Failure of fiber bundles with local load sharing

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We develop a recursion-relation approach for calculating the failure probabilities of a fiber bundle with local load sharing. This recursion relation is exact, so it provides a way to test the validity of the various approximate methods. Applying the exact calculation to uniform and Weibull threshold distributions, we find that the most probable failure load coincides with the average strength as the size of the system $N\to\infty$.

I. INTRODUCTION

Fracture and failure of materials are common but mostly unwanted phenomena. Under external load materials undergo a degradation process that inevitably leads to failure if not stopped. Also the statistical properties of the strength of materials with stochastically distributed elements are important in applications. To analyze the failure in heterogeneous materials, several simple models were proposed, among which the fiber-bundle model[1,2] has in the past few years drawn much attention.[3–12] Although these models are very simple, they seem to contain many of the key properties of some real fracture and failure processes. The fiber-bundle model consists of total number N of fibers whose strengths, or failure thresholds are drawn from continuous distribution,

$$\mathrm{Prob}(t_i<t)=P(t)=\int_0^t p(u)du,$$

where t_i is the threshold of an individual fiber, say, fiber i, and $p(u)$ is called the threshold distribution function (or sometimes the probability density function).

There is also a chain-of-bundles model for fibrous composites, on which extensive work has been done (see, e.g., Refs. 13–23). This model considers a fibrous material as a *chain* of M bundles each with N fibers. If we let $F_N(\sigma),\sigma\geqslant 0$ be the distribution function for the strength of a single bundle of N fibers, then $H_{M,N}(\sigma)$, the distribution function for the strength of the composite,[17] is just

$$H_{M,N}(\sigma)=1-[1-F_N(\sigma)]^M,\qquad \sigma\geqslant 0$$

by the *weakest link rule*, because the composite is an arrangement of satistically and structurally independent bundles and its strength is that of its weakest bundle. So the studies of the chain-of-bundles model reduce to calculating and analyzing the behavior of $F_N(\sigma)$, the probability distribution of the strength of a single bundle [In Ref. 17 it was denoted with $G_n(x)$, where n is the number of fibers in the bundle and x is the load applied.] $F_N(\sigma)$, which is our main concern in this paper, is nothing but the failure probability of a bundle of N fibers when a load σ is applied. And $F_N(\sigma)$ is in fact the static failure probability. For the time-dependent failure problem, which is not of concern in this paper, see, e.g., Refs. 19, 20, and 23.

When certain external load is applied to the fiber bundle, some weak fibers may break. As each fiber breaks, the load is redistributed among surviving fibers. According to the load redistribution rules, the models can be divided into two types. One type of fiber-bundle model is the equal load-sharing model,[1,7] in which external load distributed equally among all surviving fibers. The other type of fiber bundle is local load-sharing model,[8,15] in which the load of a failed fiber is taken up by the nearest surviving neighbors of the failed fiber. The equal load-sharing model is analytically solvable,[7,9,10] while the local load-sharing model seems more difficult to be treated analytically. In this paper, we focus on a local load-sharing fiber-bundle model and study the failure probabilities of the bundle. Actually some authors[3] have calculated the failure probabilities by using some approximate methods. We are motivated that there should be an exact approach to this problem which can test the results obtained from approximate methods.

II. THE MODEL

The system we study in this paper is an array of N fibers, or say a one-dimensional local load-sharing fiber bundle. We put up a question: What is the probability that the bundle fails when an external load σ is applied to the bundle. In this paper, we speak of load on a ''force per fiber'' basis; that is, the load is the total external force on the bundle divided by N, the total number of fibers in the bundle.

The load-sharing rules are essential to the definition of the model. If the load on the bundle is σ (please remember that the total external force on the bundle is $N\sigma$), a surviving fiber carries load $K\sigma$, where K is called a load concentration factor.[23] It can be seen that for the equal load-sharing model

$$K=\frac{N}{N_s},$$

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where N_s is the number of surviving fibers and N is the total number of fibers, failed and surviving, in the bundle.

For the local load-sharing model the load concentration factor varies from one fiber to another. For a surviving fiber we have

$$K=1+\frac{r}{2}, \tag{2.1}$$

where r is the number of consecutive failed fibers immediately adjacent to this surviving fiber (counting on both sides). The local load-sharing rules means that the load of a failed fiber is redistributed in equal portions onto its two nearest surviving neighbors, one on each side.

To define the model, the boundary conditions also need to be specified. Two possible boundary conditions may apply the model. One is the cyclic boundary condition, or periodic boundary condition, which considers fiber 1 and fiber N as nearest neighbors for each other. This version of fiber bundle consists of N fibers mounted evenly on a circle. The other boundary condition, which is our main concern in this paper, can be called closed boundary conditions. In this case, two imaginary fibers, fiber 0 and fiber $N+1$, bound the whole bundle. The two imaginary fibers are of infinite strength, which means they can stand any load transferred from their failed neighbors and thus they never break.

III. THE FAILURE PROBABILITIES

We have put up a question: What is the probability $F_N(\sigma)$ that a bundle of N fibers fails under load σ? We first note that for each fiber there are two possibilities: failed or surviving. Thus there are 2^N possible configurations for a bundle of N fibers. Hereafter we use a 1 to denote a surviving fiber and a 0 to denote a failed fiber. We can then easily write out all the 2^N configurations in the form $(\cdots\ 0\ 1\ 1\ \cdots\ 0\ 1\ \cdots)_N$. The subscript N outside the right parentheses indicates the size of the system, while the 0's and 1's in the parentheses indicate the status of each fiber, failed or surviving. As examples, we list all the configurations for $N=1$, 2, and 3. For $N=1$, there are two configurations $(0)_1$ and $(1)_1$. For $N=2$, the four configurations are: $(00)_2$, $(01)_2$, $(10)_2$ and $(11)_2$. For $N=3$, there are eight configurations: $(000)_3$, $(001)_3$, $(010)_3$, $(011)_3$, $(100)_3$, $(101)_3$, $(110)_3$, and $(111)_3$. If there is at least one survival fiber in the bundle after load σ is applied, we say that the bundle survives the load σ. In general there are 2^N-1 survival configurations for a bundle of N fibers. The failure configuration is the case that all fibers in the bundle fail, which can be denoted by $(0000\cdots000)_N$. Each of these 2^N-1 survival configurations is independent of each other, so the probability $S_N(\sigma)$ that a bundle of N fibers survives load σ can be obtained by adding up the probabilities of all survival configurations, that is

$$S_N=\sum s(\text{configuration}), \tag{3.1}$$

where s is the probability of an individual survival configuration.

