Model for a random-matrix description of the energy-level statistics of disordered systems at the Anderson transition

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We consider a family of random-matrix ensembles (RME's) invariant under similarity transformations and described by the probability density $P(\mathbf{H})=\exp[-\mathrm{Tr}V(\mathbf{H})]$. Dyson's mean-field theory (MFT) of the corresponding plasma model of eigenvalues is generalized to the case of weak confining potential, $V(\epsilon)\sim(A/2)\ln^2(\epsilon)$. The eigenvalue statistics derived from MFT are shown to deviate substantially from the classical Wigner-Dyson statistics when $A<1$. By performing systematic Monte Carlo simulations on the plasma model, we compute all the relevant statistical properties of the RME's with weak confinement. For $A_c\approx 0.4$ the distribution function of the energy-level spacings (LSDF) of this RME coincides in a large energy window with the LSDF of the three-dimensional Anderson model at the metal-insulator transition. For the same A_c, the variance of the number of levels, $\langle n^2\rangle-\langle n\rangle^2$, in an interval containing $\langle n\rangle$ levels on average, grows linearly with $\langle n\rangle$, and its slope is equal to 0.32 ± 0.02, which is consistent with the value found for the Anderson model at the critical point.

I. INTRODUCTION

Random-matrix theory (RMT) was introduced by Wigner[1] and Dyson[2] to provide a statistical description of the quantized energy levels of heavy nuclei, and since then it has been applied to a great variety of complex systems, quantum and classical.[3] Gor'kov and Eliashberg[4] suggested that the Wigner-Dyson (WD) statistics, derived from RMT, could be used to describe the energy levels of small metallic particles at low temperature, in connection with the study of their electromagnetic properties. Here, a statistical description is made necessary by the presence of disorder and irregularities in the shape of the particles. For the case of disordered conductors, one can resort to powerful field-theoretical techniques, which have allowed Efetov[5] and Altshuler and Shklovskii[6] to show analytically that the WD statistics are more than a simple phenomenological conjecture and describe exactly the local fluctuations of the energy levels in metals in a certain regime.

The WD statistics are characterized by strong energy-level correlations, giving rise to the phenomenon of the level repulsion. These are the correlations that typically exist among the eigenvalues of a Gaussian ensemble (GE) of random Hermitian matrices $\mathbf{H}$, that is, an ensemble of matrices randomly distributed with probability density $P(\mathbf{H})\propto\exp[-\mathrm{Tr}\mathbf{H}^2]$.

The GE's (and, therefore, the WD statistics) do not bear any hint of the spatial dimensionality d of a physical system. Furthermore, they are, by definition, invariant under similarity transformations and thus there is no basis preference in them. This means that they can be applied only to particular regimes of a physical system where (1) all the normalized linear combinations of the eigenstates have similar properties; (2) the dimensionality is, in some sense, irrelevant.

For a disordered electronic system this is just the ergodic regime of the metallic state. In the metallic phase, the eigenstates are extended structureless objects. If we further assume that all the relevant times are larger than the ergodic time $\tau_D=L^2/D$ (D being the diffusion coefficient), any diffusive particle can completely and homogeneously fill the total sample volume $V=L^d$ during its trajectory (ergodic regime) and, thus, does not feel the space dimensionality. Alternatively, the ergodic time defines a natural energy scale, $E_c=\hbar/\tau_D$ known as Thouless energy. The ergodic regime, and therefore RMT, is valid within energy intervals $\epsilon\ll E_c$, or within an interval containing a number of levels $N<E_c/\Delta\sim g$, where Δ is the mean level spacing. The quantity g is known as the dimensionless conductance in units of $e^2/\hbar$. When $\epsilon>E_c$, the level statistics depends on the dimensionality and is different from WD distribution (Altshuler-Shklovskii regime).

The nonergodic regime is never reachable in the metallic phase in the thermodynamic limit, in any energy interval containing a large but *finite* number of levels, since the dimensionless conductance g diverges in the limit $L\to\infty$. Therefore, the WD statistics describe exactly the energy-level correlations of the metallic state in the $L\to\infty$ limit that exists for $d>2$ at relatively small disorder.

For $d=1,2$, no metallic state exists in the thermodynamic limit, if any disorder is present, the system being always an insulator. For $d=3$, with disorder increasing, the system goes through the Anderson transition[7] to the insulating state, where all states are localized.[8] The level statistics in these situations obviously cannot be described by the a-dimensional, classical RMT of the GE. In particular, such simple $U(N)$ invariant random-matrix ensembles (RME's) cannot be an appropriate description of a Hamiltonian matrix, the eigenvectors of which undergo the phenomenon of localization, since one can construct extended states by a linear combination of localized states. In contrast, the proper $P(\mathbf{H})$ distribution should contain eigenvector-eigenvalue correlations or a basis preference, which exclude those unitary transformations that would lead to the formation of such extended states.

Random banded matrices[9,10] are an example of noninvariant RME's and are perhaps more realistic for describing Hamiltonians of quasi-one-dimensional disordered electronic systems[9] and other quantum chaotic systems.[10] Their statistical properties exhibit a crossover from the WD to the Poisson statistics as a function of the parameter b^2/N (b is the band width), which is similar to what happens in quasi-one-dimensional systems upon decreasing the ratio ξ/L, ξ being the localization length.

Another model worth mentioning is represented by the ensemble of sparse random matrices. This model has been shown to be closely related to the Anderson model on a *Bethe lattice*,[11] which is known to possess a localization transition.[12] On the basis of this similarity, sparse random-matrix ensembles should also display an Anderson transition from a localized to a delocalized regime upon increasing the "mean connectivity parameter."[11] The energy-level statistics in the delocalized regime have been proven to belong to the Wigner-Dyson universality class.[11]

While the RMT description of the quasi-one-dimensional disordered systems can be provided by random banded matrices, an analogous description of the energy statistics near the critical point for $d>2$ is still missing. This problem has recently become a very outstanding one, after intense study, both numerical[13–20] and analytical,[21,22] has shown that the spectral correlations at the metal-insulator transition are also universal and very different from the WD and the Poisson statistics.

There is not yet full consensus on the exact nature of the critical energy-level statistics. The main finding of the analytical treatment based on a scaling analysis[21,22] is the existence of a power-law decay of the two-level correlation function with a nontrivial critical exponent. All numerical simulations show that at the critical point, the statistical fluctuations of the energy levels are scale invariant. However, while some authors[15,16,20] claim a good agreement of their numerical results with the analytical predictions, others[13,17–19] suggest that the critical statistics constitute more simply a "hybrid" between the WD and the Poisson distributions without any nontrivial critical exponent.

The search of a RMT description of the critical statistics has prompted the investigation of physically motivated RME's which exhibit nontrivial deviations from the WD statistics. One important generalization has been obtained in Refs. 23 and 24, starting from the GE and introducing a symmetry breaking term of the form

$$P(\mathbf{H}) \propto e^{-\mathrm{Tr}\mathbf{H}^2} e^{-h^2 N^2 \mathrm{Tr}([\mathbf{\Lambda},\mathbf{H}][\mathbf{\Lambda},\mathbf{H}]^\dagger)}. \tag{1.1}$$

The h-dependent term breaks the $U(N)$ invariance and tends to align $\mathbf{H}$ with a symmetry breaking unitary matrix $\mathbf{\Lambda}$, thus setting the basis preference. It was shown in Ref. 24 that even after averaging over $\mathbf{\Lambda}$ the implicit presence of the symmetry breaking term causes a dramatic change in the level correlations of the resulting ensemble, which exhibit a crossover from the WD to the Poisson statistics.

On the other hand, a lot of work[25] has been devoted to the analysis of $U(N)$ invariant generalizations of the GE, of the kind

$$P(\mathbf{H}) \propto e^{-\mathrm{Tr}V(\mathbf{H})}. \tag{1.2}$$

Such ensembles arise from a *global maximum entropy ansatz* of RMT, in which an information entropy is maximized by the distribution.[25] The function $V(\mathbf{H})$ acts like a generalized Lagrange multiplier and it is determined, e.g., by requiring that the density of eigenvalues is some given function $\rho(\epsilon)$, taken directly from the microscopic system being investigated:

$$\langle \mathrm{Tr}[\,\delta(\epsilon-\mathbf{H})]\rangle = \rho(\epsilon). \tag{1.3}$$

For a long time it was believed that the local statistical properties of the eigenvalues of these ensembles are completely independent of V and identical to those of the GE. This hypothesis, so far supported only by numerical evidence, has been proved more rigorously very recently[26,27] for a large class of functions V. On the other hand, it has also been demonstrated[28–31] that there exists another class of functions V, known as *weak confining potentials*, for which the eigenvalue statistics display very strong deviations from the universal WD behavior. Such potentials are characterized by a very slow asymptotic growth:

$$V(\epsilon) \sim A\ \ln^2|\epsilon|, \quad |\epsilon| \to \infty. \tag{1.4}$$

It is important to emphasize that this asymptotic behavior of V has been inferred from numerical studies of *random transfer matrix* models for disordered conductors, through the maximum entropy ansatz.[25,32,33] In that case, one considers the eigenvalues of some combination of the transfer matrix, which are directly related to the conductance and become larger and larger, namely less confined, when the disorder increases.

It has been shown[28–31] that the local eigenvalue fluctuations of the RME's, with confining potential (1.4), exibit a crossover from the WD statistics to a more Poisson-like behavior when the parameter A is decreased. Since it is a common belief that there is a connection between the statistics of eigenvalues and eigenstates of a RME, this breakdown of the WD universality seems to contradict the argument (presented above for the GE) that a $U(N)$ invariant ensemble cannot exhibit Poissonian statistics, typical of localized states. In Ref. 31, it has been suggested that Poissonian behavior in such RME's is a remarkable phenomenon, due to the spontaneous breakdown of the $U(N)$ symmetry at the transition from a power-law potential, $V(\epsilon) \sim |\epsilon|^\alpha$, to the logarithmic potential of Eq. (1.4) when $\alpha \to 0$. A crucial point in reaching this conclusion was the observation that the two-level correlation function of the invariant RME's with weak confinement and the one of the RME's with symmetry breaking are identical in a certain range of the parameters. Having established that these generalized RME's with soft confinement belong to a new universality class, characterized by nonclassical correlations that interpolate from the WD to the Poisson statistics, there remains the important question of whether or not their properties are related to the critical energy-level statistics of the Anderson model.

In this paper, we address this question through a careful study of the RME's with distribution given by Eq. (1.3). First, we shall show that the local statistical properties of a generalized RME of this sort can be correctly and easily determined using an extension of Dyson's functional derivative formalism for the corresponding one-dimensional (1D)

plasma model for the eigenvalues. This formalism has been used successfully by Beenakker[39] to investigate the random-matrix theory of the transmission eigenvalues in disordered conductors. In particular, we prove that a modified mean-field theory (MFT) of the plasma model is able to yield the nonuniversal asymptotic behavior of the two-level correlation function for weak confinement.

Second, we perform extensive Monte Carlo simulations of the one-dimensional Coulomb plasma of the eigenvalues and calculate all the relevant quantities that describe short-range and long-range statistical properties of the RME's. The comparison with the MFT and other analytical results shows that this Monte Carlo method provides very accurate answers for this problem and can be used to study more complicated RME's where no analytical results are known.

Finally, we critically compare the level statistics of these RME's with the results recently obtained for the 3*d* Anderson model at the metal-insulator transition. Our analysis shows that, while the asymptotic correlations of the RME's do not agree with the analytical result of Refs. 21 and 22, two other statistical properties, namely, the distribution function of the level spacings and level-number variance, are remarkably close to those found numerically in exact diagonalizations of the Anderson model.

The paper is organized in the following way. In Sec. II, we set up formalism and notation and review Dyson's derivation of the effective plasma model for the eigenvalue distribution. In Sec. III, we develop a MFT of this model, generalized to the weak confining potentials. The MFT study of the two-level correlation function is carried out in Sec. IV. Section V is devoted to the study of Monte Carlo simulations. In Sec. VI we discuss and compare the results of RMT with the analytical and numerical results of the energy-level statistics of the Anderson model at the metal-insulator transition. Summary and conclusions are presented in Sec. VII.

