Criticality and multifractality of the Potts ferromagnetic model on fractal lattices

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The critical and multifractal properties of the local and global magnetizations of the zero-field ferromagnetic q-state Potts model on hierarchical lattices of several distinct fractal dimensions (d_f) are obtained and studied by an exact recursion procedure. The critical exponents α, β, and ν, the correlation length, and thermodynamic functions as specific heat and global magnetization are calculated for general q and related with the Hölder exponent (α_H) that describes the multifractal structure (or spectra) of the local-order parameter. The hyperscaling law was successfully tested for a family of lattices confirming the relation $d_f = 2-\alpha$. The $f(\alpha)$-multifractal spectra of the local magnetization at the critical point of the diamond hierarchical lattices family is numerically obtained and studied for general q and lattice connectivity. The domain boundaries of the α_H-Hölder exponent ($\alpha_{H\min}, \alpha_{H\max}$) were analytically calculated recovering the numerical figures.

I. INTRODUCTION

The large majority of works on spin systems in dimensions greater than 1 are obtained either by numerical or through approximated analytical methods. The exact results available in dimensions 2 or 3 (or greater) are very few and are always of recognized importance, being used many times as a reference when some approximation, in similar problems, are made. Exact results are very important when obtained in Bravais lattices, as the Onsager solution for the Ising square lattice,[1] the Baxter solution for the six-vertex problem,[2] and others (see Baxter[3]). Otherwise, exact solutions on non-Bravais lattices are also important tools for the knowledge of many complicated points in the phase diagrams of several models. For instance, the exact solution of spin models on the Bethe lattice are reported on the book of Baxter[3] as one of the interesting exact solutions for higher dimensional spin systems. The Bethe lattice can be viewed as a kind of hierarchical lattice (HL),[4–7] and the HL is a relevant family of non-Bravais lattices that can be considered, in many situations, as approximated lattices of some Bravais ones. In fact, when the Migdal-Kadanoff[8,9] approximation (bond-moving schemes) is used for spin systems on hypercubic lattices we obtain the same spin systems on HL's, giving results that, as have been shown in several works, present a good agreement with the results obtained for the corresponding Bravais lattices by other methods (series, numerical, etc.).[9–18] Some results (specially critical frontiers and critical exponents) are relatively simple to be obtained using this kind of lattices,[19–21] and therefore they can be regarded as a forward step after mean field and Bethe latticelike approximations to solve models on Bravais lattices (whose solutions are usually extremely hard or even impossible to be obtained). For spin glass systems, for example, these approaches have been largely used in the past[22–27] and are still being considered since the exact solutions for these systems are far away to be reached. Nevertheless the exact calculation of thermodynamic functions (specific heat, magnetization, etc.) are much more complicated to be obtained within HL's approach and we have found in the literature, some times, the use of heuristic recipes to calculate these functions,[14,28–30] almost always with obscure points and with no general method allowing the obtention of these functions in other similar problems. We have developed in the last few years a method that, systematically, allows us to calculate exactly these functions for Ising systems on a large family of fractal hierarchical lattices.[31–33] To do this we must take in account the topology of the lattice, where the coordination number differs from site to site depending on the hierarchy of the lattice we are considering. This method was used to calculate exactly several thermodynamical functions and critical properties of the Ising model on HL's (Refs. 31–33) and, in one case, on a very specific $q=3$ Potts model.[34] Some curious and unexpected multifractal aspects have been revealed in these systems and some exact results for the multifractal function have been calculated.[31,32] Also, a very interesting *connection* among critical exponents and the Hölder exponent[35] came out in some works.[31,34]

In this work we extend the method for the general Potts model,[36,37] with an arbitrary number q of states. Several differents aspects appear, specially the coupling, in the recurrence relations, of the local magnetization with the pair cor-

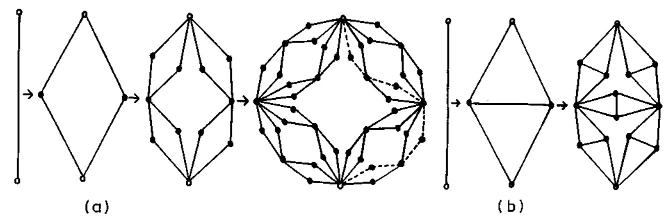

FIG. 1. Construction procedure of two hierarchical lattices: (a) up to $n=3$ for the DHL, where one of the shortest paths connecting the two roots (open sites) is explicated (dashed line); (b) Wheatstone Bridge hierarchical lattice (WBHL) up to $n=2$ level.

relation function, coupling that does not appear for the Ising model. We explain in detail how to apply this method for a large family of HL's and exactly calculate for a subset of them, some thermodynamical functions, critical exponents, critical frontiers, multifractal functions, etc. For the Potts model on these HL's, our results for these thermodynamical functions and multifractality are new ones. Otherwise, we are also capable to *prove* that the hyperscaling law is valid for all HL's we have tested, where for the dimension (in the hyperscaling law) we understand the fractal dimension of the lattice. Our generalization of the method for Potts model also indicates the way this method could be generalized to other spin systems.

II. MODEL

Let us consider a zero field Potts ferromagnetic on the so-called diamond hierarchical lattice (hereafter DHL) shown in Fig. 1(a) for $d_f=2$. At the nth level, we can assign to each site of the lattice a q-state Potts variable σ_i and to each bond (nearest neighbor at the nth level) a dimensionless coupling constant (positive) $qK(K=J/k_BT)$, T being the temperature and J the coupling constant. The dimensionless Hamiltonian, at this level, is given by

$$H_n=-qK_n\sum_{\langle ij\rangle}\delta(\sigma_i,\sigma_j);\quad \sigma_i=0,1,\ldots,q-1, \tag{1}$$

where the sum is over the nearest neighbors sites and δ is the Kronecker δ function. The partition function, at the nth level, can be formally calculated through the expression

$$Z_n=\sum_{\langle\sigma\rangle}\exp H_n=\sum_{\langle\sigma\rangle}\exp\left[-qK_n\sum_{\langle ij\rangle}\delta(\sigma_i,\sigma_j)\right]. \tag{2}$$

In order to break the global symmetry of the Potts model, we fixed (as boundary conditions) the spins at the roots of the fractal lattices (open sites in Fig. 1) at a specific state, say $\sigma=0$. Then, looking at a particular basic cell of the nth level hierarchical lattice [Fig. 2(a)], we see that if we perform a trace over all the spins of the lattice with the exception of those spins belonging to that basic cell, the partition function will be expressed as a trace over the remaining spins (those belonging to the basic cell) with the equivalent Hamiltonian H' given by [for the DHL shown in Fig. 2(a)]:

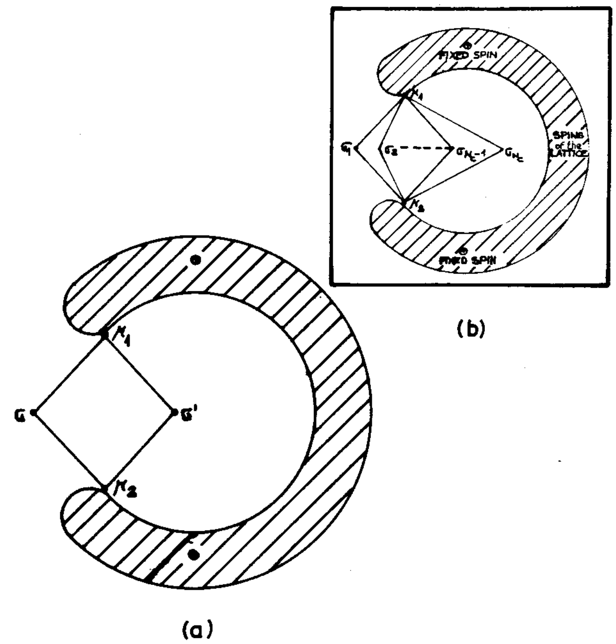

FIG. 2. Basic cell of (a) DHL; (b) generalized DHL.

$$\begin{aligned}H'=\theta&+qK[\delta(\sigma,\mu_1)+\delta(\sigma,\mu_2)]\\&+qK[\delta(\sigma',\mu_1)+\delta(\sigma',\mu_2)]+qK'\delta(\mu_1,\mu_2)\\&+qL\,\delta(\mu_1,0)\,\delta(\mu_2,0)+qH_1\delta(\mu_1,0)+qH_2\delta(\mu_2,0),\end{aligned} \tag{3}$$

where θ, K', L, H_1, and H_2 are unknown functions of K. The spins μ_1 and μ_2 connect the chosen basic cell with the rest of the lattice and σ and σ' are the internal spins of the basic cell (they are spins that appear at the nth level). The effective fields H_1 and H_2 and the coupling constants L and K' are due to boundary conditions at the roots to the connections between μ_1 and μ_2 through the rest of the lattice. With this formal equivalent Hamiltonian we can calculate the local magnetizations and the pair correlation functions of the spins μ_1 and μ_2 as functions of the parameters K', L, H_1, and H_2, and relate them with the local magnetizations and correlation functions involving the spins σ and σ' by eliminating the unknown parameters, obtaining so far the set of recursion relations.

III. RECURSION RELATIONS

We define the local magnetization of a given site spin variable σ by

$$m_i=\frac{q\langle\delta(\sigma_i,0)\rangle-1}{q-1} \tag{4}$$

and the special pair correlation function $\Delta_{\sigma_i\sigma_j}$ between sites spins σ_i and σ_j by

$$\Delta_{\sigma_i\sigma_j}\equiv q\langle\delta(\sigma_i,0)\,\delta(\sigma_j,0)\rangle-\langle\delta(\sigma_i,\sigma_j)\rangle, \tag{5}$$

where $\langle\;\rangle$ means thermodynamical average.

Now let us consider an n-level DHL and take a given connection within the basic cell of the latest hierarchy inside the lattice with internal site spin variable σ and end spin variables μ_1 and μ_2. The quantities defined by Eqs. (4) and (5) that are related to μ_1 and μ_2 can be obtained by evaluating the average quantities using the Hamiltonian given by Eq. (3). The result can be formally given by the following system:

$$\begin{aligned} \langle \delta(\mu_1,0)\rangle &= f_1(K;K',L,H_1,H_2),\\ \langle \delta(\mu_2,0)\rangle &= f_2(K;K',L,H_1,H_2),\\ \langle \delta(\mu_1,\mu_2)\rangle &= f_3(K;K',L,H_1,H_2),\\ \langle \delta(\mu_1,0)\,\delta(\mu_2,0)\rangle &= f_4(K;K',L,H_1,H_2). \end{aligned} \tag{6}$$

We can also calculate the averages involving the spins belonging to the latest level (the nth one), i.e., σ,

$$\begin{aligned} \langle \delta(\sigma,0)\rangle &= f(K;K',L,H_1,H_2),\\ \langle \delta(\sigma,\mu_i)\rangle &= g_i(K;K',L,H_1,H_2), \quad i=1,2. \end{aligned} \tag{7}$$

Inverting the system given by Eqs. (6), i.e., obtaining K', L, H_1, and H_2 as functions of the averages involving μ_1 and μ_2 and replacing these parameters in Eq. (7), we obtain recurrence relations for averages involving spins σ (latest level) as functions of averages involving spins of previous levels [one of the spins μ belongs to the $(n-1)$th level and the other one belongs to the jth level $(j=0,1,2,\ldots,n-2)$ depending on the chosen basic cell].

By performing these calculations we have succeeded to obtain a coupled set of recursion relations relating the local magnetization and the special correlation function for the site spin variables of the latest hierarchy with the ones of previous hierarchies. This is given by

$$\begin{aligned} m_\sigma &= C_1^{(n)}[m_{\mu_1}+m_{\mu_2}]+C_2^{(n)}\Delta_{\mu_1\mu_2},\\ \Delta_{\sigma\mu_i} &= C_3^{(n)}m_{\mu_i}+C_4^{(n)}\Delta_{\mu_1\mu_2} \quad (i=1,2),\\ \langle \delta(\sigma,\mu_i)\rangle &= C_3^{(n)}+C_4^{(n)}\langle \delta(\mu_1,\mu_2)\rangle \quad (i=1,2), \end{aligned} \tag{8}$$

where the expressions for the coefficients $C_i^{(n)}$ are shown in Appendix A. We can see that the first two equations of the set (8) are coupled and they are related to the third one, which depends only on its own previous levels. So, since we know the model's Hamiltonian at the zeroth level, $H_0=K_0\delta(\mu_u,\mu_d)=K_0$, where μ_u and μ_d are the Potts variables assigned to the roots "up" and "down," respectively (here they are assumed to be in the zero state), it is straightforward to calculate the averages m_{μ_u}, m_{μ_d}, Δ_{μ_u,μ_d}, and $\delta(\mu_u,\mu_d)$. Their values are (initial conditions)

$$\begin{aligned} m_{\mu_u} &= m_{\mu_d}=1 \quad (Z_0=e^{K_0}),\\ \Delta_{\mu_u,\mu_d} &= q-1,\\ \langle \delta(\mu_u,\mu_d)\rangle &= 1. \end{aligned} \tag{9}$$