With $S_N\equiv S_N(\sigma)$ known, the failure probability $F_N\equiv F_N(\sigma)$ is just

$$F_N=1-S_N. \tag{3.2}$$

It is not difficult to write out the probability of each survival configuration. Taking $(0101)_4$, for example, the probability for this survival configuration is

$$s(0101)_4=F_1W_2F_1W_1,$$

where $W_i\equiv W_i(\sigma)$ is the probability that a fiber survives when it has i failed fibers adjacent it. So we can define

$$W_i(\sigma)=1-\int_0^{(1+i/2)\sigma}p(x)dx, \tag{3.3}$$

where $p(x)$ is the threshold distribution function of the fiber.

In principle, we can calculate the probabilities for all the survival configurations. However, as N increases, the exponential increase in the number of configurations restricts the algorithm. In fact, Leath and Duxbury[3] could compute the failure probabilities for fiber bundles of $N<20$ by using this configuration-counting method. As for fiber bundles of larger N, they had to use an approximate method. In this paper, we present an exact recursion relation for calculating F_N. The expense of this method scales as N^2 while the configuration-counting method scales as 2^N. The recursion relation enables us to calculate F_N for N from 1 to 10^3 or more. The exact results of F_N can serve as a test of the previous approximate results.

Before we present the recursion relation, we note that the configurations for a bundle of $N+1$ fibers, can be easily produced from that of N. For example, $(10)_2$ is a configuration for $N=2$. By simply adding a fiber to the right end of the configuration, we get a configuration for $N=3$. The added fiber can be failed or surviving, so $(10)_2$ produces two configurations: $(100)_3$ and $(101)_3$. In general, each configuration of N produce two configurations of $N+1$. For this reason it is convenient to calculate the probabilities of $N+1$ from that of N. Using the same example, the probability for the configuration $(10)_2$ is

$$s(10)_2=W_1F_1,$$

while the probabilities for $(100)_3$ and $(101)_3$ are

$$s(100)_3=W_2F_2,$$

and

$$s(101)_3=W_1F_1W_1.$$

Since the expressions for $s(10)_2$ and $s(101)_3$ contain some common factors, it is then convenient to calculate $s(101)_3$ by using $s(10)_2$, that is

$$s(101)_3=s(10)_2W_1.$$

And $s(100)_3$ can also be calculated through $s(10)_2$

$$s(100)_3=\frac{s(10)_2}{W_1F_1}W_2F_2,$$

Another example, $(10010)_5$ is a configuration for $N=5$. It produces two configurations for $N=6$: $(100100)_6$ and $(100101)_6$. It can be seen that

$$s(10010)_5 = W_2 F_2 W_3 F_1,$$

and that

$$s(100100)_6 = W_2 F_2 W_4 F_2 = \frac{s(10010)_5}{W_3 F_1} W_4 F_2,$$

and

$$s(100101)_6 = W_2 F_2 W_3 F_1 W_1 = s(10010)_5 W_1.$$

Here we also used $s(10010)_5$ to calculate $s(100100)_6$ and $s(100101)_6$.

A special case should be noted here that the failure configuration of N $(00\cdots00)_N$ produces two configurations of $N+1$, one of which is the failure configuration $(00\cdots000)_{N+1}$ and the other one is a survival configuration $(00\cdots001)_{N+1}$. In this survival configuration only the right end fiber survives. The probability for this configuration is just

$$s(00\cdots001)_{N+1} = F_N W_N.$$

Based on a similar idea to generate probabilities for larger bundles from those of smaller bundles, Harlow and Phoenix[17] have developed a recursive analysis and arrived at matrix formulation for the problem. Now we try to obtain a recursion relation for the calculating of F_N. For a bundle of N fibers, the total number of the survival configurations is 2^N-1. The key point to get the exact recursion relation is to classify the various survival configurations properly. We classify these 2^N-1 survival configurations into different groups. Let $S_N(i,j)$ be the set of configurations that have the form

$$(\cdots 1\ \overbrace{00\cdots0}^{i}\ 1\ \overbrace{00\cdots0}^{j})_N,$$

where i and j are integers satisfying $0 \leqslant i \leqslant N-1$ and $0 \leqslant i+j \leqslant N-1$. In words, there are j consecutive 0's in the configuration right to the right-most 1, and i consecutive 0's between the right-most 1 and the next 1. No matter whether the other fibers in the bundle are failed or surviving, those configurations with the same i and j belong to the same group $S_N(i,j)$. It should be noted here that there are some survival configurations that contain only one 1, such as $(00010)_5$, $(001)_3$, and $(1000)_4$, etc. In these configurations, j is the number of 0's right to the 1, and i is the number of 0's left to the 1. So $(00010)_5$ is in group $S_5(3,1)$, $(001)_3$ is in group $S_3(2,0)$ and $(1000)_4$ is in group $S_4(0,3)$.

In general there are $N(N+1)/2$ groups of survival configurations for a given value of N. As examples, we list all the groups of configurations for $N=1$, 2, 3, and 4.

For $N=1$, the only group is $S_1(0,0)$

$$S_1(0,0) = \{(1)_1\}$$

For $N=2$ there are three groups. They are

$$S_2(0,0) = \{(11)_2\},$$

$$S_2(0,1) = \{(10)_2\},$$

$$S_2(1,0) = \{(01)_2\}.$$

For $N=3$, the six groups are

$$S_3(0,0) = \{(111)_3, (011)_3\},$$

$$S_3(0,1) = \{(110)_3\},$$

$$S_3(0,2) = \{(100)_3\},$$

$$S_3(1,0) = \{(101)_3\},$$

$$S_3(1,1) = \{(010)_3\},$$

$$S_3(2,0) = \{(001)_3\}.$$

For $N=4$, there are 10 groups of configurations

$$S_4(0,0) = \{(0011)_4, (0111)_4, (1011)_4, (1111)_4\},$$

$$S_4(0,1) = \{(0110)_4, (1110)_4\},$$

$$S_4(0,2) = \{(1100)_4\},$$

$$S_4(0,3) = \{(1000)_4\},$$

$$S_4(1,0) = \{(0101)_4, (1101)_4\},$$

$$S_4(1,1) = \{(1010)_4\},$$

$$S_4(1,2) = \{(0100)_4\},$$

$$S_4(2,0) = \{(1001)_4\},$$

$$S_4(2,1) = \{(0010)_4\},$$

$$S_4(3,0) = \{(0001)_4\}.$$

In this paper, we also use $S_N(i,j)$ to denote the sum of probabilities of all the configurations in the group $S_N(i,j)$. Then the survival probability of the bundle can be obtained by adding up all the group-probabilities, thus Eq. (3.1) becomes

$$S_N = \sum_{i=0}^{N-1} \sum_{j=0}^{N-i-1} S_N(i,j). \tag{3.4}$$

So this classification give a natural way to construct a recursion relation to calculate $S_N(i,j)$. The exact recursion relations can be written as

$$S_N(N-1,0) = F_{N-1} W_{N-1},$$

$$S_N(i,0) = \sum_{k=0}^{N-i-2} S_{N-1}(k,i) W_i, \quad \text{for } 0 \leqslant i \leqslant N-2,$$

$$S_N(i,j) = \frac{S_{N-1}(i,j-1)}{W_{i+j-1} F_{j-1}} W_{i+j} F_j,$$
$$\text{for } 0 \leqslant i \leqslant N-2 \text{ and } 1 \leqslant j \leqslant N-i-1. \tag{3.5}$$