II. EIGENVALUE STATISTICS AND EFFECTIVE ONE-DIMENSIONAL PLASMA MODEL

We consider an ensemble of random $N\times N$ matrices $\mathbf{H}$. The matrices $\mathbf{H}$ are supposed to represent, for example, the Hamiltonian of a complex system, such as a quantum disordered conductor. We take $\mathbf{H}$ to be either *real symmetric*, *Hermitian*, or *quaternion-real self-dual*.[3] This choice defines three possible ensembles corresponding to three different physical systems: (1) systems with time-reversal and rotational invariance; (2) systems with broken time-reversal symmetry; (3) systems with time-reversal symmetry, but broken rotational invariance.

According to the maximum entropy principle mentioned in the previous section,[25] we will assume that the probability distribution for the RME is defined by the density

$$P(\mathbf{H})=Z^{-1}e^{-\mathrm{Tr}V(\mathbf{H})}, \tag{2.1}$$

where Z is a normalization constant. The volume element is $d[\mathbf{H}]=\Pi_{i\geqslant j}dH_{ij}$, for real symmetric matrices, with obvious generalizations for the other two cases.[3] The probability density $P(\mathbf{H})d[\mathbf{H}]$ is evidently invariant under *orthogonal*, *unitary*, or *simplectic* transformations, respectively, according to the three possible choices of $\mathbf{H}$. The three ensembles are, therefore, specified and denoted after their internal symmetry: orthogonal ensemble (OE), unitary ensemble (UE), and symplectic ensemble (SE).[2]

The invariance of $P(\mathbf{H})$ in Eq. (2.1) implies that different matrices with the same eigenvalues have the same probability of occurring in the distribution. One can take advantage of this property and obtain the joint probability distribution (JPD), $\mathscr{P}(\{\epsilon_i\})$, for the *eigenvalues* ϵ_i, $i=1,2,\ldots,N$ of the matrices $\mathbf{H}$. For this purpose, it is necessary to express the various components of $\mathbf{H}$ in terms of the N eigenvalues ϵ_i and other mutually independent variables p_j, which together with ϵ_i form a complete set. The variables p_j can be integrated out and the final result for JPD is[3]

$$\mathscr{P}(\{\epsilon_i\})=C_{N_\beta}\exp\left(-\frac{1}{2}\beta\sum_i^N V(\epsilon)\right)\prod_{i<k}|\epsilon_i-\epsilon_k|^\beta, \tag{2.2}$$

where $\beta=1$ for the OE, $\beta=2$ for the UE, and $\beta=4$ for the SE. C_{N_β} is such that $\mathscr{P}$ is normalized to unity. The *universal* Jastrow factor $\Pi_{i<k}|\epsilon_i-\epsilon_k|^\beta$ comes form the Jacobian of the variable transformation. It is universal in the sense that (1) it is always present in the JPD of the eigenvalues, whenever the initial RME's distribution probability is of the form Eq. (2.1); (2) it is independent of the particular choice of $V(\mathbf{H})$ and depends only on the symmetry of the ensemble.

The JPD's of Eq. (2.2) characterize all the statistical properties of the eigenvalues of an invariant RME, with the internal symmetry discussed above. It describes the so-called energy-level statistics, if $\mathbf{H}$ is the Hamiltonian of the system.

Following Dyson,[34] the JPD can be rewritten in the following form:

$$\mathscr{P}(\{\epsilon_i\})=Z_{N_\beta}^{-1}\exp[-\beta\mathscr{H}(\{\epsilon_i\})], \tag{2.3a}$$

$$\mathscr{H}(\{\epsilon_n\})=-\sum_{i<j}\ln|\epsilon_i-\epsilon_j|+\sum_i V(\epsilon_i). \tag{2.3b}$$

The probability distribution (2.2) has the form of a Gibbs distribution for a classical, one-dimensional system of N ''particles'' ϵ_i, described by the ''Hamiltonian'' $\mathscr{H}$. The symmetry parameter β plays the role of the equilibrium ''temperature.''

These fictitious ''particles,'' namely, the eigenvalues ϵ_i, interact among each other through a pairwise logarithmic repulsion at any energy scale. The external one-body potential $V(\epsilon)$ keeps the system confined. $V(\epsilon)$ is the only quantity of the RME that can be related to the microscopic parameters of the original physical model through its *global* statistical property, namely, the density of eigenvalues (see below). The logarithmic repulsion does not depend on any microscopic detail of the real system, its origin being completely geometrical. From now on we will use this particle model analogy freely and call ''particles'' the eigenvalues of the RME's.

The *local* statistical fluctuations of $1\approx n\ll N$ eigenvalues are conveniently described by the n-level correlation functions, defined as

$$G_n(\epsilon_1,\ldots,\epsilon_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathscr{P}_N(\{\epsilon_i\})d\epsilon_1\ldots d\epsilon_n. \quad (2.4)$$

By definition, $G_n(\epsilon_1,\ldots,\epsilon_n)$ is the probability of finding simultaneously any n particles at positions $\epsilon_1,\epsilon_2,\ldots,\epsilon_n$, the positions of the remaining $N-n$ remaining unspecified. G_n are positive defined. In particular, $G_1(\epsilon)$ gives the density of particles at position ϵ, and it will be denoted by

$$\rho(\epsilon)\equiv G_1(\epsilon). \quad (2.5)$$

It is convenient to introduce the n-level cluster functions or cumulants. The *normalized* n-level cluster function is defined in the usual way of statistical mechanics. The first two cumulants are

$$Y_1(\epsilon)=\frac{G_1(\epsilon)}{\rho(\epsilon)}=1, \quad (2.6)$$

$$Y_2(\epsilon_1,\epsilon_2)=1-\frac{G_2(\epsilon_1,\epsilon_2)}{\rho(\epsilon_1)\rho(\epsilon_2)}. \quad (2.7)$$

We shall always consider the case of large $N\gg 1$. In this limit, the normalized cluster functions are very useful, since they tend to definite limits when the variables are written in the correct units. In taking the $N\to\infty$ limit, it is necessary to measure the particle positions in terms of the mean level spacing, Δ. If $\lim_{N\to\infty}\rho(\epsilon)=\rho_0=\text{const}$, $\Delta=\rho_0^{-1}$ and the dimensionless variables are simply

$$s_i=\epsilon_i/\Delta. \quad (2.8)$$

On the other hand, if $\lim_{N\to\infty}\rho(\epsilon)\neq$ const, we need to consider a rescaling of the ϵ_i with the local density or the more complicated unfolding procedure (see Sec. IV B).

In any case, the s_i will form a statistical model for an infinite number of particles with mean spacing equal to unity. It is only when written in terms of these rescaled variables that the local statistical properties of the eigenvalues of different RME's can be meaningfully compared.

We now come to the discussion on the explicit form of the potential $V(\epsilon)$ and how the statistics depend on it. The case of the GE, with potential $v(\epsilon)=\epsilon^2$ is the only one for which exact solutions for the density, two-point correlation function, and other statistical properties have been known exactly for a long time.[3] They are usually referred to as WD or classical statistics. One can show that in the large N limit, the particle density obeys Wigner's semicircle law, with a radius proportional to $N^{1/2}$. The two-point cluster function, in a region around the origin $\epsilon=0$, is translationally invariant, $Y_2(s_1,s_2)=Y_2(r)$, $r=|s_1-s_2|$. For $\beta=2$ [Gaussian unitary ensemble (GUE)], it has the famous form,

$$Y_2(r)=\left[\frac{\sin(\pi r)}{\pi r}\right]^2. \quad (2.9)$$

Similar expressions hold for the other two GE's.[3]

At small separations, the correlation function $G_2(r)=1-Y_2(r)$ vanishes, due to the phenomenon of level repulsion brought about by the logarithmic interaction,

$$G_2(r)\sim r^\beta, \quad r\ll 1. \quad (2.10)$$

Although the *global* statistics (such as the density) of the energy spectra of real systems do not follow the semicircle law, the *local* statistics of the level correlations of many chaotic and complex systems are very well described by Eq. (2.9). In particular, Eq. (2.9) describes the correlations in small metallic samples at low temperature,[5] as well as the correlations among a large but finite number of energy levels of a metallic system in the thermodynamic limit.[21]

Until recently, it was believed that the form of the confining potential could only affect the density of eigenstates, but not their local statistics, which would, therefore, be universal and equal to the WD statistics of the Gaussian ensembles. Such universality has been indeed proved recently in a rigorous way, for a large class of potentials, which confine the system strongly, that is, when $V(\epsilon)$ increases faster than $|\epsilon|$.[26,27]

In what follows, we will consider two kinds of non-Gaussian potentials, which confine the system weakly, with the following asymptotic behavior.

(1) Power-law potential,[30,35,36]

$$V(\epsilon)=\frac{A}{2}|\epsilon|^\alpha, \quad 0<\alpha<1, \quad |\epsilon|\to\infty. \quad (2.11)$$

(2) Squared logarithmic potential,[28,31]

$$V(\epsilon)=\frac{A}{2}\ln^2|\epsilon|, \quad |\epsilon|\to\infty. \quad (2.12)$$

It is a legitimate mathematical interest to investigate these cases and see if the WD universality is preserved. However, these RME's have also a physical interest, since they have been suggested by random transfer matrix models of disordered conductors.

III. MEAN-FIELD THEORY

We now consider a MFT analysis of the classical one-dimensional plasma model of eigenvalues $\mathscr{P}(\{\epsilon_i\})$, into which the original probability density of RME has been mapped. We first define a continuous limit of this model,[2,34] valid in the asymptotic limit of large N, by introducing the particle density $\rho(\epsilon)=\Sigma_i^N\delta(\epsilon-\epsilon_i)$. In this limit, we will assume that the Coulomb gas is a classical fluid with a continuous and smooth macroscopic density. By substituting this definition of $\rho(\epsilon)$ in Eq. (2.3b), the Hamiltonian $\mathscr{H}(\{\epsilon_n\})$ becomes an *energy* functional, $\mathscr{H}[\rho(\epsilon)]$, of $\rho(\epsilon)$

$$\mathscr{H}[\rho(\epsilon)]=-\frac{1}{2}\int_{-\infty}^{+\infty}d\epsilon\int_{-\infty}^{+\infty}d\epsilon'\rho(\epsilon)\rho(\epsilon')\ln|\epsilon-\epsilon'| + \int_{-\infty}^{+\infty}d\epsilon\rho(\epsilon)V(\epsilon). \quad (3.1)$$

When N is large but finite, the assumption of a smooth density is only an approximation. Consequently, the first

term of the right-hand side of Eq. (3.1) does not reproduce exactly the corresponding term of Eq. (2.3b), because it neglects the two-level correlations, that is, it allows the presence of the "charges" $\rho(\epsilon)d\epsilon$ and $\rho(\epsilon')d\epsilon'$ at separations $\epsilon-\epsilon'\to 0$. Since the interaction $\ln|\epsilon-\epsilon'|$ is singular, this approximation has an effect, albeit small, because N is large. One can compute the correction in a $1/N$ expansion to this term and see that it is of the form[34]

$$\delta\mathscr{H}[\rho(\epsilon)]=-\frac{1}{2}\int_{-\infty}^{+\infty}d\epsilon\rho(\epsilon)\ln(\rho(\epsilon)). \tag{3.2}$$

It is convenient to introduce a grand canonical potential $\Omega[\rho]=\mathscr{H}[\rho]-\mu\mathscr{N}[\rho]$, where $\mathscr{N}$ is the particle number functional $\mathscr{N}[\rho]=\int\rho(\epsilon)d\epsilon$ and μ is the chemical potential. The average density of particle $\langle\rho(\epsilon)\rangle$ can be expressed in terms of the functional integral,

$$\langle\rho(\epsilon)\rangle=Z^{-1}\int\rho(\epsilon)e^{-\beta\Omega[\rho]}\mathscr{D}\rho;\quad Z=\int e^{-\beta\Omega[\rho]}\mathscr{D}\rho. \tag{3.3}$$

The one-body potential $V(\epsilon)$ acts as a source term for the field $\rho(\epsilon)$ and $\langle\rho(\epsilon)\rangle$ can be expressed as

$$\langle\rho(\epsilon)\rangle=(\beta Z)^{-1}\delta Z/\delta V(\epsilon). \tag{3.4}$$

