Virial theorem for the anisotropic Ginzburg-Landau theory

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The scalar virial theorem for the Ginzburg-Landau theory is generalized to a vector virial theorem and follows from similar scaling properties of the Gibbs free-energy density. All the components of the magnetic field **H** are determined in terms of average values of the kinetic and field tensor components of the Helmholtz free-energy density. We consider two frames, the crystal's and the magnetic induction's **B**, where the scaling properties yield useful relations due to anisotropy. In the last case the scaling relations do not completely determine **H**; instead, they provide useful identities that reflect collective properties of the vortex state. We compare both the scaling and the thermodynamic methods for the particular case of straight tilted parallel vortex lines in the London limit.

I. INTRODUCTION

Some time ago a scalar virial theorem has been obtained by Doria, Gubernatis, and Rainer[1] for the Ginzburg-Landau theory through scaling arguments. It determines the scalar product between the magnetic field, **H**, and the magnetic induction, **B**, in terms of average values of the kinetic and field energies of the superconductor. This scalar virial relation has been verified first numerically[2] and then analytically near the upper critical field by Klein and Pöttinger,[3] who showed that the virial theorem provides an elegant method to understand Abrikosov's identities.[4] In this paper we generalize the previous result to a vector virial theorem, where each component of the magnetic field **H** is fully determined in terms of average values of several tensor components of the kinetic and field energies. Our formula for **H** is derived by scaling arguments applied to the anisotropic Ginzburg-Landau (AGL) theory, similar to those used to obtain the scalar virial theorem.[1] The present scaling relations for the magnetic field render relations of the form $H_i = \int d^3\mathbf{x}[h_i^2 + \delta h_i(\mathbf{D}\psi)]/\int d^3\mathbf{x} h_i$ where $i=1,2,3,$ label components along the crystal's principal axes. The local magnetic field is $\mathbf{h}$, and δh_i are just functions of the gauge invariant derivative of the order parameter, $D_j\psi$, that must vanish when the superconducting state disappears since, in this limit, $\mathbf{h}\rightarrow\mathbf{H}$.

The AGL theory is the simplest model for an anisotropic superconductor and was investigated long ago.[5] The discovery of copper-oxygen layered compounds with high transition temperatures brought a renewed interest in this theory.[6] For some of the ceramic materials, the layered structure is screened and just introducing anisotropy into the mass tensor is enough to describe many of their properties.[7] Because currents flow preferentially along the plane of lower mass, namely the copper-oxygen layers, the model has a natural frustration whenever such currents are forced to stay out of the plane. This feature has led the theory to predict some novel phenomena, like the attraction[8] between vortices which has been experimentally verified in the ceramic compound $YBa_2Cu_3O_7$.[9]

According to thermodynamics, the magnetic field is obtained through the derivative

$$\mathbf{H} = 4\pi \frac{\partial F}{\partial \mathbf{B}}. \tag{1}$$

The thermodynamic fields are also related by $\mathbf{B} = -4\pi\partial G/\partial\mathbf{H}$, where $F = \mathrm{F}/V$ and $G = \mathrm{G}/V$ are thermodynamic potentials, namely the Helmholtz and the Gibbs free-energy densities, respectively, V being the sample volume. The state of a superconductor is described at each temperature by the minimum of a thermodynamic potential, and these two potentials are the most commonly considered, the choice depending on the selected thermodynamic variables. For a fixed magnetic induction, e.g., a constant vortex density, the Helmholtz free energy, $\mathrm{F}(T,\mathbf{B})$, must be the minimum. Obviously in most experimental situations it is more interesting to consider the sample under a fixed magnetic field, thus the Gibbs free energy, $\mathrm{G}(T,\mathbf{H})/V = \mathrm{F}(\mathbf{B})/V - \mathbf{B}\cdot\mathbf{H}/4\pi$, is most convenient. The present scaling relations totally replace Eq. (1), the thermodynamic relation, which requires knowledge of the Helmholtz free energy on a certain neighborhood of **B**, because a derivative of the free energy must be calculated. The present scaling relations determine **H** just demanding knowledge of the Helmholtz free energy at a single value of **B**. This advantage of the present expressions over the thermodynamic relation is mostly useful for numeric computations of the GL theory.[2,10–13]

Equation (1) reflects the central role played by the thermodynamic fields **B** and **H** when the first law of thermodynamics is applied to a superconductor.[14,15] To understand it, take a superconductor sample subjected to an applied field and consider the work done by the sample on some far-away coils responsible for this magnetic field. Suppose that the sample temperature is lowered from above to below its critical value. Let $\mathbf{J}_{\mathbf{ext}}$ be the current density circulating in these

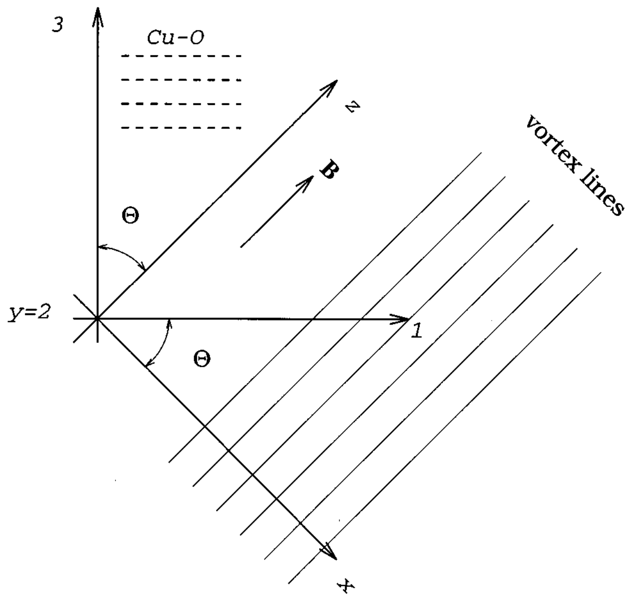

FIG. 1. The copper-oxygen layers are depicted here in the C frame, this one defined by the crystal's major axes. The B frame, which follows from the c axis upon rotation by an angle θ, has the magnetic induction $\mathbf{B}$ along one of its axes (z axis). A set of tilted straight and parallel vortex lines are pictorially represented here.

coils. When the sample becomes superconductor, an induced current density must arise in the sample in order to cancel the applied field. The local magnetic field becomes $\mathbf{h}$, distinct from the original applied field. According to Faraday's law, the expulsion of a fraction of the magnetic flux, inside the sample, leads to the presence of an electric field, $\mathbf{E}$. Therefore work is done by the sample on the coils at the rate $dW/dt = \int d^3\mathbf{x}\mathbf{E}\cdot\mathbf{J}_{\mathbf{ext}}$, when the temperature is lowered from above to below the critical value. This process must be reversible in order that the laws of thermodynamics applies. Thus it must unfold at a very slow pace to avoid irreversible phenomena, and so one neglects the displacement current and radiation effects in the Maxwell equations, resulting in $dW/dt = -(c/4\pi)\int d^3\mathbf{x}(\partial\mathbf{h}/\partial t)\cdot\mathbf{H}$. Below the critical temperature the field $\mathbf{H}$ remains uniform inside the sample only for special geometries, e.g., a long cylinder with its symmetry axis parallel to the applied field.[14,15] The only source for $\mathbf{H}$ are the external currents, $\boldsymbol{\nabla}\times\mathbf{H} = 4\pi\mathbf{J}_{\mathbf{ext}}/c$, and since we have not included the displacement current, a slight time dependence on $\mathbf{H}$ in this process is also neglected. Then the work done by the sample on the coil is simply given by

$$\frac{dW}{V} = -\frac{1}{4\pi}\mathbf{H}\cdot d\mathbf{B}, \quad \mathbf{B} = \int \frac{d^3\mathbf{x}}{V}\mathbf{h}, \tag{2}$$

in the special geometries cited before.

The scaling properties of the Gibbs free-energy density discussed in this paper yield expressions in two coordinate frames, namely the crystal's (C frame) and the magnetic induction's (B frame). We only consider here uniaxial superconductors and the B frame is defined through a rotation of the C frame: $\cos\theta = \hat{\mathbf{c}}\cdot\hat{\mathbf{B}}$ where $\hat{\mathbf{c}}$ and $\hat{\mathbf{B}}$ are along the crystal's axis of symmetry and the magnetic induction direction, respectively (see Fig. 1). In the B frame the scaling relations do not completely determine the magnetic field. Instead they give identities that must be satisfied by the collective vortex state that extremizes the free energy. The existence of such identities is one of the major results of this paper.