We note here that we have defined $F_0 \equiv 1$ when we deduce the above recursion relations. It can be seen that if we use Eq. (3.1) to calculate S_N the total number of adding is 2^N-1, while the number of adding for Eq. (3.4) is reduced

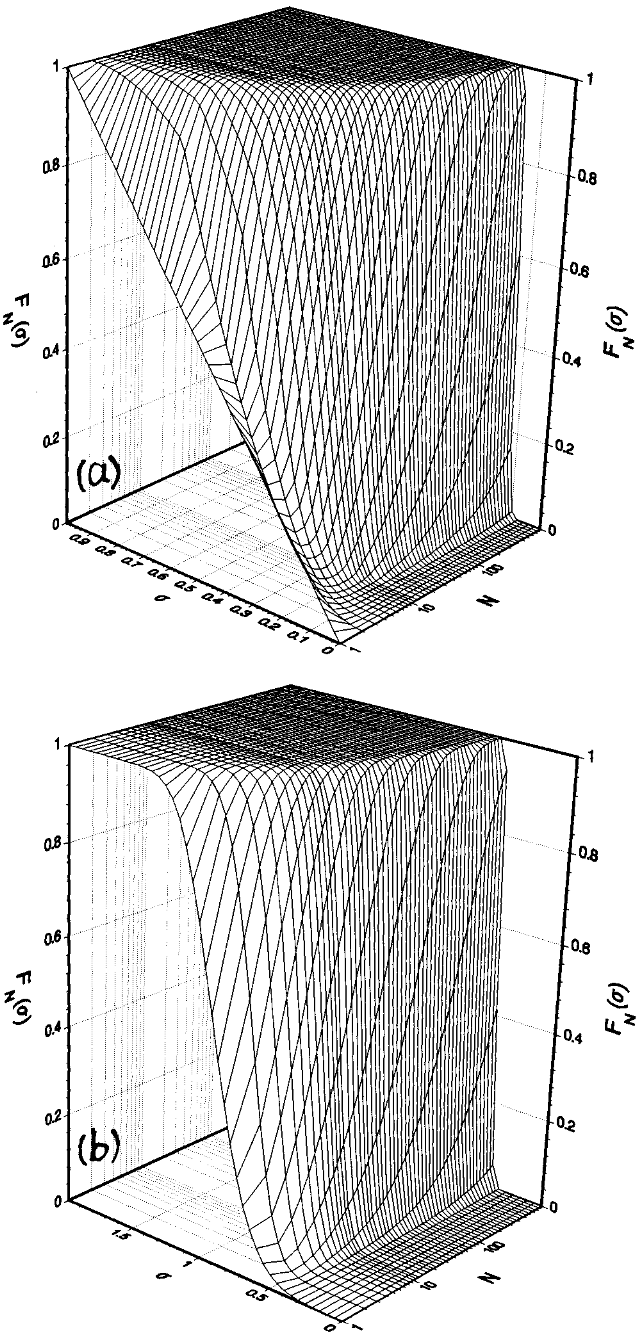

FIG. 1. A plot of $F_N(\sigma)$ as a function of σ and N. (a) for uniform threshold distribution (b) for Weibull distribution with $m=5$.

to $N(N-1)+1$. Furthermore the recursion relation Eq. (3.5) can be easily realized in a computer algorithm. The failure probabilities F_N's can then be computed successively.

Now we apply the method to specific threshold distribution. Let us consider two forms of distributions. One is the uniform threshold distribution $p(\sigma)=1$ for $\sigma\in[0,1]$ and $p(\sigma)=0$ for $\sigma>1$. This form of distribution represent a class of distributions that is in a finite region. Actually, any finite threshold region can be mapped into $[0,1]$. For the uniform distribution, the $W_i(\sigma)$ defined in Eq. (3.3) is just

$$W_i(\sigma)=\begin{cases}1-(1+i/2)\sigma, \text{ if } 0\leqslant(1+i/2)\sigma\leqslant 1,\\ 0, \text{ otherwise.}\end{cases} \tag{3.6}$$

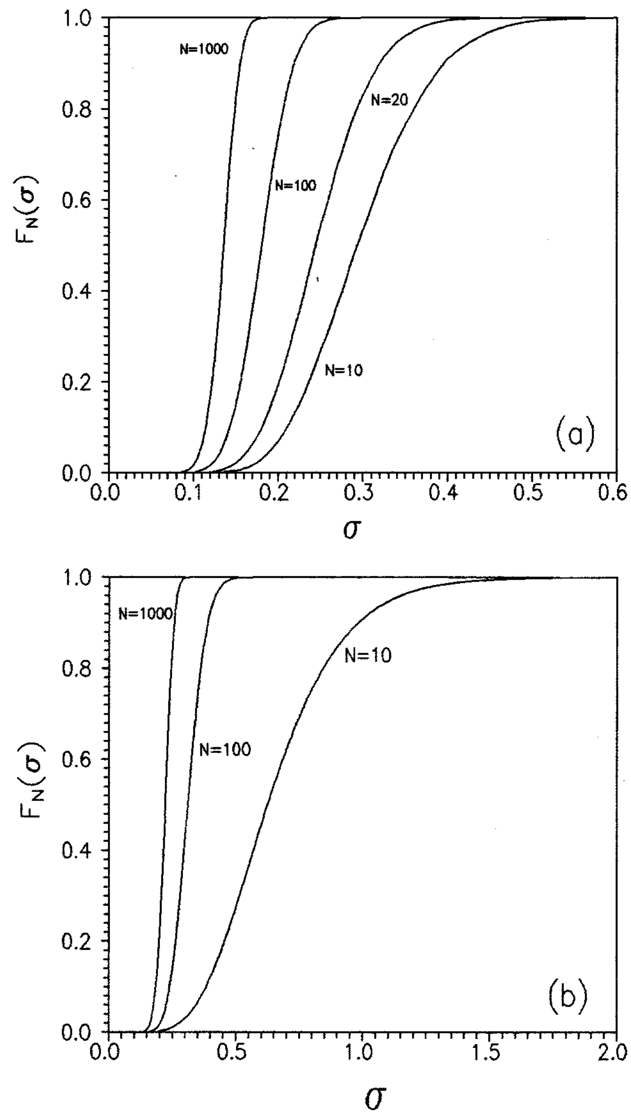

FIG. 2. For given N, $F_N(\sigma)$ as a function of σ is ever increasing with σ. The sharp increase of $F_N(\sigma)$ becomes more remarkable as N rises. (a). Uniform distribution. The four lines from the left to the right are for $N=1000$, 100, 20, and 10, respectively. (b) Weibull distribution with $m=1$. For $N=1000$, 100, and 10.