Up to now we have only assumed the existence of a smooth particle density $\langle\rho(\epsilon)\rangle$, necessary in taking the continuous limit of the Coulomb plasma, which presumably is a good approximation when N is large. The MFT, based on the continuous approximation, amounts to neglecting any entropy fluctuations about the average and using the saddle-point approximation in the integral equation for $\langle\rho(\epsilon)\rangle$. The MFT equation obeyed by $\langle\rho(\epsilon)\rangle_{\rm MF}$ is therefore[34]

$$\int_{-\infty}^{+\infty}d\epsilon'\langle\rho(\epsilon')\rangle_{\rm MF}\ln|\epsilon-\epsilon'|=V(\epsilon)-\mu, \tag{3.5}$$

where the "Lagrange multiplier" μ is to be found from the normalization condition $\int\langle\rho(\epsilon)\rangle_{\rm MF}d\epsilon=N$. The MFT interpretation of this equation is very natural, since it represents the condition of mechanical equilibrium for the charge density $\langle\rho(\epsilon)\rangle_{\rm MF}$ subject to the external potential $V(\epsilon)$. Such a MFT approximation completely disregards the entropy part $\mathscr{S}[\rho]=-\int d\epsilon\langle\rho(\epsilon)\rangle\ln\langle\rho(\epsilon)\rangle$ in the *free-energy* functional, $\mathscr{F}[\rho]=\mathscr{H}[\rho]-T\mathscr{S}[\rho]$, and is exactly applicable only for $\beta=\infty$. However, the long-range nature of the pairwise interaction in Eq. (3.1) makes the MFT approximation valid in the bulk of the spectrum, even at finite β, when $N\to\infty$. Indeed Dyson[34] has calculated the first correction to this equation in a $1/N$ expansion and shown that the more accurate equation for $\langle\rho(\epsilon)\rangle$ reads

$$\int_{-\infty}^{+\infty}d\epsilon'\langle\rho(\epsilon')\rangle\ln|\epsilon-\epsilon'|+\frac{\beta-2}{2\beta}\ln\langle\rho(\epsilon)\rangle=V(\epsilon)-\mu. \tag{3.6}$$

The second term on the left-hand side of Eq. (3.6) is the sum of two parts. The term proportional to $-T=-1/\beta$ comes clearly from the entropy contribution to the free energy. The part *independent* of β is generated in the passage to the continuous limit as discussed above [see Eq. (3.6)]. The combination of these two terms acts as an entropy contribution multiplied by an effective temperature, $T^\star(\beta)$,

$$T^\star(\beta)=\beta^{-1}-1/2, \tag{3.7}$$

which vanishes for $\beta=2$. We can write the free energy, at equilibrium, of the system in the following form:

$$\begin{aligned}F=&-\frac{1}{2}\int_{-\infty}^{+\infty}d\epsilon\int_{-\infty}^{+\infty}d\epsilon'\langle\rho(\epsilon)\rangle\langle\rho(\epsilon')\rangle\ln|\epsilon-\epsilon'|\\&+\int_{-\infty}^{+\infty}d\epsilon\langle\rho(\epsilon)\rangle V(\epsilon)\\&-T^\star(\beta)\int_{-\infty}^{+\infty}d\epsilon[-\langle\rho(\epsilon)\rangle\ln\langle\rho(\epsilon)\rangle].\end{aligned} \tag{3.8}$$

For the class of confining potentials of Eq. (2.11) and Eq. (2.12), one can show[34] that the relative contribution of the correction to the MFT equation is of order $N^{-1}\ln N$, and, therefore, vanishes in the thermodynamic limit. For the time being, we will neglect this correction and concentrate on the MFT Eq. (3.5).

A. Solution of the eigenvalue density

We now present the solution of the integral equation (3.5) for the particle density $\rho(\epsilon)_{\rm MF}\equiv\langle\rho(\epsilon)\rangle_{\rm MF}$. The solution of this equation, confined to the region $-D<\epsilon<D$, can be found using the Cauchy method[37] and is given by

$$\rho_{\rm MF}(\epsilon)=\frac{1}{\pi^2}\sqrt{D^2-\epsilon^2}\ \mathrm{Re}\int_0^D\frac{dV/d\xi}{\sqrt{D^2-\xi^2}}\frac{\xi\,d\xi}{\xi^2-\epsilon_+^2}, \tag{3.9}$$

where $\epsilon_+=\epsilon+i0$ and the band edge D is to be found from the normalization condition. We will discuss separately the two cases of a power-law and logarithmic potential.

1. Power-law confinement, $V(\epsilon)=A/2|\epsilon|^\alpha$

The integral equation for $\langle\rho(\epsilon)\rangle_{\rm MF}$ becomes

$$\rho_{\rm MF}(\epsilon)=\frac{A\alpha}{\pi^2}\sqrt{D^2-\epsilon^2}\ \mathrm{Re}\int_0^D\frac{\xi^\alpha}{\sqrt{D^2-\xi^2}}\frac{d\xi}{\xi^2-\epsilon_+^2}. \tag{3.10}$$

We need to distinguish further between the cases $\alpha\geqslant 1$ and $\alpha<1$. For $\alpha\geqslant 1$, the integral in Eq. (3.10) is divergent in the thermodynamic limit $N\to\infty$, when the band edge D also goes to infinity. The main contribution to the integral is made by the region $\xi\sim D$ and, therefore, for any fixed $\epsilon\ll D$, one can neglect the ϵ dependence in the integrand of Eq. (3.10). Then the mean level density tends to a *constant*, $\rho\to N^{1-1/\alpha}$. Thus, we reach the important conclusion that, for $\alpha\geqslant 1$, there exists translational invariance in the ϵ space in the $N\to\infty$ limit, exactly as in the case of the Gaussian ensemble. Therefore, for $\alpha\geqslant 1$, we have a condition of strong confinement, and the corresponding local statistics belong to the WD universality class, in agreement with Refs. 26 and 27. On the other hand, for $\alpha<1$, the integral in Eq. (3.10) is convergent even in the limit $D\to\infty$. For finite N (and, therefore, finite D), the integral can be calculated by transforming the original contour of integration into a sum of two pieces,

$\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 is the negative imaginary axis, and Γ_2 is a part of the positive real axis $[D{:}\infty]$. Since we are interested only in the real part contribution, the integral over Γ_2 does not contribute and we end up with

$$\rho_{\mathrm{MF}}(\epsilon)=\frac{A\alpha}{\pi^2}\sin\left(\frac{\pi\alpha}{2}\right)\sqrt{D^2-\epsilon^2}\int_0^D\frac{\xi^\alpha}{\sqrt{D^2+\xi^2}}\frac{d\xi}{\xi^2+\epsilon^2} \quad (3.11)$$

$$=\frac{A\alpha^2}{\pi D}C_{\alpha/2}\left(\frac{D}{2}\right)^\alpha\frac{\sqrt{1-z^2}}{|z|^{1-\alpha}}F\left(\frac{1}{2},\frac{1+\alpha}{2};\frac{3}{2};1-z^2\right), \quad (3.12)$$

$$C_\alpha=\frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)}=\frac{1}{2}+O(\alpha^2), \quad (3.13)$$

where $z=\epsilon/D$ and $F(a,b;c;x)$ is a hypergeometric function. The band edge is found from the normalization condition to be

$$D=2\left(\frac{N\Gamma^2(\alpha/2)}{2A\Gamma(\alpha)}\right)^{1/\alpha}. \quad (3.14)$$

The limiting function $\rho^\infty_{\mathrm{MF}}(\epsilon)=\lim_{N\to\infty}\rho_{\mathrm{MF}}(\epsilon)$ is immediately obtained from Eq. (3.12) by taking the limit $z=\epsilon/D\to 0$:

$$\rho^\infty_{\mathrm{MF}}(\epsilon)=\frac{A\alpha}{2\pi}\tan\left(\frac{\alpha\pi}{2}\right)\frac{1}{|\epsilon|^{1-\alpha}}. \quad (3.15)$$

Therefore, for $\alpha<1$, the mean density in the thermodynamic limit, $\rho^\infty_{\mathrm{MF}}(\epsilon)$, does not scale as a power of N and, especially, is not translational invariant. $\rho^\infty_{\mathrm{MF}}$ is divergent at $\epsilon=0$. However, as we will show in Sec. V, this is an artifact of the MFT solution and the exact density is sharply peaked, but finite at the center. We can say that the system undergoes a ''phase transition'' at $\alpha=1$ and the symmetry that is broken is the translational invariance of the problem in the limit $N\to\infty$.

2. Squared logarithmic confinement, $V(\epsilon)=A/2\ \ln^2(|\epsilon|)$

In order to solve the integral equation for the MFT density, it is convenient to represent this potential as a limit of combinations of the power-law potential:[38]

$$V(\epsilon)=\ln^2|\epsilon|=\lim_{\alpha\to 0}[\alpha^{-2}(|\epsilon|^\alpha-1)^2]. \quad (3.16)$$

The MFT density in this case is

$$\rho_{\mathrm{MF}}(\epsilon)=\frac{4A}{\pi D}\sqrt{1-z^2}\left[\left(\frac{D}{2}\right)^{2\alpha}C_\alpha\frac{F\left(\frac{1}{2},\frac{1}{2}+\alpha;\frac{3}{2};1-z^2\right)}{z^{1-2\alpha}}-\frac{1}{2}\left(\frac{D}{2}\right)^\alpha C_{\alpha/2}\frac{F\left(\frac{1}{2},\frac{1+\alpha}{2};\frac{3}{2};1-z^2\right)}{z^{1-\alpha}}\right], \quad (3.17)$$

where D satisfies the equation

$$N=\frac{2A}{\alpha}\left[C_\alpha\left(\frac{D}{2}\right)^{2\alpha}-C_{\alpha/2}\left(\frac{D}{2}\right)^\alpha\right]. \quad (3.18)$$

The expression for the MFT density for N finite takes a simpler expression after the limit $\alpha\to 0$ is taken,

$$\rho_{\mathrm{MF}}(\epsilon)=A\frac{\arcsin\sqrt{1-\epsilon^2/D^2}}{\pi|\epsilon|}. \quad (3.19)$$

The band edge is now an exponential function of N:

$$D=2\exp(N/A). \quad (3.20)$$

The singularity at $\epsilon=0$ is again an artifact of the MFT solution. On the other hand, we will see that in the bulk of the spectrum, the mean density is accurately described by Eq. (3.19).

IV. THE TWO-LEVEL CORRELATION FUNCTION

In this section, we study the two-level correlation function using the MFT developed in the previous section. Some of these results have been already derived for $\beta=2$, using the different method of the orthogonal polynomials.[28,31] MFT will allow us to generalize these results to any value of β.