Like the scalar virial theorem, our relation for $\mathbf{H}$ applies for an infinite superconductor, thus boundary free, and under the presence of periodic boundary conditions.[1] This is sufficient to treat the Abrikosov state and also some recently proposed states caused by disorder,[16] which can be easily introduced into the AGL theory. For the latter case, not treated here, the periodicity has no physical meaning but scaling relations are still valid provided that such a periodicity is interpreted as an artifact, such that the cell boundaries are taken very large at the end of the calculations.[1] In this paper we compare the present scaling relations for $\mathbf{H}$ to the thermodynamic method of Eq. (1), in one particular limit, where the AGL theory can be solved analytically. This is the London limit, where the density of superconducting pairs is constant everywhere. We consider the case of a vortex lattice made of tilted parallel straight lines all making an angle θ with the c axis and find that our formula for $\mathbf{H}$ and Eq. (1) give the same results in both B and C frames. Near the upper critical field H_{c2} it is also possible to seek an analytical comparison[17] between the scaling and the thermodynamic methods for $\mathbf{H}$. This comparison is not carried here and will be considered elsewhere.

This paper is organized as follows. In Sec. II, we review the anisotropic Ginzburg-Landau (AGL) theory, the variational Ginzburg-Landau equations and present our formula for $\mathbf{H}$ in both C and B frames. We also express the AGL theory in terms of the superelectron density and current density in order to consider the London limit. In Sec. III the scaling properties of the minimum of the Gibbs energy are explored in order to derive our scaling relations for $\mathbf{H}$. In Sec. IV the periodic array of tilted parallel lines in the London limit is studied. For this particular array of vortex lines, we obtain in the next two subsections the magnetic field $\mathbf{H}$ from both the thermodynamic relation, Sec. IV A, and the scaling relations, Sec. IV B, and show that they give the same results. In order to help the reader we have included a special section, Sec. IV C, where the isotropic limit of the vector virial relation is taken and several results of the previous sections are discussed once more. Finally we conclude in Sec. V and leave for the appendices some side discussions regarding properties of certain matrices upon rotation, Appendix A, and the reciprocal space vectors, Appendix B. In Appendix C we provide more detailed information about some mathematical proofs that complete the derivations carried in Sec. IV B.

II. THE THERMODYNAMIC FIELD H FOR THE ANISOTROPIC GINZBURG-LANDAU THEORY

In this section we review the AGL theory, present this paper's expressions for $\mathbf{H}$ that replace the thermodynamic relation of Eq. (1), and introduce a frame (B frame) obtained upon rotation. When it is necessary to distinguish tensor components in the C and B frames, we use a subscript index, taking values $i=1,2,3$ and $i=x,y,z$, respectively. The Helmholtz free energy of an anisotropic superconductor is expressed in terms of a complex order parameter, $\psi = \sqrt{\rho}$ exp

$(i\chi)$, and of the local magnetic potential, $\mathbf{A}$, such that the local magnetic field is $\mathbf{h}=\nabla\times\mathbf{A}$. One obtains

$$F(\mathbf{B})=\int\frac{d^3\mathbf{x}}{V}\left(\alpha_o(T-T_c)|\psi|^2+\frac{\beta}{2}|\psi|^4+\frac{1}{2\bar{M}}[\mathbf{D}\psi]^\dagger\cdot\mathbf{m}'\cdot[\mathbf{D}\psi]+\frac{\mathbf{h}^2}{8\pi}\right),\tag{3}$$

where the covariant derivative is $\mathbf{D}=(\hbar/i)\nabla-(q/c)\mathbf{A}$. In the crystal's frame the mass tensor is

$$\mathbf{M}=\begin{pmatrix}M_a & 0 & 0\\ 0 & M_a & 0\\ 0 & 0 & M_c\end{pmatrix}.\tag{4}$$

The dimensionless tensors, $\mathbf{m}$ and $\mathbf{m}'$, extensively used in this paper, are

$$\mathbf{m}\equiv\frac{\mathbf{M}}{\bar{M}},\quad \mathbf{m}'\equiv\mathbf{m}^{-1},\tag{5}$$

where $\bar{M}=\sqrt[3]{M_a^2M_c}$ stands for the average mass. Hence, in the crystal's frame, $\mu_a=1/m_a$ and $\mu_c=1/m_c$. The variational equations obtained from the above free-energy density are the so-called Ginzburg-Landau (GL) equations,

$$\frac{1}{2\bar{M}}[\mathbf{D}^\top\cdot\mathbf{m}'\cdot\mathbf{D}]\psi=(\alpha_o(T_c-T)\psi-\beta|\psi|^2\psi),\tag{6}$$

$$\nabla\times\mathbf{h}=\frac{4\pi}{c}\mathbf{J},\quad \mathbf{J}=\frac{q}{2\bar{M}}(\psi^*\mathbf{m}'\cdot\mathbf{D}\psi+\text{c.c.}).\tag{7}$$

We claim in this paper that scaling properties of the Gibbs free-energy density, discussed in the next section, give that the magnetic field is

$$H_1=4\pi B_1^{-1}\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_1^2}{4\pi}+\frac{1}{2\bar{M}}[-\mu_a|D_1\psi|^2+\mu_a|D_2\psi|^2+\mu_c|D_3\psi|^2]\right),\tag{8}$$

$$H_2=4\pi B_2^{-1}\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_2^2}{4\pi}+\frac{1}{2\bar{M}}[\mu_a|D_1\psi|^2-\mu_a|D_2\psi|^2+\mu_c|D_3\psi|^2]\right),\tag{9}$$

$$H_3=4\pi B_3^{-1}\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_3^2}{4\pi}+\frac{1}{2\bar{M}}[\mu_a|D_1\psi|^2+\mu_a|D_2\psi|^2-\mu_c|D_3\psi|^2]\right).\tag{10}$$

The fields ψ and $\mathbf{A}$ in the above formulas must be solutions of the GL equations in order that the right-hand side (RHS) of the above equations yield the magnetic field $\mathbf{H}$. Similarly in order that Eq. (1) determines the magnetic field it is also necessary that the free energy be extremized with respect to all parameters other than $\mathbf{B}$.

Similar to the above scaling expressions for $\mathbf{H}$, Eqs. (8)–(10), there is also another equation that demands the solution of the GL equations in order to be valid. It is well known and we call it the *integrated* equation,

$$\int\frac{d^3\mathbf{x}}{V}\left(\alpha_o(T-T_c)|\psi|^2+\beta|\psi|^4+\frac{1}{2\bar{M}}[\mathbf{D}\psi]^\dagger\cdot\mathbf{m}'\cdot[\mathbf{D}\psi]\right)=0.\tag{11}$$

This equation is obtained by direct integration of a GL equation, Eq. (6), multiplied by ψ. A surface term is abandoned in the derivation of the above equation, since its contribution vanishes because of the special boundary conditions assumed here. A simple way to see the limitation to an infinite superconductor made of lattice cells is to study the case of no applied field. Then the scaling relations reduce to $\int d^3\mathbf{x}|\partial\psi/\partial x_i|^2=0$ along each of the crystal's axes $(i=1,2,3)$. The only possible solution to such relations is a constant order parameter everywhere. Thus the scaling relations do not take into account the possibility of interfaces, e.g., superconductor-insulator barriers, where the order parameter must vary over a distance characterized by the coherence length. For this case of no applied field and no interfaces, the *integrated* equation just fixes the modulus of the order parameter: $|\psi|=\sqrt{\alpha_o(T_c-T)/\beta}$.