The other form of the threshold distribution is Weibull distribution $p(x)$ that is

$$\int_0^\sigma p(x)dx=1-\exp[-(\sigma/\sigma_s)^m]. \tag{3.7}$$

This distribution has two adjustable parameters. σ_s is the scale strength, which sets the size of the typical strength in the distribution and, m is the Weibull modulus, which determines the scatter in the distribution of the fiber thresholds. We simply set $\sigma_s=1$, then the Weibull distribution becomes

$$\int_0^\sigma p(x)dx=1-\exp(-\sigma^m). \tag{3.8}$$

One advantage of the Weibull distribution is that it has no strict upper cutoff and it has been broadly applied in some fields. For this form of distribution

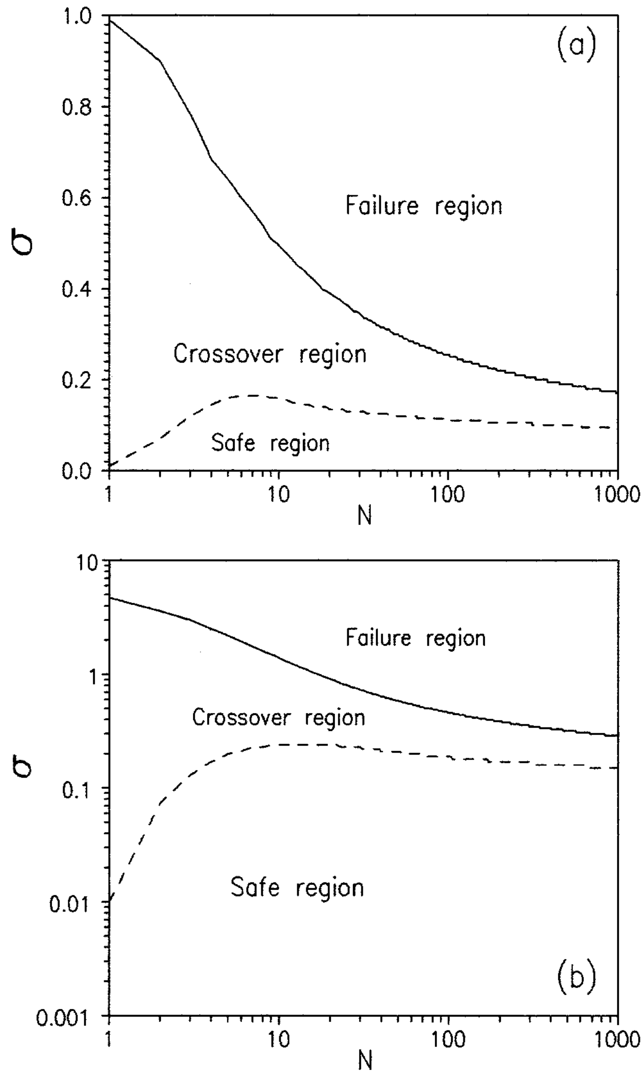

FIG. 3. $\sigma_0^*(\epsilon)$ and $\sigma_1^*(\epsilon)$ divide the (N,σ) space into three regions: failure, crossover, and safe regions. The safe index ϵ is set to be 0.01. As N becomes larger, the cross region becomes narrower and narrower. (a) Uniform distribution and (b) Weibull distribution with $m=1$. In both cases the solid line is for $\sigma_1^*(\epsilon)$ while dashed line for $\sigma_0^*(\epsilon)$.

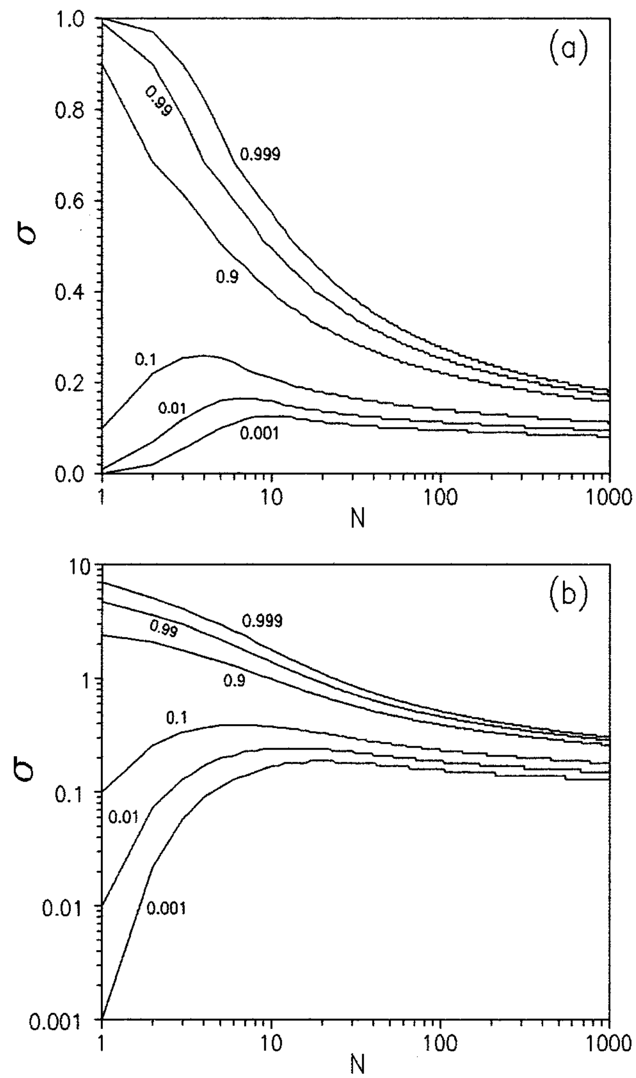

FIG. 4. The iso-failure-probability lines in the (N,σ) space for $F_N(\sigma)=0.001$, 0.01, 0.1, 0.9, 0.99, and 0.999. The number beside each line indicates the value of $F_N(\sigma)$. (a) Uniform distribution. (b) Weibull distribution, $m=1$. This figure shows how the three regions defined in Fig. 3 change as the safe index ϵ is changed.

$$W_i(\sigma)=e^{-[(1+i/2)\sigma]^m}. \tag{3.9}$$

With $W_i \equiv W_i(\sigma)$ known, noticing that $S_1(0,0)=W_0$ and $F_1=1-W_0$, we can get from the recursion relation (3.5) the failure probabilities $F_N(\sigma)$'s for any N and any σ.

IV. THE EXACT RESULTS

Using the exact method we calculated the failure probabilities $F_N(\sigma)$ for both the uniform and Weibull threshold distributions. The results are shown in Fig. 1, which is a three-dimensional drawing. From Fig. 1 we get a general idea how $F_N(\sigma)$ varies with changing σ and N.

