Thermal residual stress in composites with anisotropic interphases

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(Received 20 March 1995; revised manuscript received 30 November 1995)

A theoretical model for built-in thermal residual stresses in three concentric, transversely isotropic, cylinders is presented. Marked differences between Nairn's earlier isotropic result [Polym. Compos. **6**, 123 (1985)] and the current extension are demonstrated by means of examples.

I. INTRODUCTION

Consider a model composite (Fig. 1) made up of either two or three cylinders. Perfect interfacial bonding is assumed. The cylinders are assembled together at a relatively high temperature T_{ref}, and then cooled down to temperature T. The problem consists in determining the stresses that develop in the cylinders as the temperature decreases progressively, down to temperature T, given that they possess different elastic constants and coefficients of thermal expansion. Various authors have addressed this problem.[1–3] Here we develop a model that is more general than Nairn's results[3] as all cylinders (not only the central cylinder) possess transversal isotropy. This type of anisotropy is a specialized case of orthotropy,[4] namely, the (r,θ) plane is a plane of isotropy. The issue of optimizing material characteristics to minimize residual stresses in the cylinders, is addressed here in selected examples, and more fully so elsewhere.[5]

A. One cylinder

Consider first a *single* hollow cylinder with free ends, subjected to the following conditions: (i) the axial strain is constant, so that plane sections perpendicular to the axis remain plane during straining, (ii) every cross section perpendicular to the axis undergoes radial strains only. For a linear elastic, isotropic cylinder under internal pressure P_i and external pressure P_o, the classical solution[6] is

$$\sigma_{rr}=-P_i\,\frac{(b/r)^2-1}{(b/a)^2-1}-P_0\,\frac{1-(a/r)^2}{1-(a/b)^2}, \tag{1}$$

$$\sigma_{\theta\theta}=P_i\,\frac{(b/r)^2+1}{(b/a)^2-1}-P_o\,\frac{1+(a/r)^2}{1-(a/b)^2}, \tag{2}$$

$$\sigma_{zz}=\xi\ \ (\text{a constant}), \tag{3}$$

where σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} are the radial, hoop, and longitudinal stresses at a distance r from the symmetry axis, and a and b are the internal and external radii of the cylinder.

B. Two cylinders

Next, consider two concentric cylinders, where the internal cylinder is solid rather than hollow, and where both cylinders are transversely isotropic. The strain-stress relationship takes the Duhamel-Neumann form (in cylindrical coordinates):[6]

$$\begin{pmatrix}\epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\end{pmatrix}=\begin{pmatrix}\frac{1}{E_r} & -\frac{\nu_{\theta r}}{E_r} & -\frac{\nu_{zr}}{E_z}\\ -\frac{\nu_{\theta r}}{E_r} & \frac{1}{E_r} & -\frac{\nu_{zr}}{E_z}\\ -\frac{\nu_{zr}}{E_z} & -\frac{\nu_{zr}}{E_z} & \frac{1}{E_z}\end{pmatrix}\begin{pmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\end{pmatrix}+\begin{pmatrix}\alpha_r\\ \alpha_r\\ \alpha_z\end{pmatrix}\Delta T, \tag{4}$$

where ν is the Poisson ratio, E is Young's modulus, and $\Delta T=T-T_{\mathrm{ref}}$. As necessary, there are four independent elastic constants and two thermal expansion coefficients. Transverse isotropy is a convenient case since the form of the stresses in the cylinders is the same as for the isotropic case[3] (this is not true for other specialized cases of orthotropy). For the fiber, the internal pressure and radius are zero. Equations (1)–(3) become

$$\sigma_{rr}^f=\sigma_{\theta\theta}^f=A^f,\quad \sigma_{zz}^f=C^f, \tag{5}$$

where A^f and C^f are constants. On the other hand, the pressure on the outer surface of the external cylinder (the matrix) is zero and Eqs. (1)–(3) become

$$\sigma_{rr}^m=A^m+B^m\left(\frac{R_1}{r}\right)^2,\quad \sigma_{\theta\theta}^m=A^m-B^m\left(\frac{R_1}{r}\right)^2,\quad \sigma_{zz}^m=C^m, \tag{6}$$