We also propose here another set of scaling relations besides Eqs. (8)–(10). The new relations are obtained by scaling of the B frame's axes. We choose the magnetic induction to be $\mathbf{B}=B_1\hat{\mathbf{x}}_1+B_3\hat{\mathbf{x}}_3$ in the C frame, with no loss of generality because only uniaxial superconductors are studied here. The rotation axis is $2\equiv y$ and let θ be the angle defining this particular rotation, such that a point with coordinates (x_1,x_2,x_3), in the C frame, has coordinates (x,y,z) in this new frame, where $x=\cos\theta x_1+\sin\theta x_3$, $y=x_2$, and $z=-\sin\theta x_1+\cos\,\theta x_3$ [see Fig. (1)]. Such a rotation is represented by the matrix

$$\mathbf{R}=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}.\tag{12}$$

In the new frame the mass matrix becomes $\mathbf{M}(\theta)=\mathbf{R}^\top(\theta)\cdot\mathbf{M}\cdot\mathbf{R}(\theta)$, and the dimensionless matrices, previously defined, transform in the same way, rendering

$$\mathbf{m}=\begin{pmatrix}m_{xx} & 0 & m_{xz}\\ 0 & m_a & 0\\ m_{xz} & 0 & m_{zz}\end{pmatrix},\quad \begin{aligned}m_{xx}&=m_a\cos^2\theta+m_c\sin^2\theta,\\ m_{zz}&=m_a\sin^2\theta+m_c\cos^2\theta,\\ m_{xz}&=(m_a-m_c)\sin\theta\,\cos\theta,\end{aligned}\tag{13}$$

and

$$\mathbf{m}'=\begin{pmatrix} \mu_{xx} & 0 & \mu_{xz} \\ 0 & \mu_a & 0 \\ \mu_{xz} & 0 & \mu_{zz} \end{pmatrix}, \quad \begin{aligned} \mu_{xx}&=\mu_a\cos^2\theta+\mu_c\sin^2\theta, \\ \mu_{zz}&=\mu_a\sin^2\theta+\mu_c\cos^2\theta, \\ \mu_{xz}&=(\mu_a-\mu_c)\sin\theta\,\cos\theta. \end{aligned} \tag{14}$$

The B frame corresponds to a choice of angle θ such that the magnetic induction is along the z axis, $\mathbf{B}=B_z\hat{\mathbf{z}}$. From Fig. (1) we see that

$$\tan\theta=\frac{B_1}{B_3}, \quad B_z=\sqrt{B_1^2+B_3^2}. \tag{15}$$

The scaling relations in the B frame are given below:

$$\frac{H_xB_x}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_x^2}{4\pi}+\frac{1}{2M}[-\mu_{xx}|D_x\psi|^2+\mu_a|D_y\psi|^2 + \mu_{zz}|D_z\psi|^2]\right)=0, \tag{16}$$

$$\frac{H_yB_y}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_y^2}{4\pi}+\frac{1}{2M}[\mu_{xx}|D_x\psi|^2+\mu_{xz}((D_z\psi)^*(D_x\psi) + \text{c.c.})-\mu_a|D_y\psi|^2+\mu_{zz}|D_z\psi|^2]\right)= 0, \tag{17}$$

$$\frac{H_zB_z}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_z^2}{4\pi}+\frac{1}{2M}[\mu_{xx}|D_x\psi|^2+\mu_a|D_y\psi|^2 - \mu_{zz}|D_z\psi|^2]\right). \tag{18}$$

Notice that the above equations can only determine H_z, the magnetic field $\mathbf{H}$ component along the magnetic induction $\mathbf{B}$. However the RHS of Eqs. (16) and (17) must still be satisfied by the solution $(\psi,\mathbf{A})$ that extremize the free energy. In the next section we show that they are valuable tools to provide information on the properties of the collective state of vortices.

The two sets of scaling relations introduced in this section, Eqs. (8)–(10) and Eqs. (16)–(18), are in agreement with the scalar virial relation[1] shown below.

$$\frac{\mathbf{H}\cdot\mathbf{B}}{4\pi}=F_{\text{kin}}+2F_{\text{field}}, \tag{19}$$

$$F_{\text{kin}}=\frac{1}{2M}\int\frac{d^3\mathbf{x}}{V}[\mathbf{D}\psi]^\dagger\cdot\mathbf{m}'\cdot[\mathbf{D}\psi], \quad F_{\text{field}}=\int\frac{d^3\mathbf{x}}{V}\frac{\mathbf{h}^2}{8\pi}. \tag{20}$$

In the next section we consider the London limit of the AGL theory, and for this reason we find it convenient to write the AGL free-energy density, the *integrated* equation, and the scaling relations for $\mathbf{H}$ in terms of the superelectron density, $\rho=|\psi|^2$, and the supercurrent density, $\mathbf{J}$. In order to do so, we first notice that, for any of the two frames previously defined, $\psi^*D_j\psi=(\hbar/2i)\nabla_j\rho+\rho(\hbar\nabla_j\chi-qA_j/c)$. The imaginary part of this expression multiplied by the matrix $\mathbf{m}'$ is

$$\frac{\hbar}{2iM}\mathbf{m}'\cdot\boldsymbol{\nabla}\rho=\frac{1}{2M}(\psi^*\mathbf{m}'\cdot\boldsymbol{\nabla}\psi-\text{c.c.}). \tag{21}$$

The sum of the above equation to the expression for $\mathbf{J}$, Eq. (7), gives that

$$\psi^*\mathbf{D}\psi=\frac{\hbar}{2i}\boldsymbol{\nabla}\rho+\frac{\overline{M}}{q}\mathbf{m}\cdot\mathbf{J}. \tag{22}$$

Now we cast the AGL Helmholtz free-energy density into this new variable formulation:

$$F(\mathbf{B})=\int\frac{d^3\mathbf{x}}{V}\left(\alpha_o(T-T_c)\rho+\frac{\beta}{2}\rho^2 + \frac{\hbar^2}{8M\rho}[\boldsymbol{\nabla}\rho]^\top\cdot\mathbf{m}'\cdot[\boldsymbol{\nabla}\rho]+\frac{\overline{M}}{2\rho q^2}\mathbf{J}^\top\cdot\mathbf{m}\cdot\mathbf{J}+\frac{\mathbf{h}^2}{8\pi}\right). \tag{23}$$

Using this $(\rho,\mathbf{J})$ representation the scaling relations for $\mathbf{H}$, in both frames, become

$$\frac{H_jB_j}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\left(\frac{h_j^2}{4\pi}+\frac{\hbar^2}{8M\rho}[\boldsymbol{\nabla}\rho]^\top\cdot\mathbf{V_j}\cdot[\boldsymbol{\nabla}\rho] + \frac{\overline{M}}{2\rho q^2}\mathbf{J}^\top\cdot\mathbf{W_j}\cdot\mathbf{J}\right), \tag{24}$$

where *no* summation over repeated indices is understood. The two sets of matrices, $\mathbf{V_j}$ and $\mathbf{W_j}$, are discussed in Appendix A for both frames. Lastly the *integrated* equation becomes

$$\int\frac{d^3\mathbf{x}}{V}\left(\alpha_o(T-T_c)\rho+\beta\rho^2+\frac{\hbar^2}{8M\rho}[\boldsymbol{\nabla}\rho]^\top\cdot\mathbf{m}'\cdot[\boldsymbol{\nabla}\rho] + \frac{\overline{M}}{2\rho q^2}\mathbf{J}^\top\cdot\mathbf{m}\cdot\mathbf{J}\right)=0. \tag{25}$$

In the next section the scaling relations for $\mathbf{H}$, presented in this section, namely Eqs. (8)–(10) and Eqs. (16)–(18), are derived by use of the scaling properties of the Gibbs free energy.

III. SCALING PROPERTIES OF THE GIBBS ENERGY

According to thermodynamics the Gibbs free-energy density of the superconducting state,

$$G(\mathbf{H},T)=\int\frac{d^3\mathbf{x}}{V}\left(\alpha_o(T-T_c)|\psi|^2+\frac{\beta}{2}|\psi|^4 + \frac{1}{2M}[\mathbf{D}\psi]^\dagger\cdot\mathbf{m}'\cdot[\mathbf{D}\psi]+\frac{\mathbf{h}^2}{8\pi}-\frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}\right), \tag{26}$$

is a minimum under the variation of all parameters for fixed temperature and magnetic field. The relevant parameters that extremize the above Gibbs free energy are usually taken to be the fields, ψ and $\mathbf{A}$. The new assumption here is that the local coordinates (x,y,z) [or (x_1,x_2,x_3)] are also parameters that must extremize the Gibbs energy. In the same way that extremizing the Gibbs energy with respect to the fields yields the GL equations, extremizing the Gibbs energy with respect to the local coordinates should lead to new equations, which are our scaling relations for **H**.