A. MFT integral equation for the two-level correlation function

Within the continuous formalism of the previous section, let us consider the following definition of the density-density correlation function $R_2(\epsilon,\epsilon')$:

$$R_2(\epsilon,\epsilon')=\frac{\langle\rho(\epsilon)\rho(\epsilon')\rangle}{\langle\rho(\epsilon)\rangle\langle\rho(\epsilon')\rangle}-1. \quad (4.1)$$

Using the definition of $\rho(\epsilon)$, one sees that $R_2(\epsilon,\epsilon')$ differs from the two-level *cluster* function $Y_2(\epsilon,\epsilon')$ defined in Eq. (2.9) of Sec. I by the singular self-correlation,

$$Y_2(\epsilon,\epsilon')=\frac{1}{\langle\rho(\epsilon)\rangle}\delta(\epsilon-\epsilon')-R_2(\epsilon,\epsilon'). \quad (4.2)$$

The correlation function $R_2(\epsilon,\epsilon')$ can be easily expressed in terms of a functional derivative of $\langle\rho(\epsilon)\rangle$ with respect to $V(\epsilon)$ in Eq. (3.3). By using the relation $\delta\Omega/\delta V(\epsilon)=\rho(\epsilon)$, one obtains[39]

$$R_2(\epsilon,\epsilon')=-\frac{\beta^{-1}}{\rho(\epsilon)\rho(\epsilon')}\frac{\delta\langle\rho(\epsilon)\rangle}{\delta V(\epsilon)}. \quad (4.3)$$

Within the MFT, it is possible to write down an integral equation for the two-particle correlation function.[39] By taking the functional derivative $\delta/\delta V(\epsilon)$ in Eq. (3.5) and using Eq. (4.3), one obtains

$$\int_{-\infty}^{+\infty}d\epsilon''\rho(\epsilon'')_{\mathrm{MF}}\rho(\epsilon)_{\mathrm{MF}}R_2(\epsilon,\epsilon'')\ln|\epsilon'-\epsilon''|=-\beta^{-1}\delta(\epsilon-\epsilon')+\beta^{-1}\delta\mu/\delta V(\epsilon). \quad (4.4)$$

This important equation is sometimes used to claim and justify, within the approximation of MFT, the universality of the correlations in RMT, or rather their independence of the potential $V(\epsilon)$. The argument is usually the following. In the large N limit, one implicitly assumes that, at least in the region of interest, the average density scales like N and goes

to a constant, $\langle\rho(\epsilon)\rangle=\rho_0$. In this case, since the two-body interaction is translational invariant, the two-particle correlation function must be translational invariant as well. In particular, the variational derivative of the chemical potential, which is the only term that depends explicitly on the confining potential and is not translationally invariant, must vanish.[40] If we now introduce dimensionless variables $s=\epsilon/\Delta$, rescaled by the mean level spacing $\Delta=\rho^{-1}$, one obtains an equation completely independent of $V(\epsilon)$,

$$\beta\int_{-\infty}^{+\infty} ds R_2(s-s'')\ln|s'-s''|=-\delta(s-s'). \tag{4.5}$$

From this one concludes that the limiting correlation function $R_2(\epsilon-\epsilon')$ cannot depend on the choice of $V(\epsilon)$ and it is universal. Its asymptotic behavior is easily found by solving Eq. (4.5) by Fourier transformation. The result is of course the universal WD behavior of the Gaussian ensembles,

$$R_2(s-s')\sim-\frac{1}{\pi^2\beta}\frac{1}{(s-s')^2}. \tag{4.6}$$

This reasoning is crucially based on the assumption that the average density tends to a constant in the thermodynamic limit, so that it can be simply rescaled away. But we have seen that there are cases in which, if the potential is weak enough, the density is not at all a constant in the $N\to\infty$ limit and, in fact, it decreases steeply with ϵ. The system is not translationally invariant and we cannot carry out the same simple rescaling as before. Thus, the proof of the universality of the correlations is not applicable, and one may expect $R_2(\epsilon,\epsilon')$ to be different from the WD form. It is often said that in this case, MFT breaks down and its equations become invalid. This statement is not completely correct. The MFT still gives reasonable and, in fact, accurate results for the particle density in the bulk of the spectrum (as the comparison with numerical simulations will show). Therefore, this theory should work also for the correlations, when properly applied. Of course, we can no longer disregard the presence of a nonconstant density, which is responsible for the deviations from universality. Below we suggest a simple way out of this problem through the notion of *spectrum unfolding*.

B. Spectrum unfolding. Modified MFT equation for weak confinement

On comparing the correlation functions of different ensembles, it is necessary to choose energy units such that the mean level spacing is equal to one. This is trivial when the average density is a constant ρ, since in that case, $\Delta=\rho^{-1}$ and the convenient variable is $s=\epsilon/\Delta=\epsilon\rho$. The presence of a nonuniform density in the thermodynamic limit makes this linear rescaling impossible, even locally if the density is a rapidly varying function of the energy, as in the case of the logarithmic confinement. This is a common situation in the study of complex energy spectra, where one needs to subtract off the unwanted effects of a nonconstant average density, in order to analyze the fluctuations around the average density itself. This difficulty is handled by the so-called ''unfolding procedure,'' namely, the introduction of the variable $s=s(\epsilon)$, in terms of which the density becomes a constant. The variable that serves this purpose is the integrated density of states,

$$s(\epsilon)=\int_0^{\epsilon}\langle\rho(\epsilon)\rangle d\epsilon. \tag{4.7}$$

By particle conservation,

$$\langle\rho(\epsilon)\rangle d\epsilon=\langle\tilde{\rho}(s)\rangle ds, \tag{4.8}$$

where $\langle\tilde{\rho}(s)\rangle$ is the average density in the variable s. From Eqs. (4.7) and (4.8), it follows that

$$\langle\tilde{\rho}(s)\rangle=1\,,\quad\langle s\rangle=1\,. \tag{4.9}$$

The definition of the two-particle correlation function in the unfolding variables becomes

$$R_2(s,s')=\frac{\langle\rho(\epsilon_s)\rho(\epsilon_s')\rangle}{\langle\rho(\epsilon_s)\rangle\langle\rho(\epsilon_s')\rangle}-1\,, \tag{4.10}$$

where ϵ_s is the inverse function of $s(\epsilon)$. Using these definitions in Eq. (4.4), the integral equation for $R_2(s,s')$ is[41]

$$\int_{-\infty}^{+\infty} ds'' R_2(s,s'')\ln|\epsilon_{s''}-\epsilon_{s'}|=-\beta^{-1}\delta(s-s'). \tag{4.11}$$

The MFT equation for $\langle\tilde{\rho}(s)\rangle$ reads

$$\int_{-\infty}^{+\infty} ds'\langle\tilde{\rho}(s')\rangle\ln|\epsilon_s-\epsilon_{s'}|=V(\epsilon_s)-\mu. \tag{4.12}$$

Equations (4.11) and (4.12) together mean that $\beta R_2(s,s')$ is the solving kernel of[39]

$$\int_{-\infty}^{+\infty} ds\,\psi(s')\ln|\epsilon_s-\epsilon_{s'}|=\varphi(s)+\text{const}, \tag{4.13}$$

that is,

$$\psi(s)=\int_{-\infty}^{+\infty} ds\beta R_2(s,s')\varphi(s'). \tag{4.14}$$

The additive constant in Eq. (4.13) has to be chosen such that $\int_{-\infty}^{+\infty} ds\,\psi(s)=0$, since the variations in $\langle\tilde{\rho}(s)\rangle$ must occur at constant N.[39]

Equation (4.11) is the integral equation for the correlation function $R_2(s,s')$ in case of *weak confinement*. From this equation we can see that, in contrast with Eq. (4.4) valid for strong confinement, now $R_2(s,s')$ depends implicitly on the original confining potential through the unfolding variable $s(\epsilon)$. Thus it will not, in general, be universal.

C. Solution of the MFT integral equation

The solving kernel of the integral equation (4.13) must be found separately for each unfolding function $\epsilon(s)$, which in turn depends on the confining potential. We are interested in the correlations in the bulk of the spectrum, where MFT works and, thus, we can use the MFT expression for the density in order to obtain the unfolding function.

We first consider the case of the power-law potential with $0<\alpha<1$, which is the simplest case of weak confinement

where, in principle, nonclassical correlations can arise. Using Eqs (3.15) and (4.7), the unfolding function in the bulk of the spectrum in the large N limit is

$$s(\epsilon)=\lambda\ \mathrm{sgn}(\epsilon)\int_0^{|\epsilon|} d\epsilon'\,\frac{1}{\epsilon'^{(1-\alpha)}}=\lambda|\epsilon|^{\alpha}\mathrm{sgn}(\epsilon),$$

$$\lambda=\frac{A}{2\pi}\tan\left(\frac{\alpha\pi}{2}\right), \tag{4.15}$$

or

$$\epsilon(s)=\lambda^{-1/\alpha}|s|^{1/\alpha}\mathrm{sgn}(s). \tag{4.16}$$

In order to simplify the derivation, we choose $\alpha=1/(2k+1)$, $k=1,2,\ldots$. In this case,

$$\epsilon(s)=\left(\frac{s}{\lambda}\right)^{(2k+1)}. \tag{4.17}$$

The integral equation Eq. (4.13) is now

$$\int_{-\infty}^{+\infty} ds\,\psi(s')\ln|s^{(2k+1)}-s'^{(2k+1)}|=\varphi(s)+\mathrm{const.} \tag{4.18}$$

We change variables $s^{(2k+1)}=x, s_x=x^{1/(2k+1)}$, and we define

$$\tilde{\psi}(x)=\frac{1}{2k+1}\psi(s_x)x^{2k/(2k+1)}, \tag{4.19}$$

$$\tilde{\varphi}(x)=\varphi(s_x), \tag{4.20}$$

$$\tilde{R}_2(x,x')=\frac{1}{(2k+1)^2}(xx')^{-2k/(2k+1)}R_2(s_x,s_{x'}), \tag{4.21}$$

in terms of which Eqs. (4.13) and (4.14) become

$$\int_{-\infty}^{+\infty} dx'\,\tilde{\psi}(x')\ln|x-x'|=\tilde{\varphi}(x), \tag{4.22}$$

$$\tilde{\psi}(x)=\int_{-\infty}^{+\infty} dx'\,\beta\tilde{R}_2(x,x'),\tilde{\varphi}(x'), \tag{4.23}$$

and the condition on $\tilde{\psi}(x)$ is now $\int_{-\infty}^{+\infty} dx\,\tilde{\psi}(x)=0$. This is exactly the familiar case of pure logarithmic interaction with strong confinement considered above [see Eq. (4.5)]. The solution for the solving kernel $\tilde{R}_2(x,x')$ is the WD universal correlation given in Eq. (4.6). Using the relation between $\tilde{R}_2(x,x')$ and $R_2(x,x')$ and going back to the original variables, s and s', we obtain

$$R_2(s,s')=\frac{(2k+1)^2}{\pi^2\beta}\frac{(ss')^{2k}}{(s^{2k+1}-s'^{2k+1})^2}. \tag{4.24}$$

This correlation function is not translational invariant. However, we restrict ourselves to the case in which both s and s' are in the bulk and $\Delta s=|s-s'|\ll s$. In this limit, we can expand the previous expression in power of $\Delta s/s$. The correlation becomes translationally invariant, but what is even more important is that we get back exactly Eq. (4.6), namely, the universal WD correlation function. Thus we conclude that, at the MFT level, the power-law ensemble has completely classical statistics, even when the translational invariance is broken in the thermodynamic limit and the unfolding procedure is necessary.

We now consider the case of the double logarithmic potential. The unfolding function in the bulk of the spectrum, $|\epsilon|\gg 1$ is now

$$s(\epsilon)=\frac{A}{2}\,\mathrm{sgn}(\epsilon)\int_0^{|\epsilon|} d\epsilon'\,\frac{1}{\epsilon'}=\frac{A}{2}\ln|\epsilon|\mathrm{sgn}(\epsilon), \tag{4.25}$$

or

$$\epsilon(s)=e^{2|s|/A}\mathrm{sgn}(s)\approx 2\,\sinh\left(\frac{2s}{A}\right). \tag{4.26}$$

We plug this function into Eq. (4.13) and we perform the change of variable, $\sinh(2s/A)=x$, with the corresponding redefinition of ψ, φ, and R_2. Following the same procedure used for the power-law potential, we arrive at

$$R_2(s,s')=-\frac{1}{\pi^2A^2\beta}\frac{\cosh\left(\frac{2s}{A}\right)\cosh\left(\frac{2s'}{A}\right)}{\sinh^2\left(\frac{s-s'}{A}\right)\cosh^2\left(\frac{s+s'}{A}\right)}. \tag{4.27}$$

Again, the correlation function is not translational invariant. If s and s' are both in the bulk, $|s'|,|s|\gg 1$ and have the same sign, we obtain the translational invariant expression,

$$R_{2(n)}(s,s')=-\frac{1}{\pi^2A^2\beta}\frac{1}{\sinh^2\left(\frac{s-s'}{A}\right)}, \tag{4.28}$$

where the subscript (n) stands for "normal" part. If $|s-s'|>A$, the argument of sinh cannot be expanded and A does not scale away. Thus the correlations for the double logarithmic potential are no longer universal and in contrast with the power-law behavior of the WD class, they decrease exponentially in agreement with the exact solution for $\beta=2$ by Muttalib *et al.*[28]

However, there is one more great surprise, pointed out in Ref. 31 for $\beta=2$, which is the reappearance of strong correlations at $s'\approx -s$. In fact, when s and s' are in the bulk but have different sign, $s'=-s+\Delta s$, we obtain

$$R_{2(a)}(s,s')=-\frac{1}{\pi^2A^2\beta}\frac{1}{\cosh^2\left(\frac{s+s'}{A}\right)}, \tag{4.29}$$

where the subscript (a) stands for "anomalous" part. This anomalous part of the correlation function breaks dramatically the translational invariance. Its remarkable property is a narrow correlation hole at $s'\approx -s$ with a depth, controlled by A, that does not decrease when $|s-s'|\simeq 2|s|\to\infty$. Notice also that the two regions, where $R_{2(n)}(s,s')$ and $R_{2(a)}(s,s')$ are nonzero, are separated by a very large distance when s and s' are in the bulk of the spectrum and N is large.

