Cylindrical Korteweg–de Vries solitons in the vortex dynamics of an ultraclean type-II superconductor

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A theory for weakly nonlinear and dispersive wave propagation in an Abrikosov vortex lattice in a type-II superconductor of cylindrical symmetry is presented. A continuum treatment of the London equation with vortex term is used, allowing nonlocal lattice elasticity. Vortex inertia is included, but pinning is ignored. A dynamical regime is derived where the cylindrical Korteweg–de Vries (CKdV) equation governs the evolution of the first-order field corrections. Fundamental properties of the CKdV equation are briefly recalled and a prototypical soliton solution is given and discussed. Dynamical system analogies are mentioned.

Recently the dynamics of Abrikosov vortices[1] in an ultraclean type-II superconductor was examined in a certain weakly dispersive and nonlinear regime.[2] With a single spatial rectangular coordinate dependence for the field variables, the Korteweg–de Vries (KdV) equation was derived. This paper investigates a type-II superconductor[3] of cylindrical symmetry in the mixed state. It is found that the cylindrical KdV (CKdV) equation[4]

$$\frac{\partial^3 v}{\partial \xi^3}+3v\,\frac{\partial v}{\partial \xi}+2\,\frac{\partial v}{\partial \eta}+\frac{v}{\eta}=0 \tag{1}$$

governs the evolution of the first-order field corrections. The CKdV equation is well known to be completely integrable,[5–9] possessing N-soliton solutions.[8] Before briefly reviewing some of the properties and physical occurrences of this nonautonomous equation, the nature of an ultraclean superconductor is described.

In an ultraclean material, the quasiparticle mean free path l exceeds the coherence length ξ times the ratio of Fermi energy ϵ_F to the magnitude of the order parameter $|\Delta|$. In these superconductors vortex drag is negligible and pinning can be very small; the Hall force dominates the dynamics.[10] In the high-T_c superconductors, the ratio $\epsilon_F/|\Delta(0)|\sim 1/20$ is small, making it rather easy to achieve the ultraclean regime. Another way of characterizing an ultraclean material is that $\omega_c\tau\gg 1$, where $\hbar\omega_c$ is the low-level energy spacing for bound vortex core states and τ is the lifetime of quasiparticles in the core. In at least one high-T_c superconductor, $YBa_2Cu_3O_{7-\delta}$, the quasiparticle lifetime can be so long that $\omega_c\tau$ is estimated as ~ 14 for temperatures $T<15$ K.[10] Hall angle evidence strongly indicates the ultraclean regime for this superconductor.[10]

This paper concentrates on the limit of zero Hall force. When the Hall coefficient α_H is nonzero, the corresponding derivation is considerably more complex, requiring results through the third order of perturbation theory, where the nonlinear Schrödinger equation appears.[11] Therefore in either case a soliton equation is obtained. This paper presents the more tractable situation with $\alpha_H=0$, giving an exact result for the coupled nonlinear electrodynamic equations. It is outside the scope to go into the details of how the dispersion relation characterizes the rf response functions.[12]

The CKdV equation seems to have been first written by Maxon and Viecelli[4] in the context of ion acoustic waves in a two-component plasma. The equation was derived for long-wavelength, small-amplitude waves, which is usual. Cylindrical solitons of the CKdV type have been realized in double-plasma and other plasma experiments, and the agreement with theory is generally good.[13,4] For a review of the CKdV equation in plasma physics, through the early 1980s, see Ref. 14. The CKdV equation was investigated by Miles[15] for the classical water-wave problem with an incompressible irrotational fluid bounded above by a free surface and below by a rigid horizontal surface.

The inverse scattering transformation (IST) seems to have been first applied to the CKdV equation by Calogero and Degasperis.[5] They also found an infinite number of conservation laws by this technique.[6] Bäcklund transformations have been found for the equation,[7] together with more straightforward means of deriving the conservation laws.[16,14] Similarity solutions have been considered by many authors,[17] and, consistent with the other properties, the CKdV equation passes the Painlevé test.[9]

The CKdV equation is mathematically important as a nonautonomous generalization of the KdV equation which is itself completely integrable. This conclusion can be verified by Painlevé analysis,[9] the singularity exponent being $\alpha=-2$ and the resonances $r=-1,4,6$, as for the KdV equation. In this regard the work of Grimshaw[9] should be noted. The Painlevé-property compatibility constraints for a certain variable-coefficient KdV equation appeared earlier in his approach of finding an explicit, invertible mapping to the KdV equation.[9]

I consider an isotropic, isothermal type-II superconductor at or near absolute zero. In this instance a normal current density contribution is ignored; the total current density **J** is the supercurrent density. In this study the displacement current density is neglected for simplicity; frequencies well below the superconducting gap frequency are assumed. Within mesoscopic London theory a continuum description is employed using a vortex areal density $n(\mathbf{x},t)$.[18] Nonlocal vortex interaction is accounted for, and in fact is critical to the wave propagation of interest.