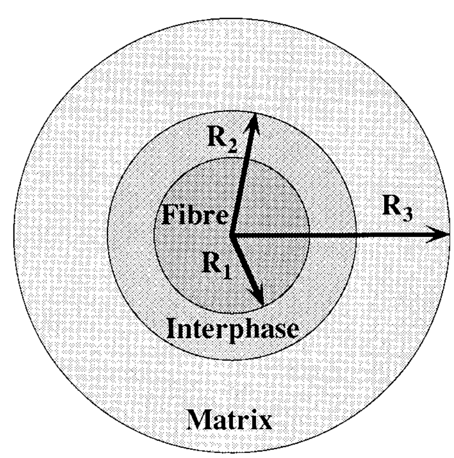

FIG. 1. Cross-sectional view of the concentric cylinder model.

where A^m, B^m, and C^m are constants, and the notations for the internal and external radii are now R_1 and R_2, instead of a and b. The problem is reduced to the determination of the five constants A^f, C^f, A^m, B^m, and C^m. The radial stress boundary conditions are associated with continuity of tractions at interfaces: $\sigma_{rr}^m=0$ at $r=R_2$, $\sigma_{rr}^m=\sigma_{rr}^f$ at $r=R_1$. A force balance in the longitudinal direction yields $\sigma_{zz}^m\phi_m+\sigma_{zz}^f\phi_f=0$, where $\phi_f=(R_1/R_2)^2$, and $\phi_m=1-\phi_f$. Using Eqs. (5) and (6), the above conditions yield

$$B^m=-\frac{A^m}{\phi_f},\quad A^f=-A^m\,\frac{\phi_m}{\phi_f},\quad C^f=-C^m\,\frac{\phi_m}{\phi_f}. \tag{7}$$

There are thus only two unknowns, namely, A^m and C^m. They can be determined by applying the strain-stress relations 4 with the interfacial no-slip conditions: $\epsilon_{zz}^f=\epsilon_{zz}^m$ at $r=R_1$, and $u_r^f=u_r^m$ at $r=R_1$ (u denotes the displacement), which, since $u_r=r\epsilon_{\theta\theta}$, is equivalent to $\epsilon_{\theta\theta}^f=\epsilon_{\theta\theta}^m$ at $r=R_1$. These conditions, combined with the thermoelastic relations 4, yield two simultaneous equations with two unknowns:

$$\begin{aligned} &A^mK_1+C^mK_2+M_1(T-T_{\mathrm{ref}})=0,\\ &A^mK_3+C^mK_4+M_2(T-T_{\mathrm{ref}})=0, \end{aligned} \tag{8}$$

where

$$K_1=2\left(\frac{\nu_{zr}^m}{E_z^m}+\frac{\nu_{zr}^f}{E_z^f}\,\frac{\phi_m}{\phi_f}\right), \tag{9}$$

$$K_2=-\left(\frac{1}{E_z^m}+\frac{1}{E_z^f}\,\frac{\phi_m}{\phi_f}\right), \tag{10}$$

$$K_3=-\left[\frac{1-\nu_{\theta r}^f}{E_r^f}\,\frac{\phi_m}{\phi_f}+\frac{1-\nu_{\theta r}^m}{E_r^m}+\frac{1+\nu_{\theta r}^m}{E_r^m}\,\frac{1}{\phi_f}\right], \tag{11}$$

$$K_4=\frac{\nu_{zr}^m}{E_z^m}+\frac{\nu_{zr}^f}{E_z^f}\,\frac{\phi_m}{\phi_f}, \tag{12}$$

and

$$M_1=(\alpha_z^f-\alpha_z^m),\quad M_2=(\alpha_r^f-\alpha_r^m). \tag{13}$$

If the matrix is isotropic, the K_i's reduce to the corresponding matrix elements previously given by Nairn (see Table 3 in Ref. 3). Solving Eqs. (8) yields

$$\begin{aligned} A^m&=-\frac{K_4M_1-K_2M_2}{K_1K_4-K_2K_3}\,(T-T_{\mathrm{ref}}),\\ C^m&=-\frac{K_1M_2-K_3M_1}{K_1K_4-K_2K_3}\,(T-T_{\mathrm{ref}}). \end{aligned} \tag{14}$$

By inserting these results into Eqs. (7), the residual thermal stresses in both the fiber and the matrix may be determined from Eqs. (5) and (6).