Consider the original coordinates, (x,y,z), changed to new ones, (x',y',z'), which are obtained by the scaling transformation

$$x=\lambda x',\quad y=y',\quad z=z', \tag{27}$$

where λ is an arbitrary parameter. We claim that the effect of such a change is to move the Gibbs energy density away from its extreme value where the fields $\psi(x,y,z)$ and $\mathbf{A}(x,y,z)$ in Eq. (26) are the solutions of the GL equations. To determine the new value of the Gibbs energy density, substitute the original coordinates by the new ones into the solutions of the GL equations, $\psi(\lambda x',y',z')$, $\mathbf{A}(\lambda x',y',z')$. Other changes must be taken into account. The derivative operator is scaled according to $\nabla_x=\lambda^{-1}\nabla'_x$, $\nabla_y=\nabla'_y$, $\nabla_z=\nabla'_z$ and the volume element remains unchanged, $\int d^3\mathbf{x}/V=\int d^3\mathbf{x}'/V'$. Notice that scaling is not introduced into the limits of integration since we are integrating throughout the whole space. Finally we assume that the Gibbs energy density increases by this scaling procedure and this leads to the extreme condition

$$\left.\frac{dG(\lambda)}{d\lambda}\right|_{\lambda=1}=0. \tag{28}$$

The above condition is just a different way to see the same principle that has led to the scalar virial theorem of Ref. 1. The only difference between the two cases is that scaling takes place in all of the three coordinates simultaneuously for the scalar virial theorem, whereas here the coordinates are scaled independently. Therefore we shall obtain three scaling relations instead of only one.

We discuss the minimization procedure for the B frame only since results for the C frame are easily retrieved taking $\theta=0°$ at the end of the present calculation.

New fields must be introduced, $\psi'\equiv\lambda^{-\nu}\psi$, $A'_x\equiv\lambda A_x$, $A'_y\equiv A_y$, and $A'_z\equiv A_z$, and such fields are explicit functions of λ whereas the original ones are not. We shall write the Gibbs energy in terms of the new fields, and thus it is more convenient to think in terms of the following replacement:

$$\begin{array}{ll}\psi(x,y,z)=\lambda^{\nu}\psi'(x',y',z'), & \\ A_x(x,y,z)=\lambda^{-1}A'_x(x',y',z'), & D_x=\lambda^{-1}D'_x,\\ A_y(x,y,z)=A'_y(x',y',z'), & D_y=D'_y,\\ A_z(x,y,z)=A'_z(x',y',z'), & D_z=D'_z.\end{array} \tag{29}$$

We shall find that the value of the parameter ν is totally irrelevant for our purposes. Notice that for the local magnetic field, we did not introduce such an arbitrary paramenter since **A** must transform like the gradient operator, $\boldsymbol{\nabla}$. In this way the covariant derivative transfoms in a single way, as expressed above, and gauge invariance is preserved. It is straightforward to determine the way the local magnetic field, $\mathbf{h}'=\nabla'\times\mathbf{A}'$, scales:

$$\begin{aligned} h_x(x,y,z)&=h'_x(x',y',z'),\\ h_y(x,y,z)&=\lambda^{-1}h'_y(x',y',z'),\\ h_z(x,y,z)&=\lambda^{-1}h'_z(x',y',z'). \end{aligned} \tag{30}$$

All the elements necessary to obtain the scaling of the original Gibbs energy, Eq. (26), have been already discussed and one obtains

$$G(\mathbf{H},T)=\int\frac{d^3\mathbf{x}'}{V'}\Bigg\{\lambda^{2\nu}\alpha_o(T-T_c)|\psi'|^2+\lambda^{4\nu}\frac{\beta}{2}|\psi'|^4+\frac{\lambda^{2\nu}}{2}\left(\frac{D'_x\psi'}{\lambda}D'_y\psi' D'_z\psi'\right)^*\begin{pmatrix}\mu_{xx} & 0 & \mu_{xz}\\ 0 & \mu_a & 0\\ \mu_{xz} & 0 & \mu_{zz}\end{pmatrix}\begin{pmatrix}\frac{D'_x\psi'}{\lambda}\\ D'_y\psi'\\ D'_z\psi'\end{pmatrix}$$
$$+\frac{1}{8\pi}\left[h'^2_x+\frac{h'^2_y}{\lambda^2}+\frac{h'^2_z}{\lambda^2}-2\left(H'_xh'_x+\frac{H'_yh'_y}{\lambda}+\frac{H'_zh'_z}{\lambda}\right)\right]\Bigg\}. \tag{31}$$

Hence we carry on the condition of Eq. (28), which contains four distinct contributions,

$$\frac{dG(\mathbf{H})}{d\lambda}=\int d^3\mathbf{x}'\left(\frac{\delta G}{\delta\mathbf{A}'(\mathbf{x}')}\frac{\delta\mathbf{A}'(\mathbf{x}')}{\delta\lambda}+\frac{\delta G}{\delta\psi'(\mathbf{x}')}\frac{\delta\psi'(\mathbf{x}')}{\delta\lambda}+\frac{\delta G}{\delta\psi'^*(\mathbf{x}')}\frac{\delta\psi'^*(\mathbf{x}')}{\delta\lambda}\right)+\frac{\partial G(\lambda)}{\partial\lambda}. \tag{32}$$

At this point we recall that the original fields, **A** and ψ, are solutions of the GL equations:

$$\left.\frac{\delta G}{\delta\mathbf{A}'}\right|_{\lambda=1}=0,\quad \left.\frac{\delta G}{\delta\psi'}\right|_{\lambda=1}=\left.\frac{\delta G}{\delta\psi'^*}\right|_{\lambda=1}=0. \tag{33}$$

Including the above consideration into the minimization principle of Eq. (28), one obtains that $\partial G(\lambda)/\partial\lambda|_{\lambda=1}=0$. This condition contains two independent terms, one proportional and the other not proportional to ν, which are, respectively,

$$2\nu\int\frac{d^3\mathbf{x}}{V}\Bigg(\alpha_o(T-T_c)|\psi|^2+\beta|\psi|^4+\frac{1}{2M}[\mu_{xx}|D_x\psi|^2+\mu_{xz}((D_z\psi)^*(D_x\psi)+\text{c.c.})+\mu_a|D_y\psi|^2]\Bigg)=0,$$

$$\int\frac{d^3\mathbf{x}}{V}\Bigg(\frac{1}{2M}[-2\mu_{zz}|D_z\psi|^2-\mu_{xz}((D_z\psi)^*(D_x\psi)+\text{c.c.})]+\frac{1}{4\pi}(h_yH_y+h_zH_z-h_y^2-h_z^2)\Bigg)=0. \quad (34)$$

The ν dependent term is just the *integrated* equation, Eq. (11), and the other one is

$$\frac{H_yB_y+H_zB_z}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\Bigg(\frac{1}{2M}[2\mu_{zz}|D_z\psi|^2+\mu_{xz}((D_z\psi)^*(D_x\psi)+\text{c.c.})]+\frac{h_y^2+h_z^2}{4\pi}\Bigg). \quad (35)$$

Similar independent scaling on the other two coordinates ($x=x'$, $y=\lambda y'$, $z=z'$) and ($x=x'$, $y=y'$, $z=\lambda z'$) also yields two equations in each case. The ν dependent one is always the *integrated* equation, Eq. (11), and the ν independent conditions are given below:

$$\frac{H_xB_x+H_zB_z}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\Bigg(\frac{1}{2M}[2\mu_a|D_a\psi|^2]+\frac{1}{4\pi}(h_x^2+h_z^2)\Bigg), \quad (36)$$

$$\frac{H_xB_x+H_yB_y}{4\pi}=\int\frac{d^3\mathbf{x}}{V}\Bigg(\frac{1}{2M}[2\mu_{xx}|D_x\psi|^2+\mu_{xz}((D_z\psi)^*(D_x\psi)+\text{c.c.})]+\frac{h_x^2+h_y^2}{4\pi}\Bigg). \quad (37)$$

Two possible distinct operations, scaling and rotation, can yield different relations depending on the order they are applied to the Gibbs energy, Eq. (26). This is true for the anisotropic supercondutors since upon rotation, the Gibbs energy is written in terms of a new set of tensorial components. In this section we choose two sets of orthogonal axes where scaling is taken through the extreme condition of Eq. (28): the C and B frame axes. However notice that the present derivation of the scaling relations is quite general and could be used along a general set of orthogonal axes. We do not analyze this general possibility in this paper and, in the next section, restrict our goals to show that the scaling relations are truthful, at least for the C and B frames. For this we choose a special limit of the AGL free energy where the theory has an analytical solution.

