec. II, the choice of the electric fields can be shown to be unimportant on the basis of unitarity, and it is somewhat surprising that the semiclassical results for  $\overline{\sigma}(\mathbf{r}, \mathbf{r}')$  do not obey unitarity, order by order in  $\hbar$ . However, it turns out that the leading-order results for  $\overline{\sigma}(\mathbf{r}, \mathbf{r}')$  may give subleading contributions to the derivatives  $\nabla_{\mathbf{r}} \cdot \overline{\sigma}(\mathbf{r}, \mathbf{r}')$ . In order to clarify these issues, we briefly reconsider the example of a cavity with ideally “random” scattering, which can be described by the circular orthogonal ensemble (COE) of RMT.<sup>6</sup> As described in the following paragraphs, this system can be observed to display all of the above surprising aspects (i.e., a nonunitary leading-order approximation which still reproduces the quantum corrections

to the conductivity correctly, if the Kubo formula is used rather than the Landauer formula), without invoking any semiclassical considerations.

Take the  $n \times n$  scattering matrix,  $S_{i,j}$ , which is associated with such a “random” cavity, connected to leads with a total of  $g_L + g_R = n$  conducting channels. The dimensionless conductance is given in terms of this scattering matrix by the Landauer formula  $g_C = \sum_{i=1}^{g_L} \sum_{j=g_L+1}^n |S_{i,j}|^2$ . Taking  $S_{i,j}$  to be a random member of the COE, one may calculate various averages of its matrix elements for any  $n$ . For example, the fact that  $\langle |S_{i,j}|^2 \rangle = (1 + \delta_{i,j})/(n+1)$  implies that  $\langle g_C \rangle = g_L g_R / (n+1)$ . Expansion in  $1/n$  gives  $\langle g_C \rangle \approx g_L g_R (n^{-1} - n^{-2})$ , in agreement with the semiclassical results for an ergodic cavity, Eqs. (27) and (38). Evaluation of the variance of the conductance is slightly more complicated, although straightforward:<sup>6</sup>

$$\begin{aligned}
\langle (g_C)^2 \rangle_{\text{Var}} &= g_L g_R \langle (|S_{i,j}|^2)^2 \rangle_{\text{Var}} \\
&\quad + g_L g_R (n-2) \langle (|S_{i,j}|^2)(|S_{i,k}|^2) \rangle_{\text{Var}} \\
&\quad + g_L g_R (g_L g_R - n + 1) \langle (|S_{i,j}|^2)(|S_{k,l}|^2) \rangle_{\text{Var}} \\
&= g_L g_R \left( \frac{2}{n(n+3)} - \frac{1}{(n+1)^2} \right) \\
&\quad + g_L g_R (n-2) \left( \frac{1}{n(n+3)} - \frac{1}{(n+1)^2} \right) \\
&\quad + g_L g_R (g_L g_R - n + 1) \left( \frac{n+2}{n(n+1)(n+3)} \right. \\
&\quad \left. - \frac{1}{(n+1)^2} \right) \\
&\approx 2g_L^2 g_R^2 / n^4. \tag{69}
\end{aligned}$$

Here the indices  $i, j, k, l$  which appear in the different type of covariance term are all taken to be different from each other. The result is again in agreement with the semiclassical result of Eq. (45) for an ergodic cavity. Note that it is obtained due to a cancellation of the first two contributions with each other, and that knowledge of moments of  $S_{i,j}$  to the second subleading order in  $1/n$  turns out to be necessary. Obviously, if the leading-order expressions for the moments of  $S_{i,j}$  were used in the Landauer formula, only the classical conductance could have been calculated correctly.

Consider now the freedom of using the known property of unitarity of the scattering matrices,  $\sum_j |S_{i,j}|^2 = 1$  for each row or column. This allows us to replace the contribution of each row of the scattering matrix to the Landauer formula,  $\sum_{j=g_L+1}^n |S_{i,j}|^2$  by the combination  $(1 - \alpha)(1 - \sum_{j=1}^{g_L} |S_{i,j}|^2) + \alpha \sum_{j=g_L+1}^n |S_{i,j}|^2$ , with any value of  $\alpha$ . As the self-consistent classical electric field for an ergodic cavity is concentrated in the leads (cf. Sec. II), it is possible to follow a line of derivation equivalent to that used above for the SCA simply by choosing an appropriate value of  $\alpha$ , specifically  $\alpha = g_L/n$ . Repeating this replacement for each row and each column gives the “Kubo formula” for the conductance,

$$g_C = \frac{g_L g_R}{n} - \sum_{i,j=1}^n E_i E_j |S_{i,j}|^2, \tag{70}$$

with the “classical electric field” factors

$$E_i = \begin{cases} g_R/n & \text{if } 1 \leq i \leq g_L \\ -g_L/n & \text{if } g_L+1 \leq i \leq n \end{cases} \tag{71}$$