C. Three cylinders

Now consider three concentric, transversely isotropic cylinders (Fig. 1). From Eqs. (1)–(3), the stresses take the following form, noting that for the middle cylinder both internal and external pressure exist:

$$\text{In the fiber:}\quad \sigma_{rr}^f=\sigma_{\theta\theta}^f=A^f,\quad \sigma_{zz}^f=C^f, \tag{15}$$

$$\begin{aligned} &\text{In the interphase:}\quad \sigma_{rr}^i=A^i+B^i\left(\frac{R_2}{r}\right)^2,\\ &\sigma_{\theta\theta}^i=A^i-B^i\left(\frac{R_2}{r}\right)^2,\quad \sigma_{zz}^i=C^i, \end{aligned} \tag{16}$$

$$\begin{aligned} &\text{In the matrix:}\quad \sigma_{rr}^m=A^m+\frac{B^m}{r^2},\quad \sigma_{\theta\theta}^m=A^m-\frac{B^m}{r^2},\\ &\sigma_{zz}^m=C^m. \end{aligned} \tag{17}$$

TABLE I. Coefficients of the unknowns in the system (22) of simultaneous equations.

$$Q_1=\frac{2\phi_m}{\phi_m-1}\left(\frac{\nu_{zr}^i}{E_z^i}-\frac{\nu_{zr}^f}{E_z^f}\right)$$

$$Q_2=-\frac{\phi_m}{E_z^f\phi_f}$$

$$Q_3=-2\left(\frac{\nu_{zr}^f}{E_z^f}\,\frac{\phi_i}{\phi_f}+\frac{\nu_{zr}^i}{E_z^i}\right)$$

$$Q_4=-\left(\frac{\phi_i}{E_z^f\phi_f}+\frac{1}{E_z^i}\right)$$

$$Q_5=2\left(\frac{\nu_{zr}^i}{E_z^i}\,\frac{\phi_m}{\phi_m-1}-\frac{\nu_{zr}^m}{E_z^m}\right)$$

$$Q_6=\frac{1}{E_z^m}$$

$$Q_7=-2\,\frac{\nu_{zr}^i}{E_z^i}$$

$$Q_8=-\frac{1}{E_z^i}$$

$$Q_9=\frac{\phi_m}{\phi_m-1}\left(\frac{1-\nu_{\theta r}^f}{E_r^f}-\frac{1-\nu_{\theta r}^i}{E_r^i}\right)$$

$$Q_{10}=\frac{\phi_m}{\phi_f}\,\frac{\nu_{zr}^f}{E_z^f}$$

$$Q_{11}=\frac{\phi_i}{\phi_f}\left(\frac{1-\nu_{\theta r}^f}{E_r^f}\right)-\frac{\nu_{\theta r}^i-1}{E_r^i}+\frac{1-\phi_m}{\phi_f}\left(\frac{1+\nu_{\theta r}^i}{E_r^i}\right)$$

$$Q_{12}=\frac{\phi_i}{\phi_f}\,\frac{\nu_{zr}^f}{E_z^f}+\frac{\nu_{zr}^i}{E_z^i}$$

$$Q_{13}=\frac{1-\nu_{\theta r}^m}{E_r^m}+\frac{1}{1-\phi_m}\left(\frac{1+\nu_{\theta r}^m}{E_r^m}\right)+\frac{\phi_m}{\phi_m-1}\left(\frac{\nu_{\theta r}^i-1}{E_r^i}\right)$$

$$Q_{14}=-\frac{\nu_{zr}^m}{E_z^m}$$

$$Q_{15}=\frac{2}{E_r^i}$$

$$Q_{16}=\frac{\nu_{zr}^i}{E_z^i}$$

TABLE II. Physical properties of the fiber and of the isotropic and transversely isotropic matrices used in the two-cylinder example. TEC means thermal expansion coefficient.

Property	Fiber (AS graphite)	Isotropic matrix (3501-6 epoxy)	Anisotropic matrix
Longitudinal Young's modulus, E_z (GPa)	220	4.3	40
Transverse Young's modulus, E_r (GPa)	14	4.3	1
Longitudinal Poisson ratio, ν_{rz}	0.2	0.34	0.34
Transverse Poisson ratio, $\nu_{\theta r}$	0.25	0.34	0.2
Longitudinal TEC α_z (ppm/°C)	−0.36	40	1
Transverse TEC, α_r (ppm/°C)	18	40	160

We now have eight unknowns: A^f, C^f, A^i, B^i, C^i, A^m, B^m, and C^m. The boundary conditions are $\sigma^f_{rr}=\sigma^i_{rr}$ at $r=R_1$, $\sigma^i_{rr}=\sigma^m_{rr}$ at $r=R_2$, $\sigma^m_{rr}=0$ at $r=R_3$, and since there is no applied force, a force balance in the longitudinal direction yields $\sigma^m_{zz}\phi_m+\sigma^i_{zz}\phi_i+\sigma^f_{zz}\phi_f=0$, where $\phi_f=(R_1/R_3)^2$, $\phi_m=1-(R_2/R_3)^2$, and $\phi_i=1-\phi_f-\phi_m$. Using Eqs. (15)–(17) the above boundary conditions yield the following relationships:

$$B^m=-R_3^2A^m, \tag{18}$$

$$A^f=A^m\,\frac{\phi_m}{\phi_m-1}+B^i\,\frac{\phi_i}{\phi_f}, \tag{19}$$

$$A^i=A^m\,\frac{\phi_m}{\phi_m-1}-B^i, \tag{20}$$

$$C^f=-\left[C^m\,\frac{\phi_m}{\phi_f}+C^i\,\frac{\phi_i}{\phi_f}\right]. \tag{21}$$