IV. THE LONDON LIMIT

In the London approximation the density of superelectrons, ρ, is constant. In this limit, the GL Helmholtz free-energy density, Eq. (23), becomes

$$F=\frac{1}{8\pi}\int\frac{d^3\mathbf{x}}{V}\Bigg[\Bigg(\frac{4\pi\Lambda}{c}\Bigg)^2\mathbf{J}^\top\cdot\mathbf{m}\cdot\mathbf{J}+\mathbf{h}^2\Bigg]. \quad (38)$$

The *integrated* version of the first GL equation, Eq. (11), plays no role other than determine ρ in terms of parameters of the theory. The second GL equation, Eq. (7), is Ampère's law,

$$\nabla\times\mathbf{h}=4\pi\mathbf{J}/c,\quad \mathbf{J}=\frac{c}{4\pi}\frac{1}{\Lambda^2}\mathbf{m}'\cdot\Bigg(\frac{\Phi_0}{2\pi}\nabla\chi-\mathbf{A}\Bigg),$$

$$\Lambda^2=\bar{M}c^2/(4\pi q^2\rho), \quad (39)$$

where Λ is the average London penetration length. The current is automatically conserved, $\nabla\cdot\mathbf{J}=0$. The curl of Ampère's law gives

$$\mathbf{h}+\Lambda^2\nabla\times(\mathbf{m}\cdot\nabla\times\mathbf{h})=\frac{\Phi_0}{2\pi}\nabla\times\nabla\chi, \quad (40)$$

According to the above equation, the Meissner effect takes place around each vortex core singularity. This is the only possible situation because boundaries to nonsuperconducting regions were excluded from the present treatment, according to our previous discussion. Such singularities are described by the vorticity, $\mathbf{v}$, related to the phase, χ, according to $\mathbf{v}=(\nabla\times\nabla\chi)/2\pi$, for vortex lines $\mathbf{v}$ vanish everywhere except at their cores. For instance, the vorticity of N vortex lines, each with a structureless core, is $\mathbf{v}(\mathbf{x})=\Sigma_{j=1}^N\oint d\mathbf{r}_j\delta^{(3)}(\mathbf{x}-\mathbf{r}_j)$, where $\mathbf{r}_j$ describes the jth vortex line in space. The number of vortex lines in space, N, is also related to the assumption of single-valuedness of the wave function: $[\chi(2\pi)-\chi(0)]=2\pi N$. To see this, integrate Eq. (40) on a plane of area A, pierced by the N vortex lines (see Fig. 1), and consider that the magnetic induction is $\mathbf{B}=\int ds\mathbf{h}/A$. The periodic boundary conditions show that the supercurrent does not contribute, and this integration is taken to the contour around area A, using Stoke's theorem: $\mathbf{B}=(N/A)\Phi_0\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the normal to this plane. Even within the London approximation it is possible to include more elaborate structures[18] for the vortex core and we claim that our scaling relations remain valid in such cases. However for our comparison between the scaling and the thermodynamic $\mathbf{H}$ relations we choose vortex lines with no core structure. Introducing Ampère's law into the free energy, we find that its extreme value for a fixed distribution of vortices $\mathbf{v}$ is $F=(\Phi_0/8\pi)\int(d^3\mathbf{x}/V)\mathbf{h}\cdot\mathbf{v}$. Notice that the extremization of this free energy has not been completely done up to this point. For a given distribution of vortex lines we have determined so far, from Eq. (40), the local magnetic field in space, $\mathbf{h}$, and the corresponding free energy. It remains to determine how vortex lines, under a fixed density, N/A, are arranged in space.

The following tensors, extensively used in our considerations on London theory,

$$\mathbf{K}_{ij}=\Lambda^2\int\frac{d^3\mathbf{x}}{V}(\boldsymbol{\nabla}\times\mathbf{h})_i(\boldsymbol{\nabla}\times\mathbf{h})_j\,,\quad \mathbf{f}_{ij}=\int\frac{d^3\mathbf{x}}{V}h_ih_j \tag{41}$$

allow Eq. (38) to be written as

$$F=F_{\rm kin}+F_{\rm field}\,,\quad F_{\rm kin}=\frac{\Lambda^2}{8\pi}\mathrm{tr}(\mathbf{K}\cdot\mathbf{m}),\quad F_{\rm field}=\frac{1}{8\pi}\mathrm{tr}(\mathbf{f}). \tag{42}$$

The above free energy has been partially extremized since the kinetic energy is expressed in terms of the local magnetic field by means of Ampère's law. Since we are only interested here in the vortex configuration that extremizes the free energy, the above expression implies no loss of generality for our purposes.

In this London limit, the scaling relations of the previous section are given by

C frame:

$$H_1B_1=f_{11}+\frac{1}{2}(-m_aK_{11}+m_aK_{22}+m_cK_{33}), \tag{43}$$

$$H_2B_2=f_{22}+\frac{1}{2}(m_aK_{11}-m_aK_{22}+m_cK_{33}), \tag{44}$$

$$H_3B_3=f_{33}+\frac{1}{2}(m_aK_{11}+m_aK_{22}-m_cK_{33}), \tag{45}$$

B frame:

$$H_xB_x=f_{xx}+\frac{1}{2}(-m_{xx}K_{xx}+m_aK_{yy}+m_{zz}K_{zz})=0, \tag{46}$$

$$H_yB_y=f_{yy}+\frac{1}{2}(m_{xx}K_{xx}-m_aK_{yy}+m_{zz}K_{zz}+2m_{xz}K_{xz})=0, \tag{47}$$

$$H_zB_z=f_{zz}+\frac{1}{2}(m_{xx}K_{xx}+m_aK_{yy}-m_{zz}K_{zz}). \tag{48}$$

Next assume that the vortex lines are straight and tilted, like rods, all parallel to each other, making an angle θ with the c axis. In this case the vorticity reduces to $\mathbf{v}(\mathbf{x})=\nu_z(\mathbf{x})\hat{\mathbf{z}}$, $\nu_z(\mathbf{x})=\Sigma_{j=1}^N\delta^{(2)}(\mathbf{x}-\mathbf{r}_j)$ where $\mathbf{x}$ and $\mathbf{r}_j$ are position vectors restricted to xy plane, orthogonal to the direction of the lines (see Fig. 1). The local field $\mathbf{h}$ is first determined in the B frame, and then in the C frame, upon rotation, simply using $h_1=\cos\theta h_x+\sin\theta h_z$, and $h_3=-\sin\theta h_x+\cos\theta h_z$. In this case Eqs. (39) become

$$\frac{1}{\Lambda^2}h_x-m_{zz}\nabla_\parallel^2h_x+m_{xz}\nabla_y^2h_z=0,$$

$$\frac{1}{\Lambda^2}h_y-m_{zz}\nabla_\parallel^2h_y-m_{xz}\nabla_x\nabla_yh_z=0,$$

$$\frac{1}{\Lambda^2}h_z-m_a\nabla_x^2h_z-m_{xx}\nabla_y^2h_z+m_{xz}\nabla_\parallel^2h_x=\frac{\Phi_0}{\Lambda^2}\nu_z\,, \tag{49}$$

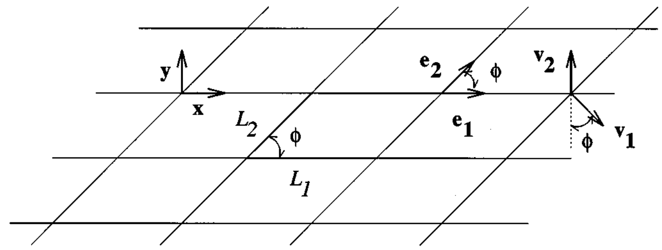

FIG. 2. On the plane xy, orthogonal to the magnetic induction $\mathbf{B}$, the parameters defining the vortex lattice are displayed, as well as the unit vectors of the unit cell, in both real and reciprocal space.