In Ref. 31, a simplified application of the method of the orthogonal polynomials, valid for $q=e^{-\pi^2 A}\ll 1$, was used to derive the two-level cluster function for the case $\beta=2$, yielding the result:

$$Y_2(s,s')=\frac{1}{\pi^2 A^2}[\sin\pi(s-s')]^2 \times\frac{\cosh\left(\frac{2s}{A}\right)\cosh\left(\frac{2s'}{A}\right)}{\sinh^2\left(\frac{s-s'}{A}\right)\cosh^2\left(\frac{s+s'}{A}\right)}. \quad (4.30)$$

The normal part of $Y_2(s,s')$,

$$Y_{2(n)}(s,s')=\frac{1}{\pi^2 A^2}\left[\frac{\sin\pi(s-s')}{\sinh\left(\frac{s-s'}{A}\right)}\right]^2, \quad (4.31)$$

is identical to the exact solution of Muttalib *et al.*,[28] for the same small values of q, and it is also identical to the exact solution of Moshe *et al.*,[24] for the case of a RME with a symmetry breaking term.

Here, using the MFT theory, we have generalized this result to any β. As in all other MFT calculations, only the asymptotic form of the R_2 is obtainable within the MFT treatment. In particular, the oscillatory function that vanishes as $|s-s'|\to 0$, and thus gives rise to a residual level repulsion, is totally out of reach. In its place we have $1/\beta$, which is the average of the oscillations. Thus, MFT and the method of orthogonal polynomials (which is exact) are completely consistent even in the case of weak confinement, where deviations from WD occur.

The appearance of anomalous correlations at $s'\approx -s$ is the result of the system trying to develop Poissonian-like correlations at $s'\approx s$, while, at the same time, complying with the $U(N)$ invariance, which forces a normalization sum rule on R_2 to be satisfied even in the large N limit.[31] The sum rule reads

$$\int_{-\infty}^{+\infty} R_2(s,s')ds'=0\,, \quad (4.32)$$

or, in terms of $Y_2(s,s')$,

$$\int_{-\infty}^{+\infty} Y_2(s,s')ds'=1\,, \quad (4.33)$$

and it must be satisfied in the case of a $U(N)$ invariant RME, because of the long-range nature of the universal logarithmic interaction always present in such ensembles.[31] The normal part alone of $Y_2(s,s')$, given in Eq. (4.31), does not satisfy Eq. (4.32). But the sum rule deficiency

$$\eta=1-\int_{-\infty}^{+\infty} Y_{2(n)}(s)ds, \quad (4.34)$$

is taken care of by the anomalous correlations:[42]

$$\int_{-\infty}^{+\infty} Y_{2(a)}(s)ds=\eta. \quad (4.35)$$

However, no matter how this is realized in practice, what is important is first the fact that the $U(N)$ invariant RME with soft confinement manages to develop the exponentially decaying two-level correlation function of Eq. (4.31) in an infinitely large energy region in the bulk of the spectrum, where the anomalous correlations are irrelevant. Second, the normal part of correlation function is exactly equal to the expression obtained for the RME (1.1), where the symmetry is explicitly broken. This occurrence has been interpreted in Ref. 31 as a signal of the spontaneous breakdown of the $U(N)$ invariance in the case of soft confinement, with the parameter η playing the role of the order parameter.

The physical interpretation of the anomalous correlations at $s'=\approx -s$ in terms of Coulomb plasma is very simple if one looks at how the logarithmic pairwise interaction is transformed in the unfolded variables,

$$f(s,s')=-\ln|\sinh(2s/A)-\sinh(2s'/A)|. \quad (4.36)$$

In the unfolded variables, the interaction is no longer translational invariant. We can try to rewrite it in a form that looks more translationally invariant in the following way:

$$f(s,s')=-\ln\left|2\,\sinh\left(\frac{s-s'}{A}\right)\cosh\left(\frac{s+s'}{A}\right)\right| =-\ln\left|2\,\sinh\left(\frac{s-s'}{A}\right)\right|-\ln\,\cosh\left[\frac{s'-(-s)}{A}\right]. \quad (4.37)$$

We see that the interaction splits into two terms: the first is an ordinary, translationally invariant, repulsion between two particles located at s' and s; the second, however, represents the interaction between a particle at position s' with the *image charge* at $-s$ of a particle at position s. In other words, a particle at position s will repel particles around it. But its image, with respect to the origin, will also repel particles around the position $-s$. Notice that the image charge term depends on A—it increases when A decreases—but it is less singular than the direct term, because of the functional dependence on $\cosh(s+s')$.

So far, we have always restricted ourselves to bulk properties of the correlations. As we will show in more detail in the next section, there are some interesting effects in the correlation function in the center of the spectrum. There, even the power-law potential for $\alpha<1$ displays deviations from the universal WD behavior.

V. MONTE CARLO SIMULATIONS

The statistical properties of the one-dimensional classical system[43] in thermodynamical equilibrium, the probability distribution of which is given by Eqs. (2.3a) and (2.3b), can be conveniently studied by carrying out Monte Carlo (MC) simulations. The MC method is useful, because it is not restricted to a particular value of β, as for the method of orthogonal polynomials, and it is very accurate in all the regions of the spectrum, in contrast with MFT, which is good only in the bulk. It also allows the evaluation of important statistical quantities like the level spacing distribution function (LSDF) and the number variance in a straightforward way. Finally, it can be used to study level statistics, where the

''particle'' interaction is more general than the simple logarithmic interaction considered in RMT. These more complicated interactions have been shown to play an important role in some disordered systems.[39]

The nature of MC simulations is best illustrated with an example. Suppose that we want to calculate the mean particle density $\rho(\epsilon)$ as a function of the position ϵ. According to Eq. (2.5), we need to perform a multidimensional integral, which up to an overall normalization is equivalent to a ''thermal'' average over an ensemble of particle configurations. The MC method replaces this ensemble average with a ''time'' average, but the time evolution is determined by equations that are artificial and chosen for convenience. To calculate $\rho(\epsilon)$, one partitions the real axis into bins with boundaries ϵ_n determined by

$$\epsilon_n = n\Delta\epsilon, \quad n = \pm 1,2,3,\ldots, \tag{5.1}$$

where $\Delta\epsilon$ is the width of the bins. At the end of each time step (to be defined below), we obtain an updated configuration $\{\epsilon_i\}$, $i=1,\ldots,N$, and we add one to a bin if a particle in this configuration lies in it. As ''time'' progresses, the number of particles in a bin will become proportional to the mean density at the position where the bin is centered. For the evolution density, we have taken a simple Metropolis algorithm, which works in the following way. At each time step or sweep, we scan through the particles and attempt to move each one. Actually, in each sweep, we pick N times one particle at random in the system, so it is possible that in one particular sweep, one particle is chosen more than once and another is not touched. The moving attempt involves picking at random any position between the particle that proceeds and the one that follows the particle that we are trying to move and taking this position as the new attempted position. The attempted move is chosen in this particular way simply to optimize the convergence rate of the algorithm. It has the important property that if we start with an order sequence of particles, $\epsilon_1 < \epsilon_2 < \ldots \epsilon_N$, the sequence remains ordered in the time evolution. To decide whether or not to accept the move, we calculate the change ΔE to the system's energy that would occur if the particle were moved to the new position. If ΔE is negative the move is accepted, and the particle is given the new position. If ΔE is positive, the move is accepted conditionally. One picks a random number between 0 and 1 and accepts the move if this number is smaller than $\exp(-\beta\Delta E)$. Before measuring any quantity, the system must reach equilibrium and this is obtained by a certain number of ''warming-up'' sweeps that reach a ''typically'' sampled configuration.

We have carried out simulations over systems with up to 200 particles. We noticed that the simulations are in all cases very stable even for smaller N, and, therefore, we have typically worked with systems of $N=100$ particles. Equilibration is usually reached very fast and we have typically used 10^5 sweeps to warm up the system. The averages are taken over 10^6 sweeps and the statistics that we are able to obtain are usually excellent.

To make sure that the method works and is able to give numerically accurate results we have first studied the density, two-particle correlation function, spacing distribution, and particle variance of the three Gaussian ensembles that are exactly known.[3] For these ensembles the MC simulations reproduce the known result very accurately. For example, in the calculation of $P(\sigma)$, the method is able to detect the small deviation that there is between the exact result and the Wigner surmise. We now proceed to discuss in detail the results for the different quantities of interest for the case of power-law and double logarithmic ensemble.

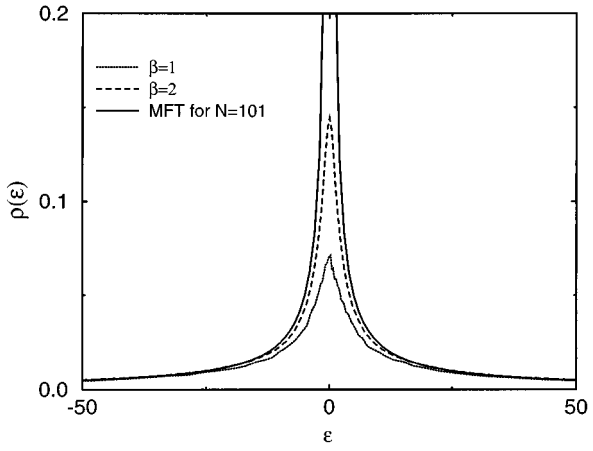

FIG. 1. Density of states for the logarithmic potential for $A=0.5$. The MC results for $\beta=1,2$ are plotted in a small region around the origin $\epsilon=0$ with the MFT density, which corresponds to $\beta=\infty$ and diverges at $\epsilon=0$. For $\beta=1,2$, and 4 (not shown), the density is finite at $\epsilon=0$. All the curves rapidly collapse on top of each other away from the origin.

A. The density of states

The MC evaluation of the average density of states $\rho(\epsilon)$ is carried out, as explained in the example above. In Fig. 1, we plot this quantity for the logarithmic potential, for $A=0.5$ and $\beta=1,2$. The case of weak power-law potential, with $0 \leqslant \alpha < 1$, is qualitatively the same.[30] The agreement between the MC result and the MF expression ρ_{MF} given in Eq. (3.19) is very good, except around the origin, where in contrast to MFT, the simulations give a sharply picked, but finite density at $\epsilon=0$. For $\beta=2$, the MC result coincides with the value obtained by the method of orthogonal polynomials.[44] The first hundred of these for the power-law potential can be easily generated numerically[44] and the density at $\epsilon=0$ obtained from them converges very fast. This fast N independence of the center of the spectrum is also seen very well with the simulations and it is an important property of the particle density for weak confinement. We refer to it as the ''incompressibility'' of the core of the particle-density distribution. In contrast to the case of strong confinement, $V(\epsilon) \sim |\epsilon|^{\alpha}, \alpha \geqslant 1$, where the density at the center of the spectrum scales with $N^{\alpha-1}$, for $\alpha<1$ the confining potential is too weak to ''compress'' the particle in the core region near the origin. After the initial formation of the sharp but finite peak at $\epsilon=0$ (which happens for $N \ll 100$), on adding more particles to the system, these always go to the ends of the distribution instead of spreading homogeneously throughout the spectrum. The particle density in the core of the spectrum is almost independent of N, but depends on the inverse temperature β. In the bulk of the spectrum, the den-

sity decreases like $1/\epsilon^{1-\alpha}$ in the $N\to\infty$ limit. Therefore, translational invariance is broken in the thermodynamic limit, in agreement with the MFT.

B. The two-level correlation function

The MC evaluation of the two-point correlation function $R_2(s,s')$, as for any other correlation function, faces the complication of the breakdown of translational invariance. Therefore, we need to carry out a numerical unfolding of the spectrum in order to compare with the classical statistics. We have considered three different unfolding procedures that can be used in different circumstances. In the simplest case, we are interested in the *bulk* correlations. Therefore, the simplest unfolding scheme consists in carrying out, for each MC configuration generated at "time" t, $\{\epsilon_i\}^t$, $i=1,\ldots,N$, the mapping

$$\{\epsilon_i\}^t\to\{s_i\}^t,\quad s_i=\mathrm{sgn}(\epsilon)\int_0^{|\epsilon|}d\epsilon\rho(\epsilon)_{\mathrm{MF}}. \tag{5.2}$$

The unfolded configurations $\{s_i\}^t$ generated in this way are then used to measure the correlations, which will be automatically in the right units.