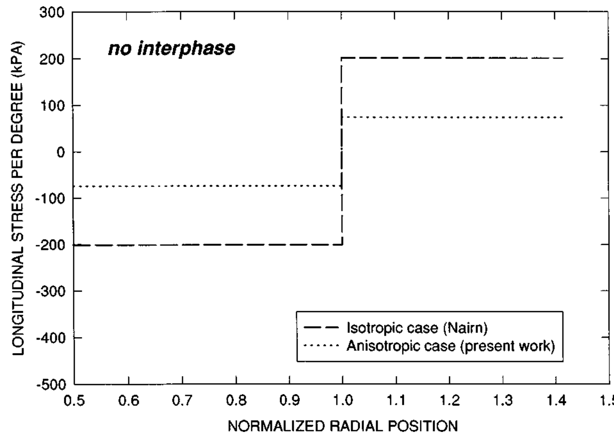

FIG. 2. The two-cylinder case: longitudinal thermal residual stresses in the fiber and the matrix as a function of normalized radial distance, using isotropic and transversely isotropic matrices (data in Table II).

There are now four unknown parameters: A^m, C^m, B^i, and C^i. They can be determined by combining Eqs. (4) with the following continuity relations: $\epsilon^f_{zz}=\epsilon^i_{zz}$ at $r=R_1$, $\epsilon^m_{zz}=\epsilon^i_{zz}$ at $r=R_2$, $u^f_r=u^i_r$ at $r=R_1$ and $u^m_r=u^i_r$ at $r=R_2$, which, since $u_r=r\epsilon_{\theta\theta}$, are equivalent to $\epsilon^f_{\theta\theta}=\epsilon^i_{\theta\theta}$ at $r=R_1$ and $\epsilon^m_{\theta\theta}=\epsilon^i_{\theta\theta}$ at $r=R_2$. The following system is obtained:

$$\begin{aligned}
&A^mQ_1+C^mQ_2+B^iQ_3+C^iQ_4+S_1(T-T_{\text{ref}})=0,\\
&A^mQ_5+C^mQ_6+B^iQ_7+C^iQ_8+S_2(T-T_{\text{ref}})=0,\\
&A^mQ_9+C^mQ_{10}+B^iQ_{11}+C^iQ_{12}+S_3(T-T_{\text{ref}})=0,\\
&A^mQ_{13}+C^mQ_{14}+B^iQ_{15}+C^iQ_{16}+S_4(T-T_{\text{ref}})=0.
\end{aligned} \tag{22}$$

The Q_i's are given in Table I, and the S_i's are $S_1=\alpha^f_z-\alpha^i_z$, $S_2=\alpha^m_z-\alpha^i_z$, $S_3=\alpha^f_r-\alpha^i_r$, and $S_4=\alpha^m_r-\alpha^i_r$. If the matrix and the interface are isotropic, the Q_i's reduce to the matrix elements calculated by Nairn (Table 2 in Ref. 3 where, in the second term of the right-hand side of Q_{13}, there is a sign difference between our expression and Nairn's, due to a typographical error in Ref. 3). The solution to the system 22 is then given by

TABLE III. Physical properties of the fiber and of the isotropic and transversely isotropic matrices and (soft) interphases used in the three-cylinder example. TEC means thermal expansion coefficient.