(the relative minus sign is due to the field being directed into the cavity in one lead, and out of it in the other). Observe that the classical conductance of the cavity is given here by a “short-range part,” which does not require knowledge of the “long-range” scattering matrix at all. It is stressed that Eq. (70) follows directly from the Landauer formula and the unitarity of  $S_{i,j}$ . In the present context it is considered to be more basic. Evaluation of the mean quantum correction to the conductance can now be performed using only the leading-order result for the corresponding moment of the matrix elements,  $\langle |S_{i,j}|^2 \rangle \approx (1 + \delta_{i,j})/n$ , despite the fact that in this approximation the scattering matrix is not unitary. Most terms cancel with each other because  $\sum_{i=1}^n E_i = 0$ , and the remaining  $-\sum_{i=1}^n E_i^2/n$  gives the WL result. The results for UCF may likewise be obtained using only the leading-order expression for the moments,  $\langle (|S_{i,j}|^2)(|S_{k,l}|^2) \rangle_{\text{Var}} \approx (\delta_{i,k} \delta_{j,l} + \delta_{i,l} \delta_{j,k})/n^2$ . The analogy with the semiclassical calculation is thus complete.

It is expected that explicitly using the self-consistent electric field could be very helpful in other calculation schemes too. For example, it should enable the calculations of Ref. 7 to be done by expanding only to the second order in the diffuson-Cooperon expansion, and not to the sixth order as was found necessary there [this is based on counting the number of ladders in the different diagrams for the “long-range part” of UCF (Ref. 14)].

In this system one can explicitly study the next-order corrections to the moments of  $S_{i,j}$  and observe how their contributions could in principle be as large as the WL and UCF terms being calculated, but vanish due to cancellations between the different terms. This cancellation can be demonstrated without actually calculating the higher-order corrections, by arguing that  $\sum_i E_i = 0$  and using the fact that the COE is insensitive to the actual values of the  $i$  and  $j$  indices. As noted in the closing paragraph of Sec. II, a similar cancellation can be demonstrated in diffusive systems, using the diagrammatic technique. More generally, one may argue on the basis of universality that additional contributions from higher-order terms should not be expected, at least in the case of systems involving ideally ergodic cavities (Sec. V). In order to verify this expectation, it would be very interesting to study explicitly the contribution of higher-order corrections which are known for semiclassical propagators,<sup>34</sup> especially those which are known to involve  $\hbar^{1/2}$  or  $\hbar^{1/3}$  corrections, i.e., the effects of diffraction and caustics.

In the field of disordered systems, higher-order corrections in  $\hbar$  are usually analyzed in terms of two distinct small parameters.<sup>35</sup> The first gives the accuracy of the description of individual scattering events, and may be written as  $\lambda_F/l$ , where  $\lambda_F$  is the Fermi wavelength and  $l$  is a microscopic length such as the mean free path (for a semiclassical analysis, an obvious microscopic length is the radius of curvature of the obstacles depicted in Fig. 1). It is often argued that

inaccuracy with respect to this parameter can be tolerated, because the actual potential in a real system is unknown anyway, and instead the mobility or the effective scattering cross section are measured directly. The second small parameter is essentially  $\hbar/\Delta t_{esc} \sim 1/g$ , where  $\Delta$  is the mean single-particle level spacing,  $t_{esc}$  is the length of time a typical electron spends in the sample before leaving through the leads, and  $g$  is the dimensionless conductance of the sample. This parameter describes the extent of lack of unitarity in the semiclassical approach as discussed above. Recently, it has been suggested that unitarity could be built into the semiclassical approximation from the outset.<sup>36</sup> It would be very interesting to analyze this approach from the point of view of two distinct small parameters. Such an analysis could imply

that the SCA becomes, when unitarity is enforced, analogous to the nonlinear  $\sigma$  model—the latter describes a disordered system with a white-noise potential to all orders in  $1/g$ , but only to leading order in  $\lambda_F/l$ .