The second method, which turns out to be particularly useful for the logarithmic potential, consists in performing the change of variable $\epsilon\to s=s(\epsilon)$ directly in the joint probability density function,

$$\mathscr{P}(\{\epsilon_i\})\to\tilde{\mathscr{P}}(\{s_i\})\propto\exp[-\beta\tilde{\mathscr{H}}(\{s_i\})], \tag{5.3}$$

$$\tilde{\mathscr{H}}(\{s_i\})=-\sum_{i,j}\ln|\epsilon_{s_i}-\epsilon_{s_j}|+\sum_i V(\epsilon_{s_i}) -\frac{1}{\beta}\sum_i\exp\{\ln[1/\rho(\epsilon_{s_i})]\}. \tag{5.4}$$

Notice that the one-body confining potential is modified by a β-dependent term coming from the Jacobian of the transformation.

Both these two unfolding schemes make use of the MFT particle density and, therefore, can only be used to study properties in the bulk. In order to study the correlation function around the origin, we need a more precise expression for $\rho(\epsilon)$. Therefore, we carry out the following procedure. The unfolded two-level correlation function at $s=0$ can be expressed as

$$R_2(0,s')=R_2(0,\epsilon_{s'}). \tag{5.5}$$

The function $\epsilon(s)$ is now obtained by numerically inverting $s(\epsilon)=\int_0^\epsilon\langle\rho(\epsilon')\rangle d\epsilon'$, with the density $\langle\rho(\epsilon)\rangle$ evaluated directly by the MC simulations.

Once we have taken care of the unfolding, the evaluation of the two-level correlation function $R_2(s,s')$ by MC is very simple: we fix a reference particle at s and then we compute the "conditional" particle density $\rho(s')|_s$ of all the remaining particles, with respect to the reference point, using a partition in bins as explained before.[45] In studying bulk properties, the reference particle can be let free to move, since in the unfolded coordinates the density is constant. In this case, s in $R_2(s,s')$ must be interpreted as $\langle s\rangle$. Because of the particular way of choosing the trial moves adopted here, the standard deviation from $\langle s\rangle$ turns out to be small once the system has reached equilibrium.

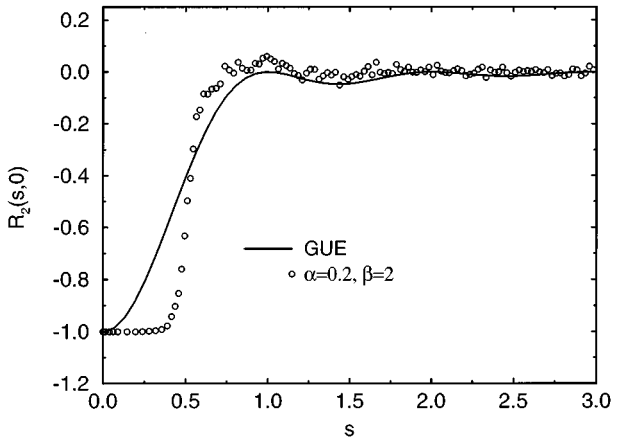

FIG. 2. The MC two-level correlation function $R_2(s'=0,s)$ vs s for the power-law potential, with $\alpha=0.2$, $\beta=2$. The solid line is the result of the GUE, $R_2(r)=-[\sin(\pi r)/\pi r]^2$. (We have omitted the δ function at the origin.)

For the power-law potential, the Monte Carlo simulations show that the two-level correlation function $R_2(s,s')$ is perfectly equal to the WD expression for any α when the reference particle is in the bulk of the spectrum, in agreement with the MFT and orthogonal polynomial results. However, this universality is broken around the origin. In Fig. 2, we plot $R_2(0,s)$ for $\alpha=0.2$ and $\beta=2$ (the δ function is not included). For small s, $R_2(0,s)$ does not follow the classical universal behavior s^β, but rather starts out like $s^{\beta/\alpha}$. Thus, we have a sort of "super-Wigner" behavior, with stronger level repulsion at short separation the smaller α is. Since the cluster function must satisfy the normalization sum rule, this implies a faster decay of $R_2(0,s)$ at large s. Using the method of orthogonal polynomials, one can show that this decay goes like $1/s^{(1+1/\alpha)}$.[44]

We now discuss the bulk properties of $R_2(s,s')$ for the logarithmic potential. As shown in Fig. 3, the MC simulations confirm fully the surprising result of the bulk breakdown of the translational invariance in $R_2(s,s')$ and the appearance of the "ghost" correlation hole at $s=-s'$. The reference particle was let free to move in the positive part of the spectrum around $\langle s\rangle=24$ (in unfolded coordinates). Besides the usual correlation hole around this position, another one appears symmetrically with respect to $s=0$, as if there was an image of the reference particle, the "ghost," located around $\langle s\rangle=-24$. The contribution of the anomalous part increases upon decreasing A. For values of the parameter A not too small, both parts of the cluster functions obtained numerically are in good agreement with the analytical results given in Eqs. (4.28) and (4.29), as seen in Fig. 4, where the normal part of R_2 is plotted. However, as one can also see from the same figure, already for $A=0.2$ the MC result for the normal part starts to deviate from the analytical formula, which in fact becomes invalid for $A<1/\pi^2\approx0.1$. In this regime, the MC simulations show that the cluster function instead of decaying exponentially at very small s starts out more and more flat and in the limit of very small A, it converges to a box or "well" of width 1/2 and depth (-1). A

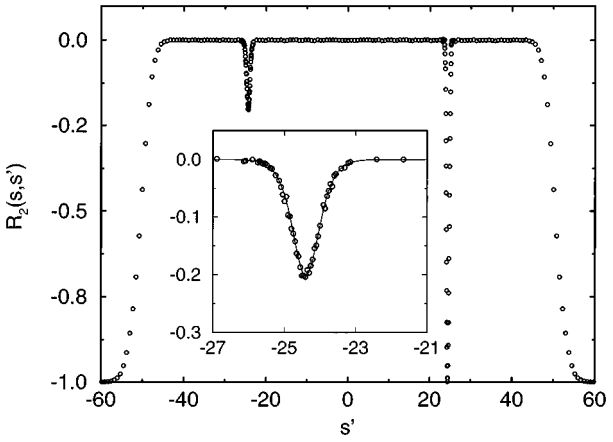

FIG. 3. MC result for the two-level correlation function for the logarithmic confinement, Eq. (2.12), showing the existence of a "ghost" hole at $s'=-s$. The simulations are performed for $\beta=2$ and $A=0.5$, with $N=101$ particles. The reference particle is mobile around $s\approx 24.4$. The solid line in the inset corresponds to Eq. (4.29).

similar behavior occurs for the anomalous part as well. Therefore, in the $A\to 0$ limit, the cluster function is composed of two rectangular wells centered at s and $-s$. This result can also be obtained from the exact solution by Muttalib *et al.*,[28] plotting their general expression for R_2, which remains valid in the regime of very small A. We conclude that the correlation function $R(s,s')$ of the double logarithmic ensemble displays a crossover from WD toward a Poissonic behavior for *intermediate* A, i.e., $1/\pi^2<A<1$, but never really becomes exactly equal to $\delta(s-s')$ as in the Poisson distribution.

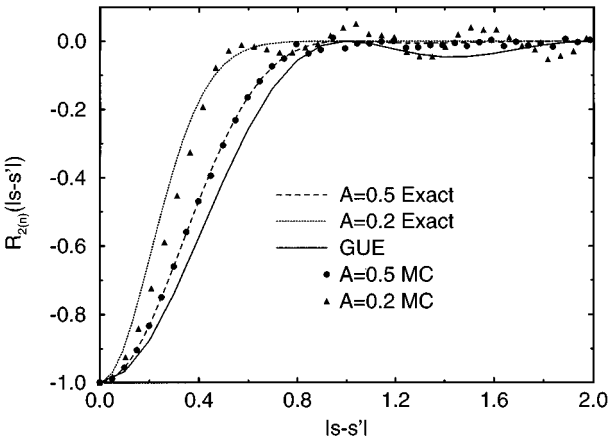

FIG. 4. MC results for the normal part of $R_{2(n)}(s,s')$ $\beta=2$, plotted together with the corresponding exact expressions, Eq. (4.28), and with the GUE curve as a comparison. Notice the good agreement at small $|s-s'|$ between numerical and exact results for $A=0.5$, which becomes worse for $A\leqslant 0.2$. The fluctuations at $|s-s'|>1$, more visible for $A=0.2$, are due to finite-size fluctuations of the exact density, which the unfolding procedure via the MFT density cannot cure. These fluctuations are much smaller for $\beta=1$.

C. The level spacing distribution function

Let us consider a sequence of successive levels $\epsilon_1\leqslant\epsilon_2\leqslant\ldots$ and let $S_1,S_2,\ldots$ be their distance apart, $S_i=\epsilon_{i+1}-\epsilon_i$. The average value of S_i is the mean level spacing Δ. We suppose for the moment that the average density and, therefore, $\Delta=\rho^{-1}=\text{const}$. We further define the relative spacings $\sigma_i=S_i/\Delta$. The nearest neighbor LSDF, $P(\sigma)$, is defined by the condition that $P(\sigma)ds$ is the probability that any σ_i will have a value between σ and $\sigma+d\sigma$.

If the energy levels are completely uncorrelated, one can immediately prove that the LSDF is the Poisson distribution,

$$P_p(\sigma)=\exp(-\sigma). \tag{5.6}$$

In contrast, for a large class of chaotic or disorder systems where the energy levels are correlated, $P(\sigma)$ is very well described by the so-called Wigner surmise,[46]

$$P_w(\sigma)=\frac{\pi\sigma}{2}\exp\left(-\frac{\pi}{4}\sigma^2\right). \tag{5.7}$$

The Wigner surmise vanishes at short separations, showing the phenomenon of level repulsion, typical, for example, for extended wave functions of a disordered conductor. The Poisson distribution, in contrast, allows level degeneracy, as in the case of an Anderson insulator, where the wave functions are localized and do not overlap. Notice that $P_w(\sigma)$ falls down faster at large σ than $P_p(\sigma)$. This is again due to level repulsion which, in a finite energy window, prevents the appearance of large energy gaps with no levels in them.

For the Gaussian ensembles one can derive exact expressions for $P(\sigma)$, which turn out to be very close but not identical to the Wigner surmise.[3] All the RME's with logarithmic interaction and strong confinement belong to the GE (or WD) universality class and thus their $P(\sigma)$ is also very close to the Wigner surmise.

The analytical determination of the LSDF is not straightforward. For $\beta=2$, the function $P(\sigma)$ can be expressed in terms of a determinant of the two-level cluster function,[3] which, in general, must be evaluated numerically.[28] Alternatively, one can use MFT,[36,40,47] but this method has not been extended yet to the potential (2.12). On the other hand, the LSDF is easily calculated by MC. In terms of the plasma model, $P(\sigma)$ is defined, once the system has been unfolded, as the probability density of finding the nearest adjacent particle at a distance s from a given reference particle. The LSDF obviously coincides with the two-level correlation for very small σ.