The averages associated with the first level can be obtained using these initial conditions in the set of Eqs. (8) together with the values of constants $C_i^{(n)}$ for $n=1$. These constants are functions of K_n (i.e., K_1), which are straightforwardly related to the Potts transmissivity variable, introduced by Tsallis and Levy[13]:

$$t_n \equiv \frac{1-\exp(-qK_n)}{1+(q-1)\exp(-qK_n)}. \tag{10}$$

This variable is well defined between 0 and 1 and its hierarchical relation, for the usual DHL, is given by

$$t_{n-1}=\frac{2t_n^2+(q-2)t_n^4}{1+(q-1)t_n^4}. \tag{11}$$

This allows us to obtain the critical transmissivity and the critical temperature. With the set of Eqs. (8), (11) and also the initial values for m_{μ_u}, m_{μ_d}, Δ_{μ_u,μ_d}, and $\delta(\mu_u,\mu_d)$ given by Eqs. (9), we are able to calculate the local magnetization of each site of the lattice for any level as function of the temperature, for general q.

IV. GLOBAL MAGNETIZATION AND CRITICAL EXPONENTS

In order to obtain the global magnetization, i.e., the average magnetization "per site" at the nth level, we need to sum over the recurrence relations for the local magnetization up to that level. The thermodynamic limit is obtained taking $n\to\infty$. Adding up the local magnetizations of the spins at the nth level [see the left-hand side (LHS) of the first equation of (8)], we will have on the RHS to sum over sites of the terms (Δ's) which are defined over bonds. Analogously, when summing the second equation of (8) for all bonds present at the nth level, we have to sum carefully the terms m_{μ_i} and $\Delta_{\mu_1\mu_2}$ that appear on the RHS of this equation. Defining the global magnetization as the average magnetization per site at the nth (L_n) level

$$m_n \equiv \left(\frac{\Sigma_{s\in L_n} m_s}{N_s^n}\right), \tag{12}$$

where the sum is performed over all sites (N_s^n) of the lattice considered at the nth level (L_n), $N_S^n=2/3(4^n+2)$, $n=0,1,2\ldots$ and defining

$$\Delta_n \equiv \frac{\Sigma_{b\in L_n}\Delta_{\sigma\nu}}{N_b^n}, \tag{13}$$

where the sum is over all bonds of the lattice at the nth level, being $N_b^n=4^n$ the total number of these bonds. We finally obtain the recurrence relations connecting these "macroscopic" quantities for an n-level DHL with their values at two previous levels. These equations can be written (for the DHL) as

TABLE I. Values takes from Bleher and Zalys (Ref. 38) at first line and ours at the second one, where N_c is the coordination number of basic cell roots.

N_c	2	4	5	10	50	100
β_{Bleher}	0.161 743	0.463 242	0.546 752	0.745 762	0.943 172	0.970 517
β_{exact}	0.161 743	0.463 241	0.546 752	0.745 754	0.943 794	0.971 526

$$m_n = \frac{N_s^{(n-1)}}{N_s^{(n)}}\left[4C_1^{(n)}+1+2\frac{C_1^{(n)}}{C_1^{(n-1)}}\right]m_{n-1} - \frac{N_s^{(n-2)}}{N_s^{(n)}}\left[4C_1^{(n)}+2\frac{C_1^{(n)}}{C_1^{(n-1)}}\right]m_{n-2} + 2\frac{N_b^{(n-1)}}{N_s^{(n)}}C_2^{(n)}\Delta_{n-1} - 4\frac{N_b^{(n-2)}}{N_s^{(n)}}C_2^{(n)}\frac{C_1^{(n)}}{C_1^{(n-1)}}\Delta_{n-2},$$

$$\Delta_n = 4\frac{N_s^{(n-1)}}{N_b^{(n)}}C_3^{(n)}m_{n-1} - 4\frac{N_s^{(n-2)}}{N_b^{(n)}}C_3^{(n)}m_{n-2} + \left[C_4^{(n)}+2\frac{C_3^{(n)}}{C_3^{(n-1)}}\frac{N_b^{(n-1)}}{N_b^{(n)}}\right]\Delta_{n-1} - 8C_4^{(n-1)}\frac{C_3^{(n)}}{C_3^{(n-1)}}\frac{N_b^{(n-2)}}{N_b^{(n)}}\Delta_{n-2}. \quad (14)$$

Since we know the values of m_0, m_1, Δ_0, and Δ_1, it is possible to calculate the global magnetization at any level. The initial values can be obtained with the aid of the first two equations of (8), using the set of initial conditions (9). In Fig. 3 we show the exact average magnetization per site versus temperature for $q=2$, $q=4$, and $q=10$ at the 100th level. We calculate also these magnetizations for other values of n, but the curves are virtually the same if n is large enough.

It is also possible to calculate exactly the critical exponent β among others. We notice that m_n and Δ_n have the same asymptotic behavior near T_c, i.e., $m_n \sim \lambda_1(\epsilon_n)^\beta$ and $\Delta_n \sim \lambda_2(\epsilon_n)^\beta$ where λ_1 and λ_2 are amplitudes, $\epsilon_n \equiv T_c - T_n/T_c$ and therefore the following equation for r_c^β can be obtained:

$$r_c^{2\beta} - \left[C_1^{(c)}+C_4^{(c)}+\frac{1}{2}\right]r_c^\beta + C_4^{(c)}\left[C_1^{(c)}+\frac{1}{2}\right] - \frac{q-1}{2}C_2^{(c)}C_3^{(c)} = 0, \quad (15)$$

where

$$r_c \equiv \left.\frac{\partial t_{n-1}}{\partial t_n}\right|_{t_c} \quad (16)$$

and $C_i^{(c)}$ ($i=1,2,3,4$) are the coefficients listed in Appendix A, calculated at the critical temperature $T_c(t_c)$. The value of r_c for DHL is

$$r_c = 2 + \frac{2(q-2)t_c^2}{2+(q-2)t_c^2} + \frac{4(q-1)t_c^4}{1+(q-2)t_c^4}. \quad (17)$$