For a given N, $F_N(\sigma)$ is a monotonic increasing function of σ. For both the uniform and Weibull threshold distributions $F_N(0)=0$; and $F_N(1)=1$ for the uniform distribution and $F_N(\sigma\to\infty)=1$ for the Weibull distribution. $F_N(0)=0$ corresponds to the situation that no external load is applied to the bundle, so that the bundle will not break anyway; $F_N(1)=1$ or $F_N(\sigma\to\infty)=1$ corresponds to the other extreme situation that a maximum load or an infinite load is applied to the system, thus the bundle will fail definitely. Typical results in Fig. 2 clearly shows that F_N has a sharp increase at a certain σ.

In principle, only when $\sigma=0$ can the failure probability $F_N(\sigma)$ be exactly zero, and only when $\sigma=1$ for the uniform distribution or $\sigma\to\infty$ for the Weibull distribution will $F_N(\sigma)$ be exactly 1. However, for practical purposes, it is convenient to introduce an "safe index" ϵ $(0<\epsilon\ll 1)$. The physical meaning of the safe index is the following. On one hand, whenever there is an $F_N(\sigma)<\epsilon$, we consider it is safe and the bundle will not fail; on the other hand, when $F_N(\sigma)>1-\epsilon$, we consider that the bundle will fail. In our calculations we set the safe index $\epsilon=0.01$.

Two interesting values of σ are $\sigma_0^*(\epsilon)$ and $\sigma_1^*(\epsilon)$. $\sigma_0^*(\epsilon)$ is the value of σ at which $F_N(\sigma_0^*)=\epsilon$. And $\sigma_1^*(\epsilon)$, on the other hand, is the value of σ that satisfy

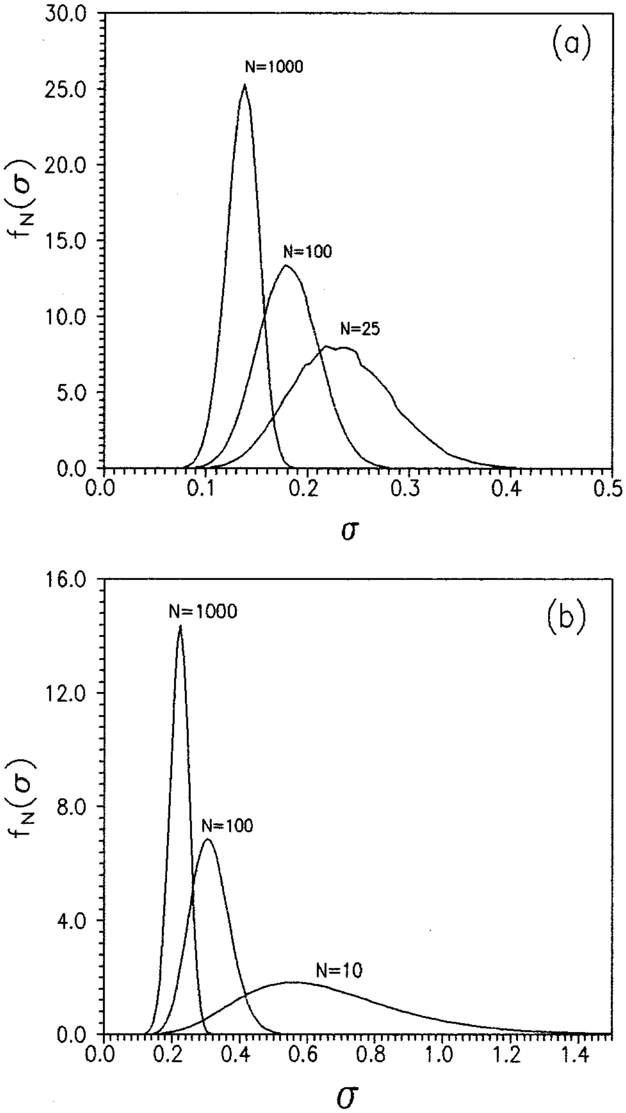

FIG. 5. Failure probability density $f_N(\sigma)$. (a) Uniform threshold distribution, $N=1000,100,$ and 25. (b) Weibull distribution with $m=1$, $N=10,$ 100, and 1000.

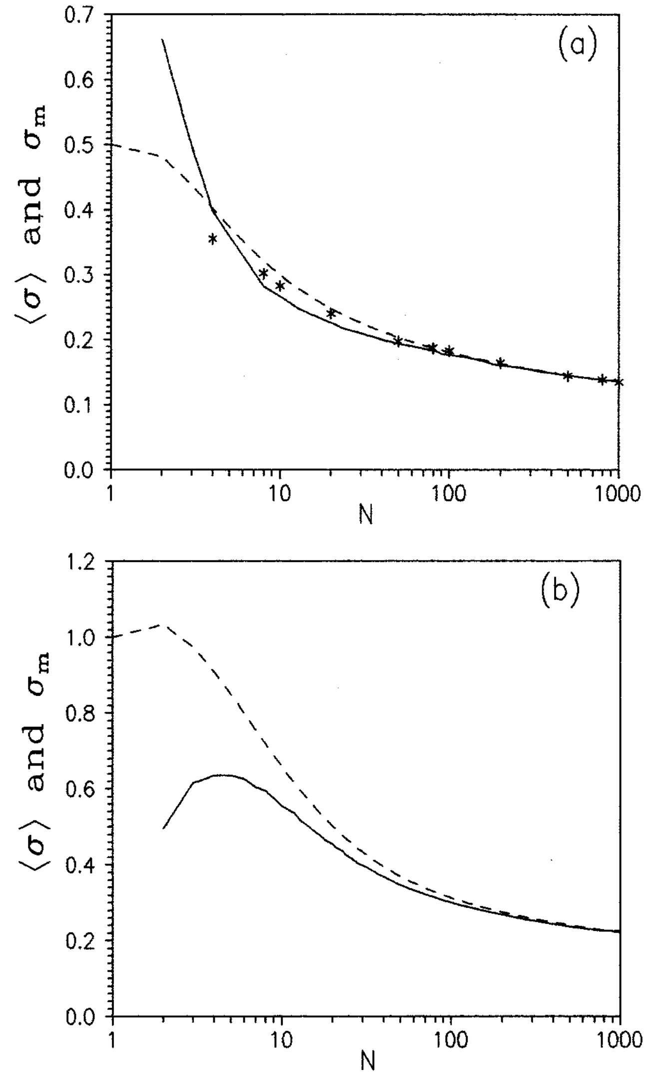

FIG. 6. The average strength and the most probable failure load are dependent of system size N. At small N the two quantities show apparent difference, but almost coincide as $N\to\infty$. (a) Uniform threshold distribution. The asterisks in the plot stands for the average strength from actual simulations of 100 samples. (b) Weibull distribution with $m=1$. In both cases the solid lines are drawn for the most probable failure load σ_m while the dashed lines for the average strength $\langle\sigma\rangle$ defined in Eq. (5.2).