where $\nabla_\parallel^2=\nabla_x^2+\nabla_y^2$. The solution is sought in momentum space,

$$\mathbf{h}(\mathbf{x})=\int\mathbf{h}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})d^2\mathbf{k}/(2\pi)^2,$$

$$h_x(\mathbf{k})=m_{xz}(\Lambda k_y)^2h_z(\mathbf{k})\Omega(\mathbf{k})/P(\mathbf{k}),$$

$$h_y(\mathbf{k})=-m_{xz}(\Lambda k_x)(\Lambda k_y)h_z(\mathbf{k})\Omega(\mathbf{k})/P(\mathbf{k}),$$

and

$$h_z(\mathbf{k})=\Phi_0\Omega(\mathbf{k})\nu_z(\mathbf{k}),$$

where we have defined $P(\mathbf{k})=1+m_{zz}(\Lambda\mathbf{k})^2$, $Q(\mathbf{k})=1+m_a(\Lambda\mathbf{k})^2$, $R(\mathbf{k})=1+m_{zz}(\Lambda k_x)^2+m_c(\Lambda k_y)^2$, and $\Omega(\mathbf{k})=P(\mathbf{k})/Q(\mathbf{k})R(\mathbf{k})$. Once the local field is determined, the currents are also obtained by similar fashion. In the case of a single tilted vortex[19] a full description of this state is provided by Eqs. (49). Some time ago[8] it was discovered that, for a special range of the tilt angle θ, the local field h_z displays a change of sign along the x axis, that is, on the plane defined by the c axis and the magnetic induction. This feature gives rise to an attractive potential along this special plane, and consequently, yields the formation of the so-called vortex chains, experimentally verified in the ceramic compound YBCO.[9] The above equations also determine the solution for N vortex lines, since the vorticity is additive and consequently the superposition principle is valid. However this does not determine the collective vortex state completely since it remains to describe how the lines are arranged in space. Interestingly in order that RHS of the scaling relations, Eqs. (43)–(48), determine their left-hand side, LHS, the vortex state must be fully determined, including the collective arrangement of vortex lines in space.

Next we consider a tilted vortex line state forming a periodic array with one vortex per unit cell. Hence the array is characterized by the unit cell parameters, namely its two sides L_1, along the x axis, and L_2, that makes an angle ϕ with L_1 (see Fig. 2). One of these three parameters can be discarded because there is a constraint among them due to the magnetic flux quantization, $L_1L_2\sin\phi=\Phi_0/B_z$ (see Appendix B). Thus there are truly only two free parameters associated with the unit cell. We find convenient, for the present purpose, to parametrize the unit cell with new variables, σ and ω, also discussed in Appendix B:

$$\sigma \equiv L/L_1, \quad L \equiv \sqrt{\frac{\Phi_0}{B_z}}, \quad \omega \equiv L_2 \cos\phi / L_1. \tag{50}$$

The local magnetic field generated by such a periodic collective array of vortices must be a periodic function under displacements defined by the basic parameters of the unit cell. This periodic field is obtained from the single vortex solution simply by restricting the set of momentum space vectors, **k**, to a smaller set consistent with the lattice, the so-called reciprocal space vectors, **g**. We describe the reciprocal space through dimensionless reciprocal space vectors, $\mathbf{G} = \Lambda \mathbf{g}$. Similarly to $\Lambda \mathbf{k}$, the vector **G** lies in the plane xy, assuring the translational symmetry of **h** along the z axis (see Appendix B).

We introduce here the following notation used extensively in the next discussions:

$$P(\mathbf{G}) = 1 + m_{zz}\mathbf{G}^2, \tag{51}$$

$$Q(\mathbf{G}) = 1 + m_a\mathbf{G}^2, \quad Q_c(\mathbf{G}) = 1 + m_c\mathbf{G}^2, \tag{52}$$

$$R(\mathbf{G}) = 1 + m_{zz}G_x^2 + m_c G_y^2, \quad R_a(\mathbf{G}) = 1 + m_{zz}G_x^2 + m_a G_y^2,$$

$$R_{ca}(\mathbf{G}) = 1 + m_c G_x^2 + m_a G_y^2, \tag{53}$$

$$\Omega(\mathbf{G}) = \frac{P}{QR}. \tag{54}$$

For future purposes we list some identities involving the above polynomials.

$$R = P + (m_c - m_{zz})G_y^2, \tag{55}$$

$$m_{zz}Q(P-R) + m_a(m_c - m_{zz})G_y^2 = -m_{xz}^2 G_y^2, \tag{56}$$

$$-m_{xz}^2 G_y^4 + \sin^2\theta R_a^2 = P^2 - \cos^2\theta R^2, \tag{57}$$

$$-m_{xz}^2 G_y^4 + \cos^2\theta R^2 = P^2 - \sin^2\theta R_a^2, \tag{58}$$

$$m_{xz}^2 G^2 G_y^2 + P^2 = P(Q+R) - QR, \tag{59}$$

$$-R^2 + P(Q+R) - QR = (m_a - m_c)\sin^2\theta (Q+R) G_y^2, \tag{60}$$

$$-R_a^2 + P(Q+R) - QR = -(m_a - m_c)\cos^2\theta (R + R_{ac}) G_y^2. \tag{61}$$

Using the previously introduced notation it becomes straightforward to express the components of the tensors **h**, **J**, **K**, and **f** in reciprocal space. This is first obtained in the B frame, and then in the C frame, with the help of some Appendix A identities, like Eq. (A3) and Eq. (A4):

B frame:

$$h_x(\mathbf{x}) = m_{xz}\Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \frac{\Omega}{P} G_y^2 \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right), \tag{62}$$

$$h_y(\mathbf{x}) = -m_{xz}\Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \frac{\Omega}{P} G_x G_y \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right), \tag{63}$$

$$h_z(\mathbf{x}) = \Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \Omega \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$\frac{4\pi J_x(\mathbf{x})}{c} = \frac{\partial h_z}{\partial y} = i\Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \Omega \frac{G_y}{\Lambda} \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$\frac{4\pi J_y(\mathbf{x})}{c} = -\frac{\partial h_z}{\partial x} = -i\Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \Omega \frac{G_x}{\Lambda} \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right), \tag{64}$$

$$\frac{4\pi J_z(\mathbf{x})}{c} = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$$

$$= -i m_{xz}\Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \frac{\Omega}{P} \mathbf{G}^2 \frac{G_y}{\Lambda} \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$K_{xx} = B_z^2 \sum_{\mathbf{G}} \Omega^2 G_y^2, \quad K_{yy} = B_z^2 \sum_{\mathbf{G}} \Omega^2 G_x^2, \tag{65}$$

$$K_{zz} = B_z^2 m_{xz}^2 \sum_{\mathbf{G}} \frac{\Omega^2}{P^2} \mathbf{G}^4 G_y^2, \quad K_{xz} = -B_z^2 m_{xz} \sum_{\mathbf{G}} \frac{\Omega^2}{P} \mathbf{G}^2 G_y^2,$$

$$f_{xx} = B_z^2 m_{xz}^2 \sum_{\mathbf{G}} \frac{\Omega^2}{P^2} G_y^4, \quad f_{yy} = B_z^2 m_{xz}^2 \sum_{\mathbf{G}} \frac{\Omega^2}{P^2} G_x^2 G_y^2, \tag{66}$$

$$f_{zz} = B_z^2 \sum_{\mathbf{G}} \Omega^2,$$

C frame:

$$h_1(\mathbf{x}) = \sin\theta \Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \frac{R_a}{PR} \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$h_2(\mathbf{x}) = h_y(\mathbf{x}), \tag{67}$$

$$h_3(\mathbf{x}) = \cos\theta \Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \Omega \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$\frac{4\pi J_1(\mathbf{x})}{c} = i\cos\theta \Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \frac{Q_c}{PR} \frac{G_y}{\Lambda} \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$J_2(\mathbf{x}) = J_y(\mathbf{x}) \tag{68}$$