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<sup>23</sup>The full form of  $F(E, t)$  does not give rise to a Fermi-surface property, as it has contributions from a range of  $E - \mu \sim \epsilon \sim \hbar/t$ , even at vanishing temperatures. For  $\omega \rightarrow 0$  one has  $\text{Re}F(E, t) = \frac{1}{2} \{ -f'_{\text{FD}}(E) + \int (d\epsilon/\pi) \sin(\epsilon t/\hbar) [\epsilon] [f_{\text{FD}}(E - \epsilon/2) - f_{\text{FD}}(E + \epsilon/2)]/\epsilon \}$ , where the first term gives the part of the conductivity which is symmetric with respect to reversal of the magnetic field and the second term gives the asymmetric part; see, e.g., H. U. Baranger and A. D. Stone, Phys. Rev. B **40**, 8169 (1989). For finite  $\omega$  there are three modifications of this formula: (a) both terms are multiplied by  $\epsilon^{i\omega t}$ ; (b) the energy integration variable is shifted, so that  $\tilde{\epsilon} + \hbar\omega$  replaces  $\epsilon$  in the Fermi functions [for example, the symmetric term has  $\tilde{\epsilon} = 0$ , i.e.,  $(f_{\text{FD}}(E - \hbar\omega/2) - f_{\text{FD}}(E + \hbar\omega/2))/\hbar\omega$  replaces  $-f'_{\text{FD}}(E)$ ]; (c) an additional term, equal to the principal part of  $-ie^{i\omega t} \int (d\tilde{\epsilon}/\pi) [\cos(\tilde{\epsilon}/\hbar)/\tilde{\epsilon}] \{ f_{\text{FD}}[E - (\tilde{\epsilon} + \hbar\omega)/2] - f_{\text{FD}}[E + (\tilde{\epsilon} + \hbar\omega)/2] \}/(\tilde{\epsilon} + \hbar\omega)$ , appears (apart from these, there is always the term of  $F(E, t)$  which cancels with the diamagnetic term, and reads  $-(i/\hbar\omega) \int (d\epsilon/2\pi) \cos(\epsilon t/\hbar) [f_{\text{FD}}(E - \epsilon/2) - f_{\text{FD}}(E + \epsilon/2)]/\epsilon$ ). As long as the whole volume of the sample is integrated over, including possible edge states, there is no need to invoke additional non-Fermi-surface terms [cf., e.g., P. Streda, J. Phys. C **15**, L717 (1982)].

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- <sup>28</sup>See the appendix of B. L. Altshuler, A. G. Aronov, A. I. Larkin, and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **81**, 768 (1981) [Sov. Phys. JETP **54**, 411 (1981)].
- <sup>29</sup>It is possible to use only backward directions, instead of only forward directions—this would arise naturally if we had switched the roles of  $\mathbf{r}$  and  $\mathbf{r}'$ , i.e., started from a description of the orbit  $\alpha$  as following the path  $\beta$  first, and then continuing along the periodic orbit  $\gamma$ . For the time-reversal symmetric systems considered here, it is obvious that the results would not change.
- <sup>30</sup>R. M. Clarke *et al.*, Phys. Rev. B **52**, 2656 (1995); I. H. Chan *et al.*, Phys. Rev. Lett. **74**, 3876 (1995); C. M. Marcus *et al.*, Semicond. Sci. Technol. **9**, 1897 (1994); and in Ref. 5.
- <sup>31</sup>Any point  $(\mathbf{r}, \mathbf{p})$  on a periodic orbit  $\gamma$  belongs in fact to many different periodic orbits, which differ from each other by their periods  $T_\gamma$ . These different periods are integer multiples of the period of the “primitive” periodic orbit which corresponds to  $(\mathbf{r}, \mathbf{p})$ . The orbits with different repetition numbers differ from each other in their value of  $n_\gamma$ , hence the necessity in principle for the time variable in  $(\mathbf{r}, \mathbf{p}, t)$  (they share the same value of  $D_\gamma$ , for example). However, for any given time  $t > t_E$  the overwhelming majority of the periodic orbits is primitive, so that using  $n(\mathbf{r}, \mathbf{p})$ , with the smallest value of the corresponding  $n_\gamma$  would introduce only a negligible error.
- <sup>32</sup>S. Iida, H. A. Weidenmuller, and J. A. Zuk, Phys. Rev. Lett. **64**, 583 (1990); Ann. Phys. (N.Y.) **200**, 219 (1990).
- <sup>33</sup>It is particularly impressive to compare the few lines of algebra required in Sec. V B with the lengthy supersymmetric calculation of comparable quantities in Ref. 32.
- <sup>34</sup>See, e.g., P. Gaspard and D. Alonso, Phys. Rev. A **47**, 3468 (1993); G. Vattay (unpublished).
- <sup>35</sup>B. L. Altshuler and A. G. Aronov, in *Electron-Electron Interaction in Disordered Systems*, edited by A. J. Efros and M. Pollak (North-Holland, Amsterdam, 1985).
- <sup>36</sup>See, e.g., M. V. Berry and J. P. Keating, J. Phys. A **23**, 4839 (1990); Proc. R. Soc. London Ser. A **437**, 151 (1992); E. B. Bogomolny, Nonlinearity **5**, 805 (1992); P. Cvitanovic and B. Eckhardt, Phys. Rev. Lett. **63**, 823 (1989); E. Doron and U. Smilansky, Nonlinearity **5**, 1055 (1992); R. Aurich, C. Matties, M. Sieber, and F. Steiner, Phys. Rev. Lett. **68**, 1629 (1992).