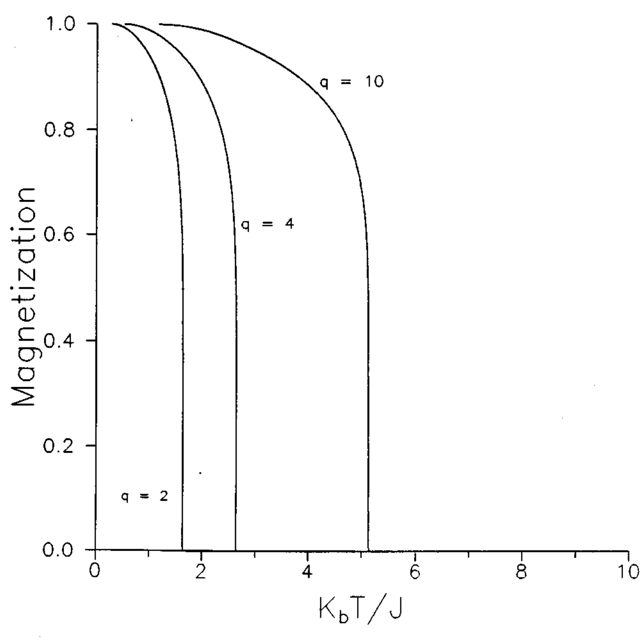

FIG. 3. DHL's magnetization versus temperature with $n=100$ and $q=2$, 4, and 10.

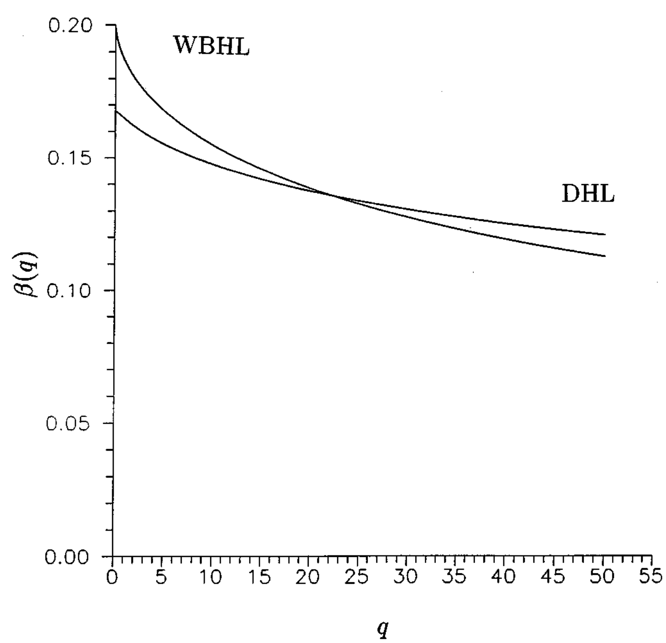

FIG. 4. Critical exponents β versus q for DHL and WBHL.

So β is easily obtained from Eq. (15) as

$$\beta=\frac{\ln[\text{solution of Eq. (15)}]}{\ln(r_c)}.$$

These exact calculations were also performed for a whole family of DHL [see Fig. 2(b) where we vary the number of connections N_c, i.e., the dimensionality of the lattice] and for the "Wheatstone Bridge hierarchical lattice" (WBHL) [see Fig. 1(b)]. In Fig. 4 we show the graph β versus q for DHL and for WBHL. Our exact result for $\beta(q)$ of WBHL is *unlike* that obtained by Caride and Tsallis[39] with exception of the point $q=2$ ($\beta_{\text{WBHL}}=0.180$). For the whole family of DHL the value of $\beta(2)$ was exactly calculated by Bleher and Zalys[38] and we show in Table I their results and ours for some members of that family for $q=2$. The discrepancy appears only due to their numerical imprecision at higher dimensions. Our results for any other value of q are new, as far as we know. The expression for the local and global magnetization recurrence relations for the WBHL and for the family of DHL are shown in Appendixes B and C, respectively.

V. INTERNAL ENERGY AND SPECIFIC HEAT

The dimensionless internal energy at zero magnetic field, the specific heat, and its critical exponent α can be exactly calculated with the aid of the third equation of the set (8). The dimensionless energy per bond is defined as

$$\frac{E}{N_b^{(n)}}=-\langle\,\delta(\mu,\sigma)\rangle,$$

recognizing that the average $\langle-\delta(\mu,\sigma)\rangle$ of two nearest neighbor spins μ and σ at nth level is the dimensionless per bond average energy (e_n),

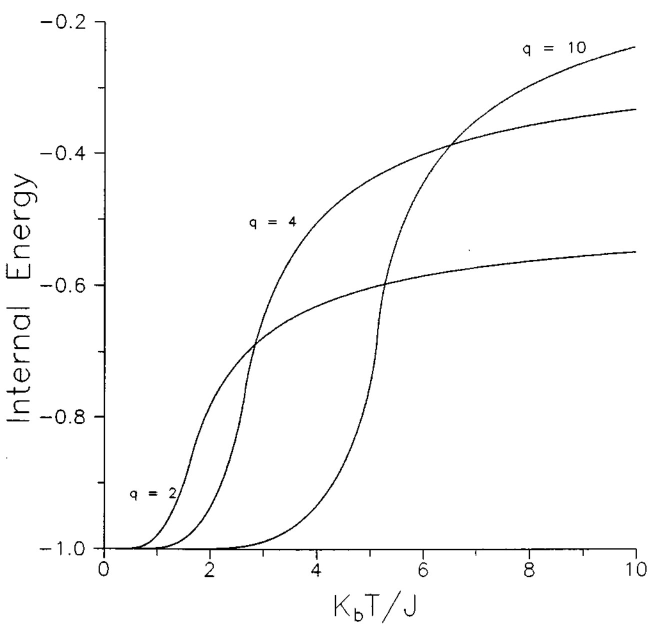

FIG. 5. Dimensionless internal energy versus temperature for DHL with $n=100$ and $q=2$, 4, and 10.