$$\frac{4\pi J_3(\mathbf{x})}{c} = -i\sin\theta \Phi_0 \frac{N}{A} \sum_{\mathbf{G}} \frac{Q}{PR} \frac{G_y}{\Lambda} \exp\left(i \frac{\mathbf{G}}{\Lambda} \cdot \mathbf{x} \right),$$

$$K_{11} = B_z^2 \cos^2\theta \sum_{\mathbf{G}} \Omega^2 \frac{Q_c^2}{P^2} G_y^2, \quad K_{22} = B_z^2 \sum_{\mathbf{G}} \Omega^2 G_x^2, \tag{69}$$

$$K_{33} = B_z^2 \sin^2\theta \sum_{\mathbf{G}} \frac{1}{R^2} G_y^2,$$

$$f_{11}=B_z^2\sin^2\theta\sum_{\mathbf{G}}\frac{\Omega R_a^2}{P^2},\quad f_{22}=B_z^2 m_{xz}^2\sum_{\mathbf{G}}\frac{\Omega^2}{P^2}G_x^2G_y^2,$$

$$f_{33}=B_z^2\cos^2\theta\sum_{\mathbf{G}}\frac{1}{Q^2}. \tag{70}$$

In reciprocal space the free-energy terms of Eq. (42) become

$$F_{\text{field}}=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}\Omega^2\left(1+\frac{m_{xz}^2}{P^2}G_y^2\mathbf{G}^2\right),$$

$$F_{\text{kin}}=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}\left[\Omega-\Omega^2\left(1+\frac{m_{xz}^2}{P^2}G_y^2\mathbf{G}^2\right)\right], \tag{71}$$

and the total free-energy density is

$$F=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}\Omega. \tag{72}$$

Notice that the unit cell parameters that extremize the free energy have not been determined up to this point. So far we have just found the local magnetic field that solves Eqs. (49) and this solution just takes that vortices form an arbitrary periodic array without any information about the optimal lattice configuration. This means that the above reciprocal space formulation of all tensors, including the total free energy, only contain a partial extremization of the problem. The complete extremization is achieved once the optimal array configuration is determined. Therefore we extremize the free energy with respect to the relevant parameters, $\partial F/\partial\sigma=0$ and $\partial F/\partial\omega=0$, respectively:

$$\sum_{\mathbf{G}}\frac{1}{(QR)^2}[-PQ(m_{zz}G_x^2-m_cG_y^2)-PRm_a(G_x^2-G_y^2)$$
$$+QRm_{zz}(G_x^2-G_y^2)]=0 \tag{73}$$

and

$$\sum_{\mathbf{G}}\frac{1}{(QR)^2}(-PQm_c-PRm_a+QRm_{zz})G_xG_y=0. \tag{74}$$

Both formulas are derived simply using $\dot{\Omega}=(-PQ\dot{R}-P\dot{Q}R+\dot{P}QR)/(QR)^2$, together with Eqs. (B10), where $\dot{\Omega}$ means $\partial\Omega/\partial\sigma$ or $\partial\Omega/\partial\omega$. The two conditions, Eq. (73) and Eq. (74), determine the unit cell parameters for the tilted straight vortex lattice limit, described by the extreme values of σ and ω. Such lattice configurations are well known in the literature at several special limits of the vortex density. At the low vortex density limit, one has the formation of chains[20,21] and, for a high vortex density, one verifies the onset of a distorted trianglular lattice.[22] Another relevant remark is that regardless of the vortex density value, there is always a distorted rectangular unit cell that extremizes the free energy. Generally speaking Eqs. (73) and (74) must admit at least two solutions, one corresponds to a local energy maximum, and the other to the absolute minimum.[21] In the isotropic limit they are the square and the triangular lattice, respectively.

A. H from thermodynamics

The Helmholtz free-energy density, Eq. (72), is a function of the magnetic induction modulus, B_z, and the angle θ between the c axis and $\mathbf{B}$: $F(B_z,\theta)$. According to the thermodynamic relation, Eq. (1), the magnetic field components are given by

$$\frac{H_1}{4\pi}=\sin\theta\frac{\partial F}{\partial B_z}+\frac{\cos\theta}{B_z}\frac{\partial F}{\partial\theta},\quad H_2=0,$$

$$\frac{H_3}{4\pi}=\cos\theta\frac{\partial F}{\partial B_z}-\frac{\sin\theta}{B_z}\frac{\partial F}{\partial\theta}. \tag{75}$$

The partial derivatives $\partial B_z/\partial B_i$ and $\partial\theta/\partial B_i$, for $i=1,2$, are obtained from Eq. (15). Direct inspection of the above relations also gives $\mathbf{H}$ in the B frame,

$$H_x=\frac{4\pi}{B_z}\frac{\partial F}{\partial\theta}=-B_z\sin\theta\cos\theta\sum_{\mathbf{G}}(m_c-m_a)\frac{Q_c}{QR^2}g_y^2, \tag{76}$$

$$H_z=4\pi\frac{\partial F}{\partial B_z}=\frac{B_z}{2}\sum_{\mathbf{G}}\frac{PQR+P(Q+R)-QR}{Q^2R^2}. \tag{77}$$

We discuss below some aspects of the explicit derivation of the free energy, Eq. (72), with respect to B_z and θ. Notice that the free energy of Eq. (72) has both an explicit and an implicit dependence on B_z, the last one due to the lattice parameters as described in Eq. (50). Such implicit dependence appears through the dimensionless reciprocal space vector, Eq. (B9), thus giving that $\partial\mathbf{G}/\partial B_z=\mathbf{G}/(2B_z)$. From the above considerations one obtains $\partial F/\partial B_z=(2/B_z)F+[1/(2B_z)]\mathbf{G}\cdot\partial F/\partial\mathbf{G}$. The derivative of the free energy with respect to θ is straightforward, the reciprocal vector $\mathbf{G}$ does not depend on θ, and only the matrix elements of the tensor $\mathbf{m}(\theta)$ do [see Eq. (13)].

Interestingly, the reciprocal space expressions of Eqs. (77) and (76) can be derived without ever having to explicitly take a derivative of the free energy. The scaling relations for $\mathbf{H}$ can do this, as shown in the next subsection.

B. H from the scaling relations

Here we show that the London limit of the scaling relations in both the C frame, Eqs. (43)–(45), and in the B frame, Eqs. (46)–(48), are satisfied by the tilted vortex lattice. We start in the B frame with the vanishing scaling relations, Eqs. (46) and (47), whose LHS are zero by definition, $B_x=B_y=0$, and consequently, their RHS must also vanish. We find advantageous to check that the sum and the difference of these two equations are fulfilled instead:

$$H_xB_x+H_yB_y=(m_{xz}K_{xz}+m_{zz}K_{zz})+(f_{xx}+f_{yy}), \tag{78}$$

$$H_xB_x-H_yB_y=(-m_{xz}K_{xz}-m_{xx}K_{xx}+m_aK_{yy})+(f_{xx}-f_{yy}). \tag{79}$$

To show that the RHS of the above equations vanish, we make use of Eqs. (65), (66), (69), (70), and write all the necessary tensor components in reciprocal space. One obtains

$$m_{xz}K_{xz}+m_{zz}K_{zz}=B_z^2m_{xz}^2\sum_{\mathbf{G}}\ \Omega^2\left(-\frac{1}{P}+\frac{m_{zz}\mathbf{G}^2}{P^2}\right)G_y^2\mathbf{G}^2$$

$$=-B_z^2m_{xz}^2\sum_{\mathbf{G}}\frac{\Omega^2}{P^2}G_y^2\mathbf{G}^2, \tag{80}$$

$$f_{xx}+f_{yy}=B_z^2m_{xz}^2\sum_{\mathbf{G}}\frac{\Omega^2}{P^2}G_y^2(G_x^2+G_y^2), \tag{81}$$

and

$$-m_{xz}K_{xz}-m_{xx}K_{xx}+m_aK_{yy}=B_z^2\sum_{\mathbf{G}}\ \Omega^2\bigg(m_a(G_x^2-G_y^2)$$

$$-(m_c-m_{zz})\frac{Q}{P}G_y^2\bigg), \tag{82}$$

$$f_{xx}-f_{yy}=B_z^2m_{xz}^2\sum_{\mathbf{G}}\frac{\Omega^2}{P^2}G_y^2(-G_x^2+G_y^2). \tag{83}$$