$F_N(\sigma_1^*)=1-\epsilon$. The physical meanings of these two values are the following. When a load $\sigma<\sigma_0^*(\epsilon)$ is applied to the system the bundle is safe and will not fail, but a load of $\sigma\geqslant\sigma_1^*(\epsilon)$ will destroy the system; under a load between the two values $\sigma_0^*(\epsilon)<\sigma<\sigma_1^*(\epsilon)$ the bundle can either fail or survive with different probabilities. Both σ_0^* and σ_1^* are functions of N. In Fig. 3 we show σ_0^* and σ_1^* versus N. The solid line stands for σ_1^* while the dashed line is drawn for σ_0^*. The two lines divide the (N,σ) space into three regions: Above the solid line is the failure region where the fiber bundle can rarely survive; Below the dashed line is the safe region, in which almost every bundle will survive; Between the two lines is the crossover region where bundles can survive or fail. We also see that crossover region becomes narrower and narrower as N increases. So as $N\to\infty$ we may expect the two lines in Fig. 3 coincide, meaning that a bundle of infinite size breaks suddenly at a point of σ, depending on N.

When $\epsilon=0$, the cross over region should cover the whole (N,σ) space, and the failure region and the survive region are only two lines in the space. So only in the sense that a nonzero safe index ϵ is introduced could the failure region and safe region occupy nonzero areas. For this reason, we draw the iso-failure-probability lines in Fig. 4, from which we may see how the three regions change as the safe index is changed.

V. THE FAILURE PROBABILITY DENSITY FUNCTION

If we define

$$f_N(\sigma)=\frac{dF_N(\sigma)}{d\sigma}, \tag{5.1}$$

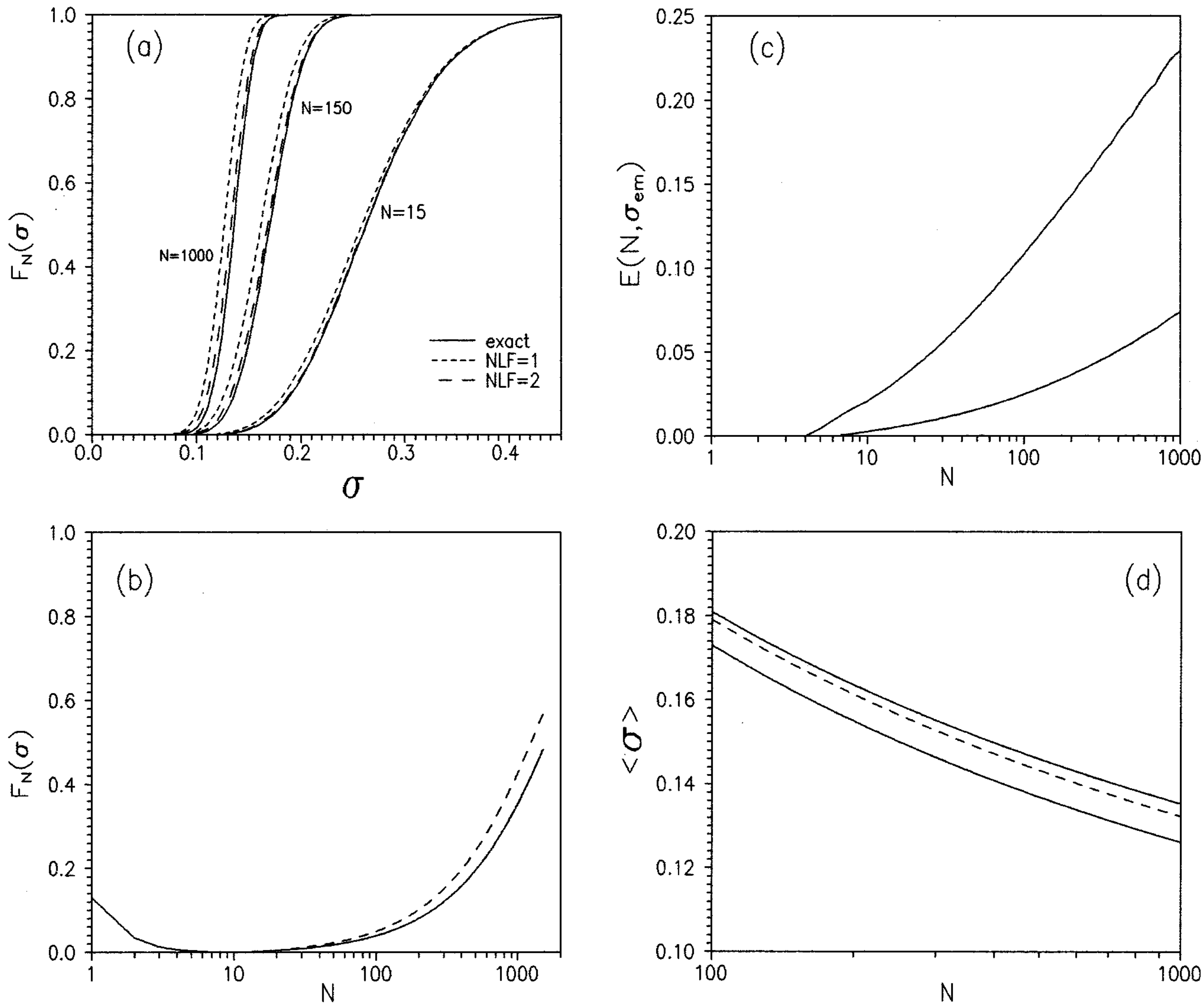

FIG. 7. Comparisons of exact results with the approximate results from the method of Leath and Duxbury for uniform threshold distribution. (a) For every given N, three lines are shown. For $N=15$, the line for NLF=2 result and that for the exact result almost coincide to the resolution of this figure. (b) At certain σ and N, NLF=2 results (dashed line) show large differences from the exact results (solid line). $\sigma=0.13$, $1\leqslant N\leqslant 1500$. (c) The maximum error $E(N,\sigma_{em})$ of the NLF approximate results. The upper line is for NLF=1 and the lower line for NLF=2. (d). Comparison of average strength. The upper solid line is for the exact result, the dashed line is for the NLF=2 result, and the lower solid line is for the NLF=1 result.

it can be easily understood that $f_N(\sigma)$ is just the failure probability density function. The average strength of the bundle, or say the mean failure load of the bundle can then be calculated as

$$\langle\sigma\rangle=\int_0^u \sigma f_N(\sigma)d\sigma, \tag{5.2}$$

where $u=1$ for the uniform distribution and $u=\infty$ for the Weibull distribution. For not too small N, the failure probability density function has a well-defined maximum at a certain value of σ as shown in Fig. 5. We denote this value of σ with σ_m, which thus satisfies

$$\left.\frac{df_N(\sigma)}{d\sigma}\right|_{\sigma=\sigma_m}=0. \tag{5.3}$$

σ_m is actually the most probable failure load, meaning that the system is most likely to break at this load. Also we found that σ_m is dependent on the system size N. In Fig. 6 we plot the average strength and the most probable failure load as functions of N, from which we can see that at small N, the average strength of the bundle is apparently different from its most probable failure load $\sigma_m(N)$; however as N increases, the two lines becomes closer and closer and at last they almost coincide. Since in Ref. 11 we have found from large amount of simulations that $\langle\sigma\rangle\sim 1/\log N$, which implies the average strength $\langle\sigma\rangle\to 0$ as $N\to\infty$, we thus can expect the most probable load σ_m also goes to 0 as $N\to\infty$. We note here that we had used the cyclic boundary conditions in Ref. 11, different from the closed boundary conditions used in this paper. But as the average strength and the most probable failure load σ_m are concerned, the two kinds of boundary conditions make little difference.