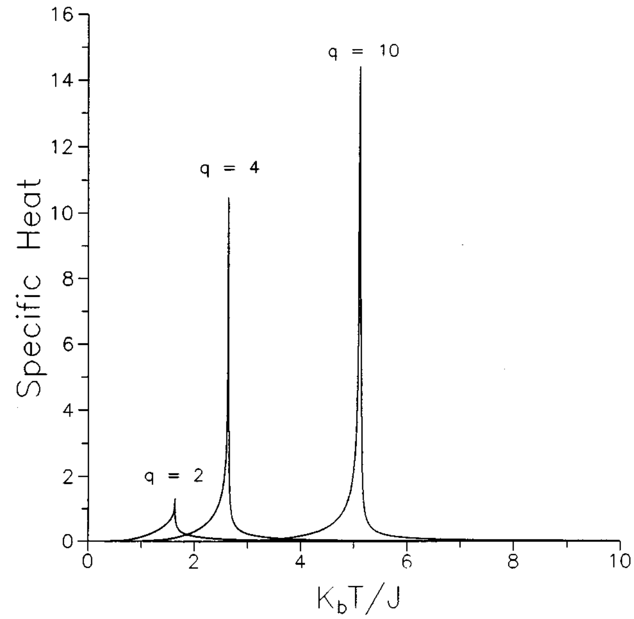

FIG. 6. Dimensionless specific heat versus temperature for DHL with $n=100$ and $q=2$, 4, and 10.

$$e_n\equiv\frac{E}{J_n}=-\frac{\langle H_n\rangle}{qJ_nN_b^{(n)}},$$

where H_n is the Hamiltonian. So the dimensionless internal energy can be rewritten as $-e_n$, where e_n can be expressed as

$$e_n=C_3^{(n)}+C_4^{(n)}e_{n-1}. \tag{18}$$

Therefore, starting with $e_0=\langle\delta(0,0)\rangle=1$, it is possible to obtain the bond average energy at any level. For n large enough the curves of the dimensionless internal energy versus temperature have the same form shown in Fig. 5 for $n=100$ and $q=2$, 4, and 10 for the DHL. The exact adimensional specific heat

$$-\frac{1}{qk_BN_b^{(n)}}\frac{\partial e_n}{\partial T}$$

can be obtained either by deriving Eq. (18) with respect to T or by numerical derivation of the curve shown in Fig. 5 and the results for $q=2$, 4, and 10 and also $n=100$ are shown in Fig. 6 for the DHL. The critical exponent α is obtained by assuming in the neighborhood of $T_c(t_c)$ that e_n can be written as $e_n=e_c+\lambda(\epsilon_n)^{\sigma}$, where $e_c=C_3^{(c)}/(1-C_4^{(c)})$ and $\epsilon_n\equiv(T_c-T_n)/T_c$. The replacement of this expression into Eq. (18) leads to

$$\sigma=\frac{\ln\left(\frac{1}{C_4^{(c)}}\right)}{\ln(r_c)}. \tag{19}$$

Since the specific heat is related with the derivative of the internal energy with respect to the temperature, we have that $\alpha=1-\sigma$.

Table II shows some values of α (for $q=1,2,4,10$) for some members of the family DHL and Table III does the

TABLE II. Critical exponents α for the DHL.

N_c	**q**=1	**q**=2	**q**=4	**q**=10
2	−1.270 559	−0.676 532	−0.248 321	0.143 778
4	−1.682 235	−1.194 450	−0.750 510	−0.281 103
10	−2.661 275	−2.363 985	−1.972 557	−1.396 827

same for the WBHL. They allow us to test the hyperscaling relations in these fractal lattices. This is possible because ν can be obtained in an independent way by the relation

$$\nu = \frac{\ln(2)}{\ln(r_c)}.$$

In all these lattices the relation[14]

$$d_f \nu = 2 - \alpha, \tag{20}$$

where d_f is the fractal dimension of the lattice, is completely satisfied for all values of q, suggesting strongly that this is the correct way the hyperscale law works for fractal lattices, as it has been conjectured before.[14]

VI. MULTIFRACTAL PROFILE

Equations (8) allow us to calculate the average magnetization of the Potts spins localized at *each* site of the lattice (*local magnetization*). One way to do this is to choose one of the shortest paths connecting the two roots of the HL and looking the local average magnetization of the spins belonging to this path. One example is shown in Fig. 1 where the chosen DHL shortest path is illustrated by the broken line. All of the shortest paths are, obviously, equivalent. These DHL shortest paths have 2^n+1 sites at the nth level. We can locate these sites in a line using a normalized coordinate

$$x_i^{(n)} \equiv \frac{i}{2^n},$$

where $i = 0,\ldots,2^n$. In Fig. 7 we present, at the critical temperature, the local (site) magnetization (in the ordinate axis) as function of the position of the site in the paths (represented by the coordinate $x_i^{(n)}$ in the abscissa), for some values of q and N_c. Although it is not clear in this figure, we observed that the total average magnetization per site of the lattice is zero in the thermodynamic limit, i.e., when n goes to infinite (at $T = T_c$). These figures appear to be self-similar and we will show that it is possible to define on then a physical measure, based on the local magnetization, that is multifractal. To do this we divide the path into boxes and define the measure of each box as the fractional magnetization of the box (i.e., the sum of the magnetization of all sites belonging to the box divided by the sum of the magnetization of all sites of the paths). We normally use the length δ of the box as

TABLE III. Critical exponents α for the WBHL.

q=1	**q**=2	**q**=4	**q**=10
−1.314 958	−0.667 034	−0.202 032	0.219 558

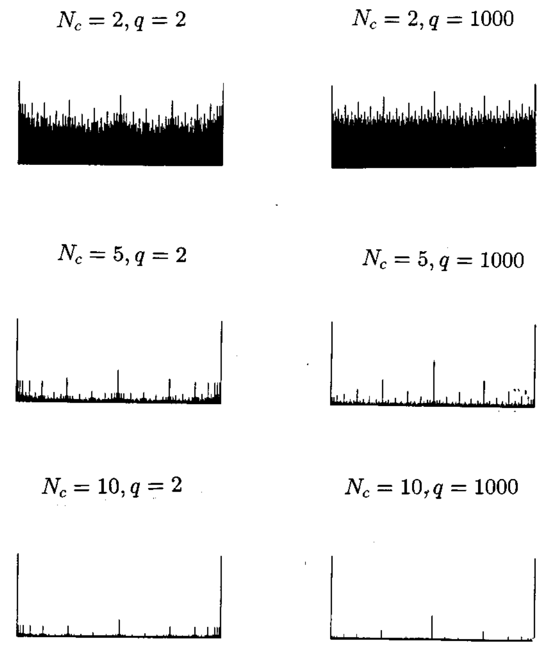

FIG. 7. Local magnetization of the spins belonging to a path of generalized DHL with $n = 10$, for some values of N_c and q. The ordinate is the local magnetization and the abscissa is the relative position to the root, starting at zero.