From the results of the two-level correlation function, we expect the LSDF for the power-law potential to be identical to the Wigner surmise in the bulk, with possible deviations at the origin. The MC simulations confirm fully these expectations. To calculate the unfolded spacing around the origin, we fix a particle at $\epsilon=0$ and we perform the unfolding by computing

$$P(\sigma)=\left[\frac{P(\epsilon)}{\langle\rho(\epsilon)\rangle}\right]_{\epsilon=\epsilon(\sigma)}, \tag{5.8}$$

where the function $\epsilon(\sigma)$ is again obtained by numerically inverting $\sigma(\epsilon)=\int_0^{\epsilon}\langle\rho(\epsilon')\rangle d\epsilon'$. The result is shown in Fig. 5, where we plot the LSDF for $\alpha=0.2$ and $\beta=2$. The classical spacing for the Gaussian orthogonal ensemble (GOE) ($\alpha=2$) is also plotted. The figure clearly shows the devia-

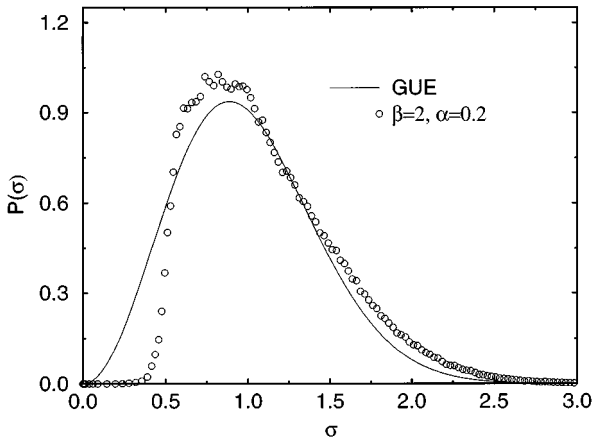

FIG. 5. MC result for the LSDF in the middle of the spectrum of the power-law potential with $\alpha=0.2$ and $\beta=2$. For $\sigma\to 0$, $P(\sigma)$ vanishes like $\sigma^{\beta/\alpha}=\sigma^{10}$. The exact GUE distribution is also plotted.

tions of $P(\sigma)$ from the classical result. In particular, for small σ, the LSDF does not follow the universal behavior σ^{β} of the Wigner surmise, but starts out like $\sigma^{\beta/\alpha}$. This is the same ''super-Wigner'' behavior already found for the two-point correlation function.

In Fig. 6, we show the *bulk* LSDF for the double logarithmic potential in the case of the orthogonal ensemble. We plot $P(\sigma)$ for several values of the parameter A, together with the distribution of the Gaussian orthogonal ensemble and Poisson distribution for comparison. We see that, for those values of $A<1$ for which the two-particle correlation function displayed a deviation from the classical GOE behavior, we have a corresponding deviation from the classical LSDF toward a more Poisson-like behavior. $P(\sigma)$ still starts out linearly at small σ, but the initial slope increases upon decreasing A, as a result of a smaller level repulsion. The peak

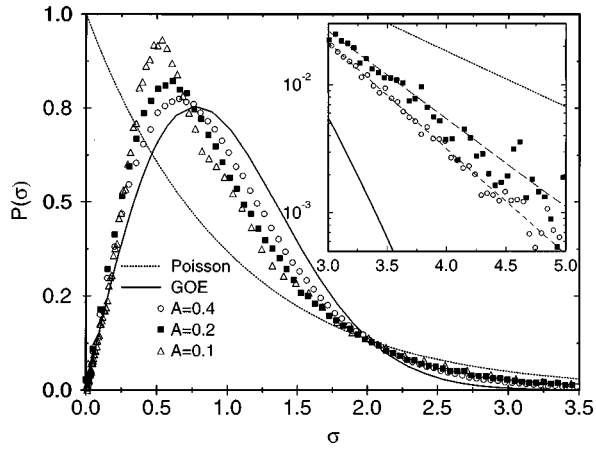

FIG. 6. MC results for the bulk LSDF of the logarithmic potential with $\beta=1$ and different values of A showing a crossover between the GOE and the Poisson distributions, also plotted. For $A<0.2$, the LSDF stops approaching the Poisson function and tends to a single δ function peak at $\sigma\approx 1$. Shown in the inset is the large σ behavior of $P(\sigma)$. Notice the logarithmic scale for the y axis. The two dashed straight lines are fitting functions of the form $\exp(-a(A,\beta)\sigma)$.

of the distribution shifts from $\sigma\approx 1$ to smaller values $\approx A$. For large separations, the decay is also slower than the GOE result. In fact, as we show in the inset, plotting $\ln P(\sigma)$ vs $\sigma\gg 1$, we can fit rather well the curves with straight lines,

$$\ln P(\sigma)\sim -a(A,\beta)\sigma, \quad \sigma\gg 1\,, \tag{5.9}$$

where the constant $a(A,\beta)>1$ decreases with increasing A. Notice that all curves cross at the same point at $\sigma\approx 2$. Similar features and deviations from the Gaussian ensemble are obtained also from the other two symmetries, unitary and symplectic.

The crossover toward the Poisson distribution stops, however, at around $A\approx 0.2$. We have shown that for $A<0.2$ the correlation function, instead of becoming closer and closer to a δ function when A is further decreased, turns around and for very small A it approaches instead a square well of width 1/2. Something similar happens for the LSDF. We can already see in Fig. 6 that for $A=0.1$ the initial slope of the distribution has stopped increasing, and the height of the peak is getting close to one. For yet smaller values of A (not shown in the picture), the initial slope starts *decreasing* and the LSDF, instead of approaching the Poisson distribution, will tend eventually to a single narrow peak of height $\gg 1$, centered at $\sigma\approx 1$.

D. The number variance

So far, we have considered the correlation functions that probe essentially the local fluctuations of a small number $n\simeq 1$ of energy levels. We now turn to the variance $\mathrm{var}(\langle n\rangle)=\langle(n-\langle n\rangle)^2\rangle=\langle n^2\rangle-\langle n\rangle^2$ of the number of levels in an energy window that contains $1\ll\langle n\rangle\ll N$ on the average. The number variance is a statistical quantity that provides a quantitative measure of the long-range rigidity of the energy spectrum.

For the Poisson distribution, the levels are uncorrelated and there are large level-number fluctuations, leading to a *linear* variance,

$$\mathrm{var}_p(\langle n\rangle)=\langle n\rangle. \tag{5.10}$$

On the other hand, the level correlations in the WD statistics make the spectrum more rigid and the number variance grows only *logarithmically*,

$$\mathrm{var}_w(\langle n\rangle)=\frac{2}{\pi^2\beta}\ln\langle n\rangle+C_\beta+O(1/\langle n\rangle), \tag{5.11}$$

where C_β is a constant of order 1, which depends on the symmetry.

The MC results for the power-law confinement show that Eq. (5.11) is perfectly satisfied, for every α, in the bulk, namely, when the energy windows do not contain the origin with its nonuniversal correlations. On the other end when the energy windows are centered at the origin, the ''super-Wigner'' correlations present at $\epsilon=0$ for $\alpha<1$ manifest themselves making the constant C_β in the number variance α dependent. As we show in Fig. 7, for $\beta=1$, C_1 decreases with α, when $\alpha<1$, because there is more level repulsion in the area of the origin and, therefore, more level rigidity in the overall spectrum. For $\alpha\geqslant 1$, the universal value of the Gaussian ensemble $C_1\approx 0.4420$ is recovered. The α depen-

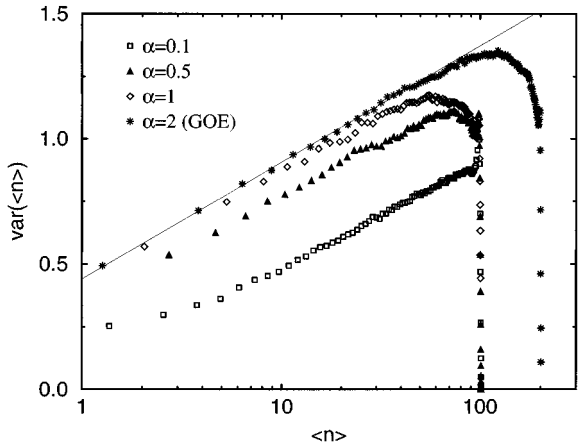

FIG. 7. The MC level-number variance vs $\langle n\rangle$ for the power-law potential, for $\beta=1$ and different values of $\alpha\leqslant 2$. The dashed curve is the GOE result given in Eq. (5.11). The energy windows containing $\langle N\rangle$ particles are centered at the origin $\epsilon=0$. The nonuniversal ''super-Wigner'' behavior of the correlations at $\epsilon=0$ is responsible for the α dependence of the constant term in the variance, which is otherwise equal to the one of GOE.

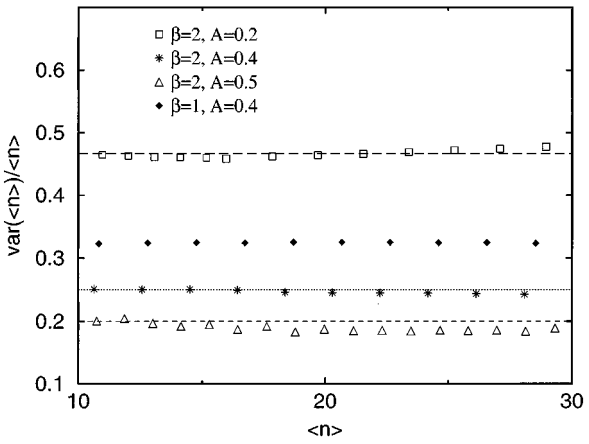

FIG. 8. The relative variance $\mathrm{var}(\langle n\rangle)/\langle n\rangle$ vs $\langle n\rangle$ for the logarithmic potential for different values of A and β. The energy windows are centered in the bulk of the spectrum and the variance is linear with $\langle n\rangle$. The constant straight lines are the slopes of the linear term of the exact solution of Ref. 29, for $\beta=2$ and corresponding A.

dence of C_β is, however, the only deviation from universality in the number variance, the logarithmic dependence being unchanged for the power-law confinement.

We now come to the case of the logarithmic potential. We have seen that the presence of the ''ghost'' correlations break translational invariance in the two-point correlation function. It was shown in Ref. 31 that, due to such a breakdown of translational invariance, the number variance for the logarithmic confinement depends on the position of the energy window in a very essential way. If the energy window does not contain the origin, then the effect of the ghost peak is not felt and the system is Poisson-like, with translationally invariant correlations (in the energy range considered) given by Eq. (4.28). In this case, the number variance is also Poisson-like and increases *linearly*, as one expects in the presence of exponentially decaying correlations. The coefficient of the linear term η is less than 1, and it is given by Eq. (4.34). Here, η is nonzero, because the normal part alone of the correlation function fails to satisfy the normalization sum rule. It increases upon decreasing A, because the ''spectral weight'' of the normal part of the cluster function (4.31) decreases. However, if the energy windows are symmetric with respect to the origin, the ghost correlations become effective and their contribution allows the sum rule to be satisfied. Therefore, the coefficient of linear term η in the variance vanishes in this case.

Indeed, the Monte Carlo simulations show a dramatic difference in the level-number variance in these two cases. In Fig. 8, we show the number variance for an energy window centered at a point in the bulk, excluding the origin. The variance grows linearly and the coefficient of the linear term is in good agreement with Eq. (4.34) and with the result obtained for the exactly soluble models by Blecken *et al.*[29] On the other hand, Fig. 9 shows the variance calculated for symmetric energy windows, containing the origin. The linear term is absent and the variance is *constant* for all integers $\langle n\rangle\gg 1$. Thus, despite the smaller level repulsion, the overall ''level'' rigidity is even higher than for the classical RMT.