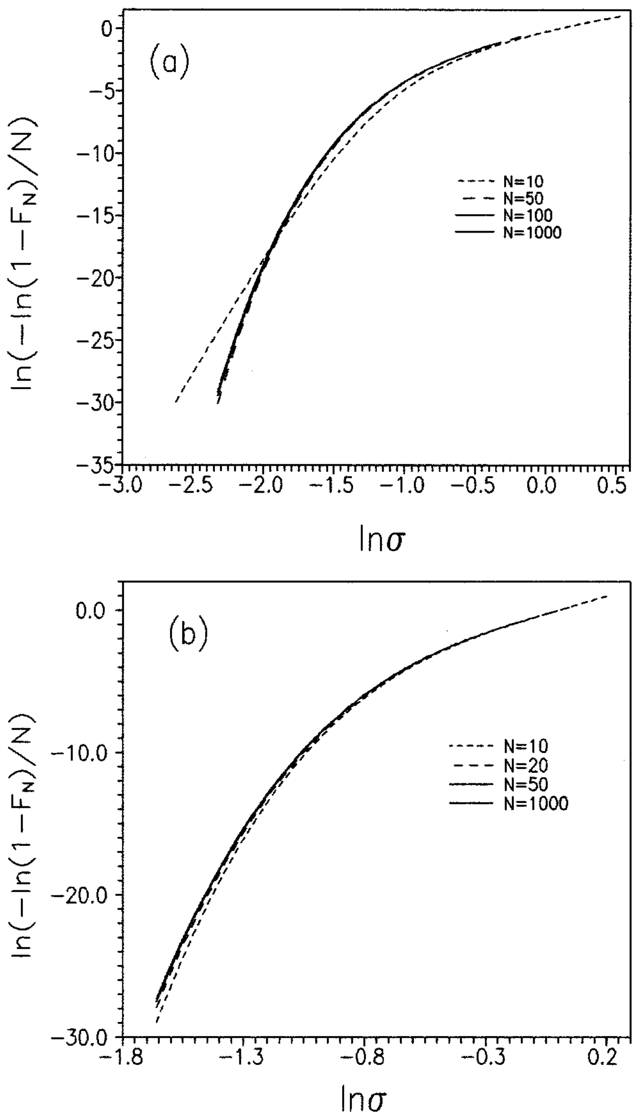

FIG. 8. Comparison of exact results to the asymptotic form for F_N [Eq. (6.3) in the text] and test of the Weibull form for the characteristic function $C(\sigma)$. (a) Weibull distribution for fiber threshold with Weibull modulus $m=2$. (b) Weibull distribution for fiber threshold with Weibull modulus $m=5$.

VI. COMPARISON WITH PREVIOUS RESULTS

First, let us compare the exact results with the approximate results in Ref. 3 to test the reliability of the approximation method. Leath and Duxbury[3] developed a recursion relation for calculating $F_N(\sigma)$. According to the *number of lone fibers* (NLF) that are taken account into their calculation, the approximate methods may be called NLF$=i$ approximation, where i is an integer. By comparing the exact results with the approximation results for $N \leqslant 20$, Leath and Duxbury concluded that NLF$=2$ approximation is accurate enough for calculating $F_N(\sigma)$. Since they did not make the comparison for larger N, the reliability of the approximate method is still unknown for $N > 20$.

Now that we have developed an exact method for the calculation of the failure probabilities, we thus can make the comparison for $N > 20$. In the following comparisons the threshold distribution is chosen to be the uniform one. In Fig. 7(a) we compare the exact results with the results from NLF$=1$ and NLF$=2$ approximations for several given values of N. It can be seen from this figure that the NLF$=2$ result is much closer to the exact one than the NLF$=1$ result is, especially when $N=15$, the NLF$=2$ result is coincident with the exact one to the resolution of this figure. However, for larger N, the difference between the exact result and the NLF$=2$ result becomes more and more apparent. We can also make the comparison for some given values of σ. We find that for certain N and σ the differences between the exact results and the approximate ones are not very small. Fig. 7(b) shows the differences between the exact result and the NLF$=2$ result for $\sigma=0.13$ and $1 \leqslant N \leqslant 1500$. The NLF$=2$ approximation method gives $F_{1500}(0.13)=0.5687$ while the exact result is $F_{1500}(0.13)=0.4828$.

If we denote $E(N,\sigma)$ as the difference of the failure probability between the NLF approximation result and the exact result. For a given N, the difference $E(N,\sigma)$ approaches a maximum value at a value of σ, which we denote by σ_{em}. The subscript em means maximum error. We found that σ_{em} is actually around σ_m. In Fig. 7(c), we show the maximum error $E(N,\sigma_{em})$ versus N. The trend is obvious that the maximum error becomes larger as N increases.

As far as the average strength is concerned, the NFL approximate method gives a weaker strength than the exact one. Figure 7(d) clearly shows that as N becomes larger the differences between the NLF results and the exact result also become larger.

From the comparisons made above we can conclude that in general the NLF$=2$ approximation is a relatively good method for the calculating of $F_N(\sigma)$; However, near to σ_m the NLF$=2$ approximation gives unsatisfactory results. Then we need turn to the exact method.

Our method of calculation has three advantages. First, it is exact so that it serves as a test of other approximate methods. Second, the exact method scales with system size N as N^2, a little less amount of calculation than the NLF$=2$ approximation methods. In our method, the number of additions for calculating $F(\sigma)$ is N^2-N+2. While the number of additions for the NLF$=2$ approximate method is $3N^2/2-9N/2+12$. Thirdly, this exact method can be easily realized in a computer algorithm. We made our calculations on a SUN workstation, and we found the exact method spends less CPU time than the NLF$=2$ approximation method. For example, the CPU time for calculating the exact results in Fig. 7(b) is about 3 400 seconds, while the NLF$=2$ calculation to obtain the dashed line in Fig. 7(b) spends about 4 800 CPU seconds. The disadvantage of our exact method is that it needs a larger amount of computer memory than the approximate methods do. Our method has only been applied to the closed boundary conditions. For other boundary conditions such as the cyclic boundary condition, some modification should be made.