$$\delta_n = 2^{-n}$$

if we are considering the nth generation of the lattice. Obviously, the sum over all boxes is equal to one, i.e., it is a probabilistic measure (more generally this measure is defined on the σ algebra generated by the partition of the interval). As the sum of the magnetization of all sites of the path is equal to the average magnetization of the lattice times the number of sites of the path (equal to 2^n+1 at the nth generation) and calling m_i the sum of the magnetization of the sites belonging to the ith box, we can write the measure (μ_i) of this box as

$$\mu_i^{(n)} \equiv \frac{m_i}{N_{\text{path}} m_n} \sim \delta_n^{\alpha_H},$$

where m_n is the average magnetization of the lattice at the nth generation and α_H is the corresponding Hölder exponent. Therefore we can associate to each box a value of α_H and collect the boxes with the same value of α_H (within a small interval $\Delta\alpha_H$) and then obtain the $f(\alpha_H)$ function. Recently, an efficient algorithm has been developed by Chhabra and Jensen[40,41] which allows us to obtain a excellent $f(\alpha_H)$ function. We used this algorithm to obtain for several values of q, at T_c, the multifractal function $f(\alpha_H)$. The plots of the $f(\alpha_H)$ functions are shown in Fig. 8. The maximum and minimum α_H values for these functions can be exactly calculated. The measures with largest α_H have the lowest values and inversely, the measures with the lowest α_H have the largest values. As the measures are defined on the magnetizations, the existence of a continuum of values of α_H implies the existence of a continuum of values of β, i.e., there are sets of sites whose magnetization scales with different exponents (a connection between α_H and β was established in

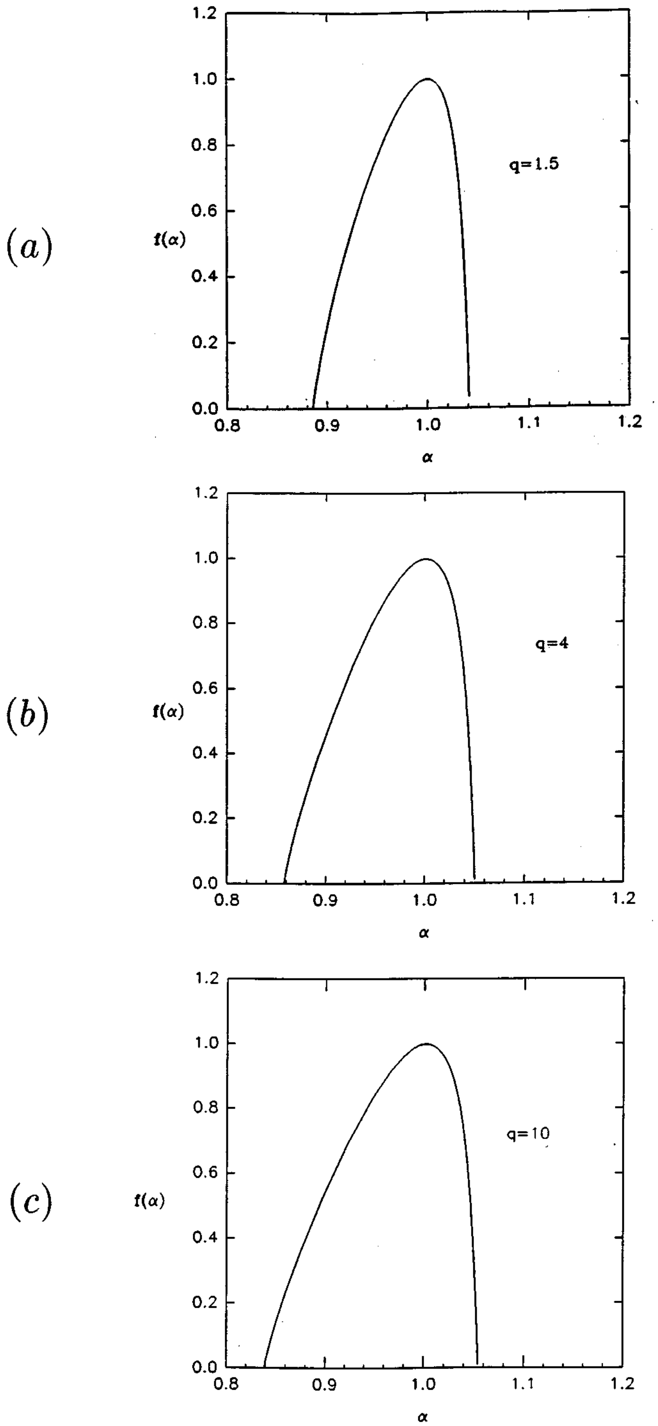

FIG. 8. $f(\alpha_H)$ multifractal function for the usual DHL ($N_c=2$) for (a) $q=1.5$; (b) $q=4$; (c) $q=10$.

Ref. 31). Therefore, the boxes containing the measures with the smallest value have a local magnetization that behaves (n large) as

$$M_{\min}^{(n)} \sim (\epsilon_n)^{\beta_{\max}},$$

i.e., with the maximum possible value of $\beta[\epsilon_n \equiv (t_n - t_c)/t_c]$. We can write the total magnetization of the profile as

$$\lambda_1 2^n (\epsilon_n)^{\beta},$$

where λ_1 is a constant and β is the magnetization critical exponent of the whole lattice. Therefore the smallest measures can be written as

$$\mu_{\min}^{(n)} \sim \delta_n^{\alpha_{\max}} \sim \frac{(\epsilon_n)^{\beta_{\max}}}{2^n (\epsilon_n)^{\beta}},$$

which implies that

$$\frac{\mu_{\min}^{(n+1)}}{\mu_{\min}^{(n)}} = 2^{-1} \left(\frac{\epsilon_{n+1}}{\epsilon_n} \right)^{\beta_{\max} - \beta}.$$

As

$$\frac{\epsilon_{n+1}}{\epsilon_n} \cong \frac{dt_{n+1}}{dt_n} \bigg|_{t_c} = 2^{-1/\nu},$$

and as $\mu_{\min}^{(0)} \equiv 1$ (a constant value), we can write

$$\mu_{\min}^{(n)} \sim [2^{-1/\nu(\beta_{\max} - \beta) - 1}]^n \sim \delta_n^{\alpha_{H^{\max}}}.$$