With these assumptions, the three basic vector equations are vortex continuity, a vortex equation of motion, and the

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London equation. The conservation of flux lines, equivalent to Faraday's law, is written as[19]

$$\frac{\partial \mathbf{B}_v}{\partial t} = -\boldsymbol{\nabla}\times(\mathbf{B}_v\times\mathbf{v}), \tag{2}$$

where $\mathbf{B}_v$, the local vortex-generated magnetic induction, is $n\phi_0\hat{\boldsymbol{\alpha}}$ and ϕ_0 is the flux quantum, $\hat{\boldsymbol{\alpha}}$ the local vortex direction.

The vortex equation of motion

$$\mu\,\frac{d\mathbf{v}}{dt} + \alpha_H\mathbf{v}\times\hat{\boldsymbol{\alpha}} = \phi_0\mathbf{J}\times\hat{\boldsymbol{\alpha}}, \tag{3}$$

where $\mathbf{v}$ is the vortex velocity and α_H the Hall coefficient, ignores pinning and drag forces. A mass μ per unit length of vortex[20] has been assumed. [Recall that $\mu=\mu(T)$ is temperature dependent, vanishing at the transition temperature.] This regime represents an ultraclean superconductor where the inertia effect could predominate. It may well be very close to those considered in vortex tunneling.[21] In addition, a very recent microscopic analysis of the Hall anomaly has found a large vortex mass coming from the core, specifically in the ultraclean limit.[22] The detection of CKdV solitons could provide a means of studying the vortex mass per unit length since the acoustic soliton speed varies as the square root of the ratio of the external magnetic induction to μ.[2] As is typical for a soliton, the amplitude, speed, and width are related; soliton measurements on a variety of materials might be able to discriminate between different vortex-mass mechanisms.[2]

As a first approximation, the Hall force is also neglected here, i.e., I consider simply the balance of inertia and the Lorentz force. The inclusion of the vector Hall term significantly alters the dispersion relation of the linearized problem, and is considered elsewhere.[11] A small viscous force may be included at the end of the treatment by the use of further perturbation theory.

The London equation may be written in the form

$$\mathbf{B}_v = \mathbf{B} - \lambda_L^2\nabla^2\mathbf{B}, \tag{4}$$

where λ_L is the London penetration depth. In the Meissner state the density n is absent and here the normal fluid or quasiparticle component does not contribute. Equation (4) takes into account nonlocal vortex interaction, over the characteristic distance λ_L. I ignore magnetic-field nonlinearity in the penetration depth, which is well justified for a wide range of field for high-T_c superconductors owing to their very large upper critical fields. The London equation is linear in this approximation.

I assume a type-II superconductor with cylindrical symmetry with a static external magnetic field along $\hat{\mathbf{z}}$, apply Ampère's law, and use the above assumptions. The magnitude of the static applied induction B_0 is assumed to satisfy $B_0/\mu_0 \gtrsim 2H_{c1}$, where H_{c1} is the lower critical field. Employing cylindrical coordinates, I let $B=B_z$, $v=v_\rho$, and $J=J_\theta$ depend only on the radial ρ (spatial) coordinate, and write Eqs. (2)–(4) as

$$\frac{\partial n}{\partial t} + \frac{1}{\rho}\frac{\partial(\rho n v)}{\partial\rho} = 0, \tag{5a}$$

$$\mu\,\frac{dv}{dt} = -\frac{\phi_0}{\mu_0}\frac{\partial B}{\partial\rho}, \tag{5b}$$

$$n\phi_0 = B - \frac{\lambda_L^2}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial B}{\partial\rho}\right). \tag{5c}$$

I perform the scaling $t'=\omega_0 t$, $\rho'=\rho/\lambda_L$, $B'=B/B_0$, $n'=n/n_0$, $v'=v/\omega_0\lambda_L$, where $\omega_0=\sqrt{\phi_0B_0/\mu_0\mu}/\lambda_L$ and $n_0=B_0/\phi_0$. For a flux density of $B_0=1$ T, $\lambda(0)\simeq 1500$ Å, and $\mu(0)\approx 10^8 m_e$/cm, the characteristic time $1/\omega_0\simeq 4\times10^{-13}$ s. These values are typical for YBCO at low temperature.