VI. DISCUSSION AND CONNECTION WITH THE CRITICAL LEVEL STATISTICS OF THE ANDERSON MODEL

Let us first summarize the main results of our analysis of the generalized RMT with soft confinement. We have seen that for the very weak logarithmic potential, $V(\epsilon)\sim A\ \ln^2|\epsilon|$, the local level fluctuations in the bulk of the spectrum display a crossover from the WD to a more Poisson-like behavior, when the parameter A is decreased. In particular, the two-level correlation function, in the bulk of the spectrum far from the origin, decays exponentially at large distances. The spacing distribution function still vanishes like s^β at short separation, but the initial slope is steeper, implying less level repulsion. The tail of the distribution decays like $\exp(-a(A,\beta)\sigma)$ with $a(A,\beta)>1$, intermediate between the WD surmise and the Poisson function. The level-number variance, when calculated within energy

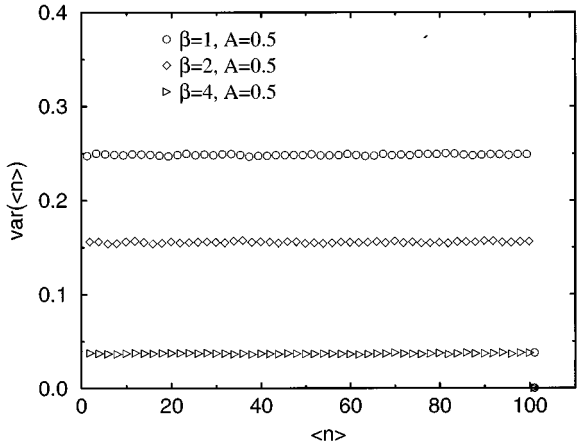

FIG. 9. The MC variance vs $\langle n\rangle$ for the logarithmic potential for the three symmetries and $A=0.5$. The energy windows are now centered at the origin. Since the sum rule Eq. (4.32) is satisfied, the linear term in the variance (see Fig. 8) vanishes and the system becomes even more rigid than the GE.

windows in the bulk of the spectrum that exclude the origin, is also Poissonian, increasing linearly with the average number of levels $\langle n \rangle$. We saw, however, that the Poisson limit cannot be reached fully within this ensemble. The nonuniversal behavior of the RME with logarithmic confining potential has been attributed to a spontaneous breaking of the $U(N)$ invariance. For steeper confining potentials, $V(\epsilon)=|\epsilon|^{\alpha}$, no deviation from the WD statistics occurs in the bulk of the spectrum, the only small deviations from universality occurring at the center.[30]

The question that we now want to address is as follows: does the nonclassical (namely, non-WD) behavior of the RME with double logarithmic confinement have anything to do with the universal energy-level statistics of the Anderson model at the critical point? The first point that needs addressing, before any comparison of the different statistical properties is attempted, is the way in which nonuniversality comes about in the invariant RMT with weak confinement. We have seen that the essential ingredient for obtaining a deviation from the WD statistics is the strong energy dependence of the averaged level density: even in the $N \to \infty$ limit, $\langle \rho(\epsilon) \rangle$ is a rapidly varying function of ϵ everywhere in the spectrum. In fact, one cannot even define a constant *local* density, since the relative variation of $\langle \rho(\epsilon) \rangle$ over an energy range equal to the mean level spacing is of order 1. This is at odds with the well-known result for the density of states in the Anderson model: in that case, $\rho(\epsilon)$ is constant over a large energy region around the center, and, moreover, it is a *noncritical* quantity, that is, it does not exhibit any drastic change at the critical point. One can reply to this serious objection by recalling the similarly well-known fact that quite often complex systems with different *global* statistical properties (such as the density of levels) have the same *local* level fluctuations[34] and vice versa. The most famous example is the GE itself: its semicircle law for the density of states is certainly not obeyed by any of the spectra of the heavy nuclei or other complex systems; yet, its correlations are very universal and describe accurately the local statistical properties of these systems. Something of this sort might happen in our case. In this respect, the existence of another ensemble [namely the broken symmetry model of Eq. (1.1)], where the level density is constant in the thermodynamic limit and nevertheless the local level correlations are the same as the RME with weak confinement, is of great importance. Thus, there exist at least two RME's, having very different global statistics, the local statistics of which belong to the same universality class. We also must emphasize again that the asymptotic logarithmic behavior for the confining potential of the invariant RME has been suggested by studies on transfer matrix models of disordered conductors through the maximum entropy principle.[25,32,33] In the transfer matrix formalism, one can express the conductance g in terms of the eigenvalues x_i of the matrix $X=TT^{\dagger}+(TT^{\dagger})^{-1}-2I$, where T is the transfer matrix and I the unit matrix

$$g=\sum_{i=1}^{N} \frac{1}{1+x_i}. \tag{6.1}$$

Localization appears in the presence of exponentially large eigenvalues x_i. Therefore, a simple maximum entropy principle can provide, through the average density $\rho(x)$, information about the localization of the wave functions, in contrast to what happens to the Hamiltonian matrix. Here, we have assumed that a $U(N)$ invariant RMT for the Hamiltonian matrix can be constructed from the confining potential derived from the corresponding transfer matrix. The hope is that such a RMT will generate the correct local energy-level statistics, despite the average energy density itself not being well represented.[48] This procedure is probably too naive, but it is clearly the simplest and we will discuss its implications.

The conjecture of the existence of universal statistical properties at the metal-insulator transition was put forward by Shklovskii *et al.*,[13] on the basis of numerical studies of the spacing distribution function (LSDF), which turned out to be scale invariant at the critical point. In a recent work, Kravtsov *et al.*[21,22] have carried out an analytical study of the critical statistics of the Anderson model. By using the analytical properties of the diffusion propagator and certain scaling relations valid at the mobility edge, they have proved that the two-level correlation function has the following asymptotic behavior:

$$R(s,s')=C\beta^{-1}|s-s'|^{-2+\gamma}, \quad |s-s'| \gg 1, \tag{6.2a}$$

$$\gamma=1-(\nu d)^{-1}<1, \tag{6.2b}$$

where C is a positive constant, while γ is a universal critical exponent related to the critical exponent ν of the correlation length ξ.

However, the level-number variance at the critical point contains two terms,[49–51]

$$\mathrm{var}(\langle n \rangle)=\eta\langle n \rangle+b\langle n \rangle^{\gamma}, \tag{6.3}$$

where $\eta<1$ and b are some universal positive constants. The power-law term originates directly from the asymptotic power-law tail in the critical two-level correlator and, thus, reflects the critical dynamics. But there is also a *linear* term,[49–51] which had been already predicted by Altshuler *et al.*[52] Formally the origin of this term is again due to the violation of the sum rule, (4.34), by the critical two-level correlation function.[49–51] The physical meaning of this term is not yet understood. Its existence, however, implies that the dominant term in the variance at the critical point is still Poissonian, albeit with coefficient less than 1. Notice that the knowledge of the two-level correlation function is not sufficient to develop a complete statistical description of the energy level at the transition. For example, the LSDF cannot be found without further statistical assumptions. By mapping the critical energy levels into a plasma model and *assuming* the existence of a particular pairwise interaction,[53] one can use the analytical result of Eq. (6.2a) for the two-level correlation function to derive explicitly the effective repulsive interaction among the levels.[40] Once the resulting interaction is known, the asymptotic form of the LSDF can be evaluated, obtaining the result[40]

$$P(\sigma) \sim \exp(-h_{\gamma}\sigma^{2-\gamma}), \tag{6.4}$$

where h_{γ} is a positive constant. Despite some numerical simulations that seem to support this finding (but see below), this approach has the serious drawback that it gives rise only to the second term of Eq. (6.3) for the level-number variance, the linear term being absent and totally unexplained.

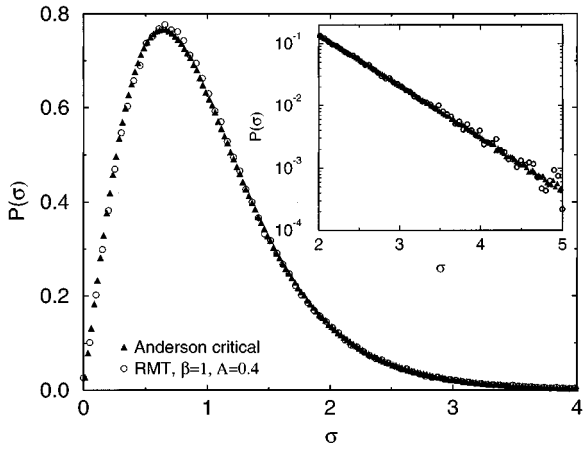

FIG. 10. LSDF of the RME with logarithmic confinement for $\beta=1$, $A=0.4$, plotted together with the energy LSDF of the three-dimensional Anderson model at the metal-insulator transition, taken from Ref. 18. The inset shows the large σ behavior in logarithmic scale for the y axis. A straight line is the best fit for both curves.

Following the work by Shklovskii *et al.*, several other groups have studied numerically the energy-level statistics at the critical point. In all cases, the statistical fluctuation property that is easiest to study numerically, namely, the LSDF, shows scale invariance and a behavior intermediate between the WD surmise and the Poisson function. There seems to be agreement also on the linear start of $P(\sigma)$ at small σ, with a slope steeper than the WD function for the metallic regime. However, the large s tail behavior is more controversial. References 13 and 18 claim that $P(\sigma)$ has a Poissonian decay at large σ,

$$P(\sigma)\sim\exp(-a\sigma), \quad \sigma\gg 1\,, \tag{6.5}$$

with $a\approx 1.9$, whereas Refs. 16 and 15 suggest a behavior in agreement with the plasma model result of Eq. (6.4). We would like to emphasize that the numerical results published in Ref. 18 explicitly show good statistics for large values of σ, and, therefore, we believe that they are reliable to extract the asymptotic behavior of $P(\sigma)$.

The second important result provided by the numerical simulations is the existence of a linear term in the variance, as in Eq. (6.3). These calculations do not exclude the presence of a power-law term of the kind $\eta\langle n\rangle^{\gamma}$, which is, however, difficult to detect and quantify because of the presence of the dominant linear term.[17] The coefficient of the linear term is shown to be $\eta\approx 0.27$ in Ref. 18 and 0.30 in Ref. 17.

The RME with weak confinement that we have considered in this paper is able to reproduce two of the main features seen in the numerical simulations of the critical statistics: the overall behavior of $P(\sigma)$—with the linear rise at $\sigma\ll 1$ and the exponential decay at $\sigma\gg 1$—and the linear dependence of the number variance. In Fig. 10, we plot the LSDF of the RME with logarithmic potential for $A=0.4, \beta=1$, together with the critical LSDF of the Anderson model from Ref. 18. The agreement between the two curves is spectacular in a very large energy range, where $P(\sigma)$ varies by five orders of magnitude. Notice in the inset of the figure the behavior for large σ of the tail of $\ln[P(\sigma)]$, which has apparently a linear slope. The parameter $A_c\approx 0.4$ identifies, among all the possible members of the family of RME with logarithmic confinement, the ensemble which has the closest LSDF to the critical statistics.

If we now compute the coefficient η of the linear term in the variance for the RME with $A_c\approx 0.4$ (see Fig. 8), we obtain $\eta\approx 0.32$, which is consistent with the numerical results from the exact diagonalizations.[17,18] Thus, the RME with logarithmic confinement is able to reproduce quantitatively the shape of the critical LSDF when $A=A_c$ and at the same time provides an accurate estimate of the leading order term of the number variance. In some sense, the ''residual Poissonian'' properties of the critical statistics are well reproduced by this generalized RMT. The RMT does not provide the asymptotic power-law behavior (6.2a) of the two-level correlation function found analytically, which, on the other hand, is also difficult to extract by direct numerical diagonalizations with reliable accuracy.[20] Clearly more work is necessary to determine if the good agreement shown here between the RME with weak confinement and the critical statistics is more than a furtuitous coincidence. It is, however, interesting and important that some of the properties of the correlations at the mobility edge can be correctly reproduced by such a simple ensemble.

VII. CONCLUSIONS

In this paper, we have studied in detail the properties of families of RME that are invariant under similarity transformations, but are characterized by a generalized level confinement. We have shown that the level statistics are affected by the confining potential when this is very soft. In particular, for a squared logarithmic potential, the statistical bulk properties are nonuniversal and deviate significantly from the Wigner-Dyson statistics of the Gaussian ensembles, exhibiting a crossover toward a more Poissonian behavior when an internal parameter is decreased. The $U(N)$ invariant RME with logarithmic confining potential belongs, together the RME with a symmetry breaking term [see Eq. (1.1)], to a new universality class, distinct from Wigner-Dyson universality of classical RMT.

We have shown that the nonuniversal behavior of the two-level correlation function for these RME's can still be obtained within Dyson's mean-field theory, generalized to the case of weak confinement. We have performed Monte Carlo simulations to calculate several important statistical properties of the generalized RME that probe both short- and long-range correlations.

The statistical properties of the RME with logarithmic confinement have strong similarities with the universal energy-level statistics of disordered conductors at the metal-insulator transition. In particular, the probability distribution of the level spacings for a three-dimensional Anderson model at the critical point can be very well fitted, throughout a wide energy range, by the corresponding RME function for one particular choice of the internal parameter. Then for the *same* value of the parameter, this RME predicts a linear behavior for the level-number variance, with a coefficient of proportionality close to the value obtained from numerical diagonalizations of the Anderson model at the critical point.

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[46] This form is for the orthogonal ensemble ($\beta=1$). For the other ensembles one can similarly define "Wigner surmises" of the form $P(\sigma)=c_1\sigma^{\beta}\exp(-c_2\sigma^2)$, where the constants c_1 and c_2 depend on β and are found by imposing the conditions that $P(\sigma)$ and its first moment are equal to 1.

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