Remembering that $\delta_n \sim 2^{-n}$ we obtain a relationship[31] among the critical exponents β, ν, the exponent $\beta_{\max}$, and the Hölder exponent α_H

$$\alpha_{H^{\max}} = 1 + (\beta_{\max} - \beta) \frac{1}{\nu}. \tag{21}$$

The boxes containing the biggest measures are associated with the local magnetizations that remain constant when $n \to \infty$. So, we can write

$$\mu_{\max}^{(n)} \sim \frac{\text{const}}{2^n (\epsilon_n)^{\beta}}.$$

Analogous to the calculation for $\mu_{\min}^{(n)}$, we obtain

$$\mu_{\max}^{(n)} \sim (2^{\beta/\nu - 1})^n \sim \delta_n^{\alpha_{H^{\min}}},$$

leading to the exact relation for the minimum values of the Hölder exponent:

$$\alpha_{H^{\min}} = 1 - \frac{\beta}{\nu}. \tag{22}$$

In Fig. 9(a) we plot $\alpha_{H^{\max}}$, $\alpha_{H^{\min}}$, ν, $\beta_{\max}$, and β as functions of q for the usual DHL ($N_c=2$). Figures 9(b) and 9(c) show the same behavior for other lattices, say $N_c=4$ and $N_c=10$. We exhibit in Fig. 8 the $f(\alpha_H)$ multifractal function for the usual DHL, for $q=1.5$, 4, and 10. We can also see that the higher the value for q the wider the $f(\alpha_H)$ function. Evidence was shown by Morgado *et al.*[31,32] that the maximum of the $f(\alpha_H)$ function reaches 1 (the dimension of the support set) when we increase the number of hierarchies used.

VII. CONCLUSIONS

We showed exact results obtained by a method that allow us to calculate recurrence relations for the local and global magnetization for the zero field Potts ferromagnetic model. Using these relations we also obtained exactly the critical exponents α, β, and ν. Our results for the critical exponents agree, for $q=2$ (Ising model) with previous results for the DHL obtained by Ref. 32 and for other lattices.[38] The possi-

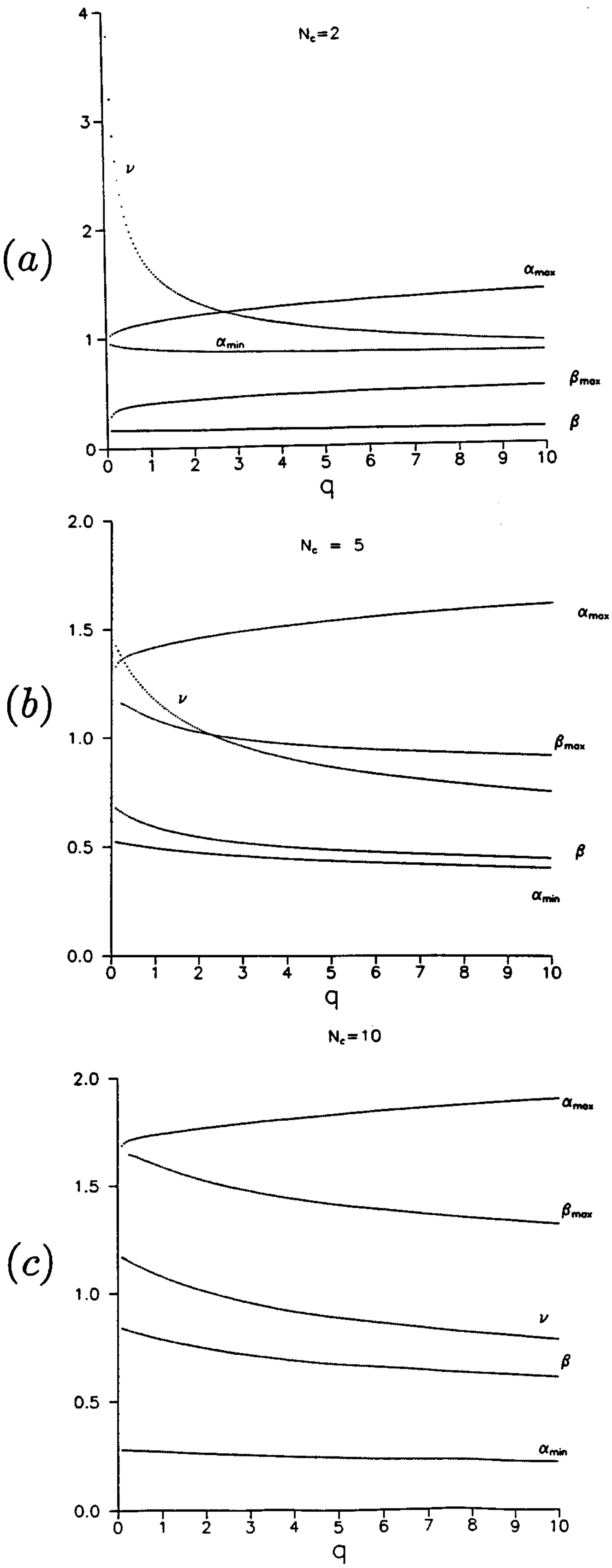

FIG. 9. Exponents $\alpha_{H^{\max}}$, $\alpha_{H^{\min}}$, ν, $\beta_{\max}$, and β as function of q for (a) DHL (N_c=2); (b) generalized DHL with N_c=5; (c) generalized DHL with N_c=10.

bility of exactly calculating the critical exponents makes it possible to verify that the hyperscaling relation is successfully obeyed by the lattices we used (for a wide variety of fractal dimensions and q values) as expected. It was also possible to exactly calculate thermodynamic functions as magnetizations, specific heat, etc. The $f(\alpha_H)$ multifractal function showed that the measure defined as the normalized local magnetization of a profile has a multifractal distribution at the critical temperature, as observed for the Ising model by Morgado *et al.*[31,32] Work to verify the validity of the other scaling laws is now in progress.