For notational simplicity, I then drop the primes to write Eqs. (5) as

$$\frac{\partial n}{\partial t} + \frac{1}{\rho}\frac{\partial(\rho n v)}{\partial\rho} = 0, \tag{6a}$$

$$\frac{dv}{dt} = -\frac{\partial B}{\partial\rho}, \tag{6b}$$

$$n = B - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial B}{\partial\rho}\right), \tag{6c}$$

where $d/dt=\partial/\partial t + v\,\partial/\partial\rho$ is the convective derivative. Equations (6) can easily be combined to yield the fourth-order equation

$$\frac{d^2 v}{\partial t\,dt} = \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial(\rho n v)}{\partial\rho} + \frac{\partial^2}{\partial t\,\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{dv}{dt}\right). \tag{7}$$

This is an exact equation, where B has been eliminated.

The dispersion relation for the linear propagation problem can be found by perturbing Eq. (7) or the system (6) about unit (scaled) vortex density and magnetic induction and zero vortex velocity. Let

$$n(\rho,t) = 1 + \delta n J_0(k\rho)e^{-i\omega t}, \tag{8a}$$

$$B(\rho,t) = 1 + \delta B J_0(k\rho)e^{-i\omega t}, \tag{8b}$$

and

$$v(\rho,t) = \delta v J_1(k\rho)e^{-i\omega t}, \tag{8c}$$

where J_ν is the Bessel function of order ν. Using the ordinary differential equation satisfied by J_0 and the relation $(d/dz)zJ_1(z)=zJ_0(z)$, the dispersion relation is given by

$$\omega^2(k) = k^2(1+k^2)^{-1}. \tag{9}$$

Thus for small k the dispersion relation is cubic, $\omega(k)\simeq k-(1/2)k^3$, and indicates, as usual, that for weak nonlinearity a KdV-type equation may arise for the first-order field corrections. This conclusion is an important consequence of the nature of the nonlocal vortex interaction, traceable back to $\lambda_L\neq 0$ in Eq. (5c). For if $n=B$, the resulting dispersion relation is simply $\omega^2=k^2$.

By making the change of independent variables $\xi=\omega(\rho-t)$, $\eta=\omega^3\rho$, Eqs. (6) become

$$-\frac{\partial n}{\partial\xi} + \frac{1}{\eta}\left(\frac{\partial}{\partial\xi} + \omega^2\,\frac{\partial}{\partial\eta}\right)(\eta n v) = 0, \tag{10a}$$

$$-\frac{\partial v}{\partial \xi}+v\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)v=-\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)B, \quad (10b)$$

$$n-B=-\frac{\omega^2}{\eta}\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)\left[\eta\left(\frac{\partial}{\partial \xi}+\omega^2\frac{\partial}{\partial \eta}\right)\right]B. \quad (10c)$$

Using the perturbation expansions in even powers of ω

$$n(\xi,\eta)=1+\omega^2 n^{(1)}+\omega^4 n^{(2)}+\cdots, \quad (11)$$

$$B(\xi,\eta)=1+\omega^2 B^{(1)}+\omega^4 B^{(2)}+\cdots, \quad (12)$$

$$v(\xi,\eta)=\omega^2 v^{(1)}+\omega^4 v^{(2)}+\cdots \quad (13)$$

in Eqs. (10) gives recursion relations for the higher-order corrections by equating coefficients of like powers of ω.