Property	Fiber (AS graphite)	Isotropic matrix (3501-6 epoxy)	Isotropic interphase	Anisotropic matrix	Anisotropic interphase
Longitudinal Young's modulus, E_z (GPa)	220	4.3	0.001	40	0.01
Transverse Young's modulus, E_r (GPa)	14	4.3	0.001	1	0.0001
Longitudinal Poisson ratio, ν_{rz}	0.2	0.34	0.5	0.34	0.5
Transverse Poisson ratio, $\nu_{\theta r}$	0.25	0.34	0.5	0.2	0.5
Longitudinal TEC, α_z (ppm/°C)	−0.36	40	170	1	50
Transverse TEC, α_r (ppm/°C)	18	40	170	160	300

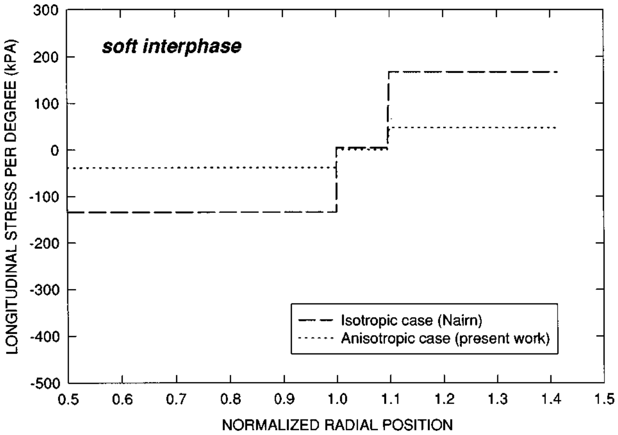

FIG. 3. The three-cylinder case: longitudinal thermal residual stresses in the fiber, the interphase, and the matrix as a function of normalized radial distance using isotropic and transversely isotropic matrices and interphases (data in Table III).

$$A^m = -\frac{D_1}{D}, \quad C^m = -\frac{D_2}{D}, \quad B^i = -\frac{D_3}{D}, \quad C^i = -\frac{D_4}{D}. \tag{23}$$

D is the determinant of the unknowns in the system (22), and D_i is the determinant obtained by replacing in D the elements of the ith column by the independent terms. Inserting Eqs. (23) into Eqs. (18)–(21), the stresses in the cylinders are determined using Eqs. (15)–(17).

II. DISCUSSION AND EXAMPLES

The solution proposed here strictly applies only to transversely isotropic cylinders. Since infinite cylinders are assumed, no shear stress develops, thus the no-slip boundary condition is not in doubt. This might not be true with finite cylinders, in which interfacial shear stresses might progressively develop at high stress, from the cylinders' ends. Finally, provided that the (anisotropic) strength of each cylindrical phase is known, the present treatment may lead to a criterion for the occurrence of instabilities, such as fiber buckling. To physically illustrate the differences between Nairn's solution[3] and the present treatment, two examples potentially relevant to fiber composites are now presented.

A. Example 1

Consider two concentric cylinders, the central one being transversely isotropic, and the other being either isotropic in Nairn's case, or transversely isotropic in the present case. The fiber properties in Table II are those of Hercules AS graphite (fiber radius$=3.5$ μm), as in Nairn;[3] The matrix properties are those of Hercules 3501-6 epoxy resin in the isotropic case, again like Nairn, and those of a hypothetical transversely isotropic matrix. Figure 2 shows the radial dependence of the longitudinal stress (this component is the most sensitive to a switch from isotropicity to anisotropicity) when the volume fraction ϕ_f is 0.5. We set $\Delta T = -1$, to calculate the stress buildup per degree of cooling (the total buildup is found by multiplying the stress per degree of cooling by the temperature change). As seen, with this transversely anisotropic matrix, both the fiber compressive and matrix tensile stresses are significantly reduced, relative to Nairn's isotropic case. This may have significant design benefit.

B. Example 2

The same conclusion is reached in the case of three concentric cylinders, in which the central one is transversely isotropic, and the two others are either isotropic in Nairn's case, or transversely isotropic. Table III lists the data used. These correspond to a "soft" interphase (the interphase moduli are lower than those of the surrounding matrix, and correspond roughly to an elastomerlike interphase). Figure 3 shows the radial dependence of the longitudinal stress when ϕ_f is 0.5 (corresponding to $R_1 = 3.5$ μm), $\phi_m = 0.4$ ($R_2 = 3.83$ μm), and $\phi_i = 0.1$ ($R_3 = 4.95$ μm), thus an interphase thickness of about 1/10 the fiber radius (Fig. 3). The presence of the transversely isotropic interphase has the desirable effect to significantly reduce the (already weak) thermal residual stresses in the fiber and the matrix. Results for a "stiff" interphase (moduli intermediate between those of the fiber and the matrix), presented in a complete study,[5] reveal a much larger interphase compressive stress in the isotropic case, but similar to the soft interphase case shown here, a significantly milder interphase stress in the transversely isotropic case. Again, this may be significant from a component design viewpoint.

ACKNOWLEDGMENT

This research was supported by the Basic Research Foundation administered by the Israel Academy of Sciences and Humanities.

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