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APPENDIX A: COEFFICIENTS OF THE DHL

Defining $A_n = e^{qKn}$, the coefficients $C_1^{(n)}$, $C_2^{(n)}$, $C_3^{(n)}$, and $C_4^{(n)}$, for the DHL, are respectively

$$C_1^{(n)} = \frac{A_n - 1}{2A_n + q - 2},$$

$$C_2^{(n)} = \left(\frac{q}{q-1}\right) \frac{(A_n-1)^2}{(A_n^2+q-1)(2A_n+q-2)},$$

$$C_3^{(n)} = (q-1) \frac{A_n}{(2A_n+q-2)},$$

$$C_4^{(n)} = \frac{A_n(A_n-1)(A_n+q-1)}{(A_n^2+q-1)(2A_n+q-2)}.$$

APPENDIX B: RECURRENCE EQUATIONS FOR THE WBHL

The recurrence relations for the "site" magnetization are

$$m_\sigma = C_{W,1}^{(n)}[m_{\mu_1} + m_{\mu_2}] + C_{W,2}^{(n)}\Delta_{\mu_1\mu_2},$$

$$\Delta_{\sigma\mu_i} = C_{W,3}^{(n)} m_{\mu_i} + C_{W,4}^{(n)}\Delta_{\mu_1\mu_2} \quad (i=1,2),$$

$$\Delta_{\sigma\sigma'} = C_{W,5}^{(n)}(m_{\mu_1} + m_{\mu_2}) + C_{W,6}^{(n)}\Delta_{\mu_1\mu_2},$$

and those for the "site" average magnetization are

$$m_n=\frac{N_s^{(n-1)}}{N_s^{(n)}}\left[6C_{W,1}^{(n)}+1+2\,\frac{C_{W,n}^{(n)}}{C_{W,1}^{(n-1)}}\right]m_{n-1}-\frac{N_s^{(n-2)}}{N_s^{(n)}}\left[6C_{W,n}^{(n1)}+2\,\frac{C_{W,1}^{(n)}}{C_{W,1}^{(n-1)}}\right]m_{n-2}+2\,\frac{N_b^{(n-1)}}{N_s^{(n)}}\,C_{W,2}^{(n)}\Delta_{n-1}$$
$$-4\,\frac{N_b^{(n-2)}}{N_s^{(n)}}\,C_{W,2}^{(n2)}\,\frac{C_{W,1}^{(n)}}{C_{W,1}^{(n-1)}}\,\Delta_{n-2},$$

$$\Delta_n=3\,\frac{N_s^{(n-1)}}{N_b^{(n)}}\,(2C_{W,3}^{(n)}+C_{W,5}^{(n)})m_{n-1}-3\,\frac{N_s^{(n-2)}}{N_b^{(n)}}\,(2C_{W,3}^{(n)}+C_{W,5}^{(n)})m_{n-2}+\frac{N_b^{(n-1)}}{N_b^{(n)}}\left[4C_{W,4}^{(n)}+C_{W,6}^{(n)}+2\left(\frac{2C_{W,3}^{(n)}+C_{W,5}^{(n)}}{2C_{W,3}^{(n-1)}+C_{W,5}^{(n-1)}}\right)\right]\Delta_{n-1}$$
$$-\frac{N_b^{(n-2)}}{N_b^{(n)}}\left[2(4C_{W,4}^{(n-1)}+C_{W,6}^{(n-1)})\left(\frac{2C_{W,3}^{(n)}+C_{W,5}^{(n)}}{2C_{W,3}^{(n-1)}+C_{W,5}^{(n-1)}}\right)\right]\Delta_{n-2},$$

where the coefficients above may be written as

$$C_{W,1}^{(n)}=\frac{A_n(A_n^2+A_n+q-2)-(3A_n+q-3)}{2A_n^3+2A_n^2+(q-2)(5A_n+q-3)},$$

$$C_{W,2}^{(n)}=\left(\frac{q}{q-1}\right)\,\frac{3A_n+q-3}{2A_n^3+2A_n^2+(q-2)(5A_n+q-3)}-\frac{A_n^2+A_n+q-2}{A_n^5+(q-1)(2A_n^2+A_n+q-2)},$$

$$C_{W,3}^{(n)}=(q-1)\,\frac{A_n(A_n^2+A_n+q-2)}{2A_n^3+2A_n^2+(q-2)(5A_n+q-3)},$$

$$C_{W,4}^{(n)}=\frac{A_n^5+A_n^2(q-2)}{A_n^5+(q-1)(2A_n^2+A_n+q-2)}-\frac{A_n(A_n^2+A_n+q-2)}{2A_n^3+2A_n^2+(q-2)(5A_n+q-3)},$$

$$C_{W,5}^{(n)}=(q-1)\,\frac{A_n^3-A_n}{2A_n^3+2A_n^2+(q-2)(5A_n+q-3)},$$

$$C_{W,6}^{(n)}=\frac{A_n^5-A_n}{A_n^5+(q-1)(2A_n^2+A_n+q-2)}+\frac{2A_n-2A_n^3}{2A_n^3+2A_n^2+(q-2)(5A_n+q-3)},$$

where $A_n=e^{qKn}$.

APPENDIX C: RECURRENCE RELATIONS FOR THE GENERALIZED DIAMOND HIERARCHICAL LATTICE

The local magnetization recurrence relations are the same for this whole family as those for the usual DHL and the average magnetizations (per hierarchy) are

$$m_n=\frac{N_s^{(n-1)}}{N_s^{(n)}}\left[2N_cC_1^{(n)}+1+N_c\,\frac{C_1^{(n)}}{C_1^{(n-1)}}\right]m_{n-1}-\frac{N_s^{(n-2)}}{N_s^{(n)}}\left[2N_cC_1^{(n)}+N_c\,\frac{C_1^{(n)}}{C_1^{(n-1)}}\right]m_{n-2}+N_c\,\frac{N_b^{(n-1)}}{N_s^{(n)}}\,C_2^{(n)}\Delta_{n-1}$$
$$-(N_c)^2\,\frac{N_b^{(n-2)}}{N_s^{(n)}}\,C_2^{(n)}\,\frac{C_1^{(n)}}{C_1^{(n-1)}}\,\Delta_{n-2},$$

$$\Delta_n=2N_c\,\frac{N_s^{(n-1)}}{N_b^{(n)}}\,C_3^{(n)}m_{n-1}-2N_c\,\frac{N_s^{(n-2)}}{N_b^{(n)}}\,C_3^{(n)}m_{n-2}+\left[C_4^{(n)}+N_c\,\frac{C_3^{(n)}}{C_3^{(n-1)}}\,\frac{N_b^{(n-1)}}{N_b^{(n)}}\right]\Delta_{n-1}-2(N_c)^2C_4^{(n-1)}\,\frac{C_3^{(n)}}{C_3^{(n-1)}}\,\frac{N_b^{(n-2)}}{N_b^{(n)}}\,\Delta_{n-2}.$$

where N_c is the number of parallel paths between the two roots and $N_s^{(n)}=[1/(2N_c-1)](2^nN_c^{n+1}+3N_c-2)$ and $N_b^{(n)}=(2N_c)^n$.

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