The lowest-order equations can be integrated with respect to ξ to yield

$$v^{(1)}=n^{(1)}+f(\eta), \quad B^{(1)}=n^{(1)}, \quad (14)$$

where f is an arbitrary function. The next-order contributions are

$$-\frac{\partial n^{(2)}}{\partial \xi}+\frac{\partial}{\partial \xi}[n^{(1)}v^{(1)}+v^{(2)}]+\frac{1}{\eta}\frac{\partial}{\partial \eta}[\eta v^{(1)}]=0, \quad (15a)$$

$$-\frac{\partial v^{(2)}}{\partial \xi}+v^{(1)}\frac{\partial v^{(1)}}{\partial \xi}=-\frac{\partial B^{(2)}}{\partial \xi}-\frac{\partial B^{(1)}}{\partial \eta}, \quad (15b)$$

and

$$n^{(2)}-B^{(2)}=-\frac{\partial^2 B^{(1)}}{\partial \xi^2}. \quad (15c)$$

These can be combined to give the cylindrical KdV equation in the form

$$\frac{\partial^3 v^{(1)}}{\partial \xi^3}+3v^{(1)}\frac{\partial v^{(1)}}{\partial \xi}+2\frac{\partial v^{(1)}}{\partial \eta}+\frac{v^{(1)}}{\eta}-f(\eta)\frac{\partial v^{(1)}}{\partial \xi}=\frac{\partial f}{\partial \eta}. \quad (16)$$

The change of coordinates

$$\xi'=\xi+\frac{1}{2}\int f\,d\eta, \quad \eta'=\eta \quad (17)$$

can be used to eliminate the last term on the left-hand side of Eq. (16).

The function $f(\eta)$ can be used to satisfy initial or boundary conditions. For a bulk superconductor with zero boundary condition at infinity, $f=0$, and then multi-soliton solutions of Eq. (16) can be written in terms of the Airy function Bi.[7,8] The single-soliton solution of the CKdV equation is

$$h(\xi,\eta)=1+h_0\int_{\xi}^{\infty}d\xi\{(6\eta)^{-1/3}\,\mathrm{Bi}[(6\eta)^{-1/3}(\xi-\xi_0)]\}^2, \quad (18)$$

where

$$v^{(1)}(\xi,\eta)=2\frac{\partial^2}{\partial \xi^2}\ln h, \quad (19)$$

and h_0 and ξ_0 are arbitrary constants. In accordance with standard cylindrical KdV theory, as a cylindrical soliton progresses inward, the amplitude grows somewhat faster than $\eta^{-1/2}$ while the width decreases somewhat faster than $\eta^{1/4}$. The square root of the peak amplitude times the width remains constant, as for one-dimensional solitons.

The first three constants of the motion,

$$c_1=\eta^{1/2}\int v^{(1)}(\xi,\eta)d\xi, \quad (20a)$$

$$c_2=\eta\int [v^{(1)}(\xi,\eta)]^2 d\xi, \quad (20b)$$

and

$$c_3=\eta^{1/2}\int v^{(1)}(\xi,\eta)\left[\frac{\xi}{2}+\eta v^{(1)}(\xi,\eta)\right]d\xi, \quad (20c)$$

can easily be verified using the CKdV equation.

Analogous dynamical systems exist in plasma physics in ion acoustic waves and in hydrodynamics.[4,15] The correspondences for the dependent variables of a two-component plasma are $n\leftrightarrow n_i$, the volume density of ions, $v\leftrightarrow v_i$, the ion velocity, and $B\leftrightarrow\phi+1$, where ϕ is the electrostatic potential. The magnetic (Lorentz) force in the vortex lattice is replaced with the electric force in the plasma. Therefore the current density in the superconductor plays a role analogous to the electric field in the plasma. In a simple fluids model,[23] there is a correspondence between the cross-sectional area of a cylinder with elastic walls and the vortex density, the fluid velocity and the vortex velocity, and between the fluid pressure and the magnetic induction.

When electron inertia is neglected in the plasma, an integration can be performed to give the electron number density explicitly in terms of the electrostatic potential. This additional nonlinearity does not occur in the vortex dynamics equations. The weak nonlinearities in the present model include bilinearity in the vortex continuity equation (2) and convective differentiation in the equation of motion (3). With the stated assumptions, the equations are completed with the London equation for the magnetic induction. The use of the continuum density n allows the modeling of tilt and compression modes of the lattice but neglects shear. The nonlocal vortex interaction is critical in obtaining the long-wave cubic dispersion relation, from Eq. (9).

The derivation here ignored pinning and drag, resulting in stringent conditions for the appearance of solitons. Limited viscosity can be included by treating a perturbed form of the CKdV equation. However, it would be of interest to find solutions of an extended CKdV or other equation with a damping term(s) comparable in size with the nonlinear and dispersive terms. Such equations will have a wider range of applicability in type-II superconductivity.

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