Secondly, we compare the exact results to some asymptotic forms resulting from previous analysis. Using a matrix formulation, Harlow and Phoenix[17] obtained the bounding distribution for bundle strength of the following form

$$F_N^{(k)}(\sigma)=1-[1-C^{(k)}(\sigma)]^N[\pi^{(k)}(\sigma)+o_N^{(k)}(\sigma)], \tag{6.1}$$

where $F_N^{(k)}(\sigma)$ with $1 \leqslant k \leqslant N$ is the bounding distribution function for bundle strength. When $k=N$, $F_N^{(N)}(\sigma)$ is just the failure probability $F_N(\sigma)$ studied in this paper. $C^{(k)}(\sigma)$ in Eq. (6.1) is the *characteristic distribution function* such that

$$\lim_{k\to\infty} C^{(k)}(\sigma)=C(\sigma), \tag{6.2}$$

where $C(\sigma)$ is called the limiting characteristic distribution function. In Ref. 17 it was denoted as $W(x)$. We do not follow their previous notation because we have used $W_i(\sigma)$ for another quantity. The "boundary term" $[\pi^{(k)}(\sigma)+o_N^{(k)} \times(\sigma)]$ in Eq. (6.1) is very close to unity as N becomes larger. So Harlow and Phoenix considered that

$$F_N(\sigma)=1-[1-C(\sigma)]^N \tag{6.3}$$

is an accurate representation of the distribution function for bundle strength (failure probability). Kuo and Phoenix[23] later developed a recursion and limit theorem which could apply separately to static strength and time-dependent failure. They also recasted this theorem into a key approximation for $F_N(\sigma)$, which was the same form as Eq. (6.3). From Eq. (6.3) we easily get

$$\ln[-\ln(1-F_N)/N]=\ln\{-\ln[1-C(\sigma)]\}. \tag{6.4}$$

The characteristic function $C(\sigma)$ is independent of N, so if we plot $\ln[-\ln(1-F_N)/N]$ versus some function of σ, the data for different N should fall onto a common line, and then the validity of Eq. (6.3) is confirmed. In addition, if $C(\sigma)$ is of the form of Weibull distribution like $C(\sigma)\sim 1-\exp(-\sigma^{\rho})$, the right-hand side of Eq. (6.4) then becomes $\rho \ln\sigma$, and the plot of $\ln[-\ln(1-F_N)/N]$ versus $\ln\sigma$ should be a straight line. In Fig. 8, we present some typical results. The fiber threshold distribution is chosen to be the Weibull distribution Eq. (3.8). Figure 8(a) is for the case Weibull modulus (or say, shape parameter) $m=2$, from which we see that the data for $N=10$ show apparent deviation from that of $N=1000$, but as N increases to 50, the line is very very close to that of $N=1000$. As N becomes larger than 100, all the data falls on a common line, indicating Eq. (6.3) is valid in this limit. We also see that the common line on which data for large N collapse is not straight overall, it is only straight at its lower part, indicating the Weibull form for $C(\sigma)$ is only valid in this region. Fig. 8(b) is for the case $m=5$, from which we notice that the line for $N=20$ is already very very close to the common line on which data for $N>50$ collapse. Here we also see that the common line in Fig. 8(b) is not straight and hence $C(\sigma)$ is not of Weibull form overall. Actually, as Harlow and Phoenix[17] pointed out, it is difficult to express $C(\sigma)$ in terms of the usual classical functions. We expect the exact results for F_N to be helpful to test some other predictions for the form of $C(\sigma)$ in Eq. (6.3).

VII. SUMMARY

In this paper, we study the failure probability of a fiber-bundle model with local load-sharing. By classifying the 2^N-1 survival configurations into different groups, we introduced an exact recursion relation for calculating the failure probability $F_N(\sigma)$. One advantage of this method is that it scales with the system size as N^2, roughly the same amount of calculating as some approximate methods. We then apply this recursion relation to two form of threshold distributions, the uniform distribution and the Weibull distribution. In both cases, the average strength of the bundle almost coincide with the most probable failure load as $N\to\infty$. Also we find that the average strength calculated from failure probability density is in good agreement with the result from actual simulations [see Fig. 6(a)]. Some comparisons to previous results are also made.

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[1] H.E. Daniels, Proc. Roy. Soc. London Ser. A **183**, 404 (1945).
[2] P.K. Sen, J. Appl. Pro. **10**, 586 (1973).
[3] P.L. Leath and P.M. Duxbury, Phys. Rev. B **49**, 14 905 (1994).
[4] P.M. Duxbury and P.L. Leath, Phys. Rev. B **49**, 12 676 (1994).
[5] D. Sornette, J. Phys. A **22**, L243 (1989). D. Sornette and S. Redner, J. Phys. A **22**, L619 (1989).
[6] Y.N. Lu and E.J. Ding, J. Phys. A **25**, L241 (1992).
[7] P.C. Hemmer and A. Hansen, J. Appl. Mech. **59**, 909 (1992).
[8] A. Hansen and P.C. Hemmer, Phys. Lett. A **184**, 394 (1994).
[9] A. Hansen and P.C. Hemmer, *Critical in Fracture: The Burst Distribution* (Research Trends, Trivandrum, India, 1994).
[10] W. Lee, Phys. Rev. E **50**, 3797 (1994).
[11] S.D. Zhang and E.J. Ding, Phys. Lett. A **193**, 425 (1994).
[12] D.G. Harlow and S.L. Phoenix, J. Mech. Phys. Solids **39**, 173 (1991).
[13] D.G. Harlow and S.L. Phoenix, J. Compos. Mater. **12**, 195 (1978).
[14] D.G. Harlow and S.L. Phoenix, J. Compos. Mater. **12**, 314 (1978).
[15] R.L. Smith, Proc. Roy. Soc. London Ser. A **372**, 539 (1980).
[16] D.G. Harlow and S.L. Phoenix, Int. J. Fracture **17**, 347 (1981).
[17] D.G. Harlow and S.L. Phoenix, Int. J. Fracture **17**, 601 (1981).
[18] R.L. Smith, Proc. Roy. Soc. London Ser. A **382**, 179 (1982).
[19] L. Tierney, Adv. Appl. Prob. **14**, 95 (1982).
[20] S.L. Phoenix and L. Tierney, Engineering Fracture Mechanics **18**, 193 (1983).
[21] R.L. Smith, Adv. Appl. Prob. **15**, 304 (1983).
[22] R.E. Pitt and S.L. Phoenix, Int. J. Fracture **22**, 243 (1983).
[23] C.C. Kuo and S.L. Phoenix, J. Appl. Prob. **24**, 137 (1987).