Direct inspection shows that the sum of Eq. (80) and of Eq. (81) vanishes and this agrees with the LHS of Eq. (78), $H_xB_x+H_yB_y=0$. However the RHS of Eq. (79) does not vanish automatically,

$$H_xB_x-H_yB_y=B_z^2\sum_{\mathbf{G}}\ [(m_aP^2-m_{xz}^2G_y^2)(G_x^2-G_y^2)$$

$$-(m_c-m_{zz})PQG_y^2]\frac{1}{(QR)^2}. \tag{84}$$

The most surprising result of this paper is that the vanishing of the RHS of the above equation has already been shown here. This is just the same condition that extremizes the free energy with respect to one of the lattice parameters:

$$H_xB_x-H_yB_y=8\pi\frac{\partial F}{\partial\sigma}=0. \tag{85}$$

We have shown that the sum and difference of H_xB_x and H_yB_y are null, consequently the RHS of Eq. (46) and of Eq. (47) vanish, and conclude that two B-frame scaling relations are valid. Further details showing that Eq. (84) and Eq. (73) are indeed the same condition can be found in Appendix C. To completely determine the magnetic field $\mathbf{H}$ in the B frame, by the scaling procedure, it remains to obtain the component H_z. Instead of Eq. (48) we consider the sum of the three scaling relations, Eqs. (46)–(48).

$$F_{\mathrm{kin}}+2F_{\mathrm{field}}=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}\frac{PQR+P(Q+R)-QR}{Q^2R^2}. \tag{86}$$

Thus we conclude that the scaling relations do determine correctly the magnetic field $\mathbf{H}$ in the B frame since $F_{\mathrm{kin}}+2F_{\mathrm{field}}=H_zB_z/4\pi$ and this is identical to a previous result, Eq. (77), derived from a thermodynamic argument. Notice that scaling in the B frame, Eqs. (46) and (47), do not determine the transverse components, H_x and H_y because $B_x=B_y=0$.

The scaling relations are also valid in the C frame as shown below. We show that the RHS of Eqs. (43)–(45) are consistent with the previous results obtained in the B frame. Notice that upon this particular rotation around the $2\equiv y$ axis, three quantities remain invariant, namely the scalar product, $H_1B_1+H_3B_3=H_xB_x+H_zB_z$, the vector component $H_2B_2=H_yB_y$, which vanishes, and also a pseudovector component, $H_1B_3-H_3B_1=H_xB_z-H_zB_x$.

The invariance of the scalar product means that the sum of two sets of RHS must be the same, namely Eq. (43) plus Eq. (45), and Eq. (46) plus Eq. (48). This condition is $f_{xx}+f_{zz}+m_aK_{yy}=f_{11}+f_{33}+m_aK_{22}$, which can be easily shown to hold, by invariance under rotation around the $2\equiv y$ axis.

We have already shown that the RHS of Eq. (47) gives $H_yB_y=0$. To show that the RHS of Eq. (44) also gives $H_2B_2=0$, we observed that the kinetic and field energies are independently invariant under this rotation: $f_{22}=f_{yy}$, $K_{22}=K_{yy}$, and $m_aK_{11}+m_cK_{33}=m_{xx}K_{xx}+m_{zz}K_{zz}+2m_{xz}K_{xz}$.

Now we proceed to show that the remaining scaling relations in the C frame are also valid, taking into account the above considerations. It is straightforward to verify that the RHS of the two scaling relations $H_1B_1+H_3B_3$ and $H_xB_x+H_zB_z$ are equal. To show that the RHS of $H_1B_3-H_3B_1$ and $H_xB_z-H_zB_x$ are the same demands some further work. To verify the above relations we start noticing that the RHS of Eq. (47) gives $H_yB_y=0$, and find that $2f_{22}=m_a(K_{22}-K_{11})-m_cK_{33}$. Introducing this expression into the two other scaling relations gives

$$H_1B_1=f_{11}+f_{22}+m_cK_{33},\quad H_2B_2=0,$$

$$H_3B_3=f_{22}+f_{33}+m_aK_{11}. \tag{87}$$

Expressing Eqs. (87) into reciprocal space gives

$$H_1B_1=B_z^2\sum_{\mathbf{G}}\frac{1}{(QR)^2}(m_c\sin^2\theta Q^2G_y^2+m_{xz}^2G_x^2G_y^2$$

$$+\sin^2\theta R_a^2), \tag{88}$$

$$H_3B_3=B_z^2\sum_{\mathbf{G}}\frac{1}{(QR)^2}(m_a\cos^2\theta Q_c^2G_y^2+m_{xz}^2G_x^2G_y^2$$

$$+\cos^2\theta R^2). \tag{89}$$

The use of the above relations allows us to compute the pseudovector component $H_3B_1-H_1B_3=H_3B_3\tan\theta-H_1B_1/\tan\theta$. This is our starting point to show that the thermodynamic relation for H_x, given by Eq. (76), can be obtained by the scaling method. The goal is to show that the RHS of $H_1B_3-H_3B_1$, calculated from the scaling relations, is identical to H_xB_z, obtained from thermodynamics. Appendix C provides more details in this proof that the scaling method does lead to the thermodynamic H_x expression of Eq. (76). We have just found a way to determine H_x from C-frame scaling information. H_x cannot be directly determined in the B frame from scaling because $B_x=0$.

In summary, we have shown, in this section, that the $\mathbf{H}$ components can be obtained either by the thermodynamic relations, Eq. (76) and Eq. (77), or by the scaling method. We also found that Eqs. (78) and (79) supply useful information about the collective properties of the vortex state. Our comparison is restricted to the situation of straight tilted vortex lines forming a periodic array with one vortex per unit cell. A better insight into the results obtained so far is found in the next section where we revisit the isotropic superconductor.

C. The isotropic limit

In the isotropic limit there is only one mass parameter identical to the average mass, $\underline{M}=M$ and, consequently, the dimensionless tensors become the identity, $\mathbf{m}=\mathbf{I}$ and $\mathbf{m}'=\mathbf{I}$. No new results are obtained in this subsection, the only goal is to make the previous results more transparent to the reader, in particular the scaling identites $H_xB_x=0$ and $H_yB_y=0$, given by Eqs. (46) and (47), and the lattice conditions, $\partial F/\partial\sigma=0$, and $\partial F/\partial\omega=0$, given by Eqs. (73) and (74). In the last subsection we found that such scaling identities demand that the lattice conditions be fulfilled and according to Eq. (85) one must have $H_xB_x-H_yB_y=8\pi\partial F/\partial\sigma=0$. It is well known that the square and the triangular lattices[23] are the two configurations that extremize the isotropic vortex state and therefore they must satisfy the above identities and conditions.

We stress that the scaling relations for the anisotropic GL theory cannot be read off from the isotropic case, unless in the special limit of large κ and high vortex density.[24] Long ago Klemm and Clem[25] have introduced a scaling procedure, different from the present one, that maps the anisotropic London theory into the isotropic one. This transformation has been used recently[7] to obtain several results in the anisotropic case from the isotropic theory. This map is limited to the high vortex density limit, where the average London penetraton length is much larger than the distance between two consecutive vortices on the lattice.

In the isotropic limit all the reciprocal space polynomials of Eq. (54) collapse into a single one, $P=Q=Q_c=R=R_a=R_{ca}$, such that

$$\Omega'(\mathbf{G})=\frac{1}{1+\mathbf{G}^2}. \tag{90}$$

Then one gets in the B frame that

$$h_x(\mathbf{x})=h_y(\mathbf{x})=0,\quad h_z(\mathbf{x})=\Phi_0\frac{N}{A}\sum_{\mathbf{G}}\Omega'\exp\left(i\frac{\mathbf{G}}{\Lambda}\cdot\mathbf{x}\right), \tag{91}$$

$$\frac{4\pi J_x(\mathbf{x})}{c}=i\Phi_0\frac{N}{A}\sum_{\mathbf{G}}\Omega'\frac{G_y}{\Lambda}\exp\left(i\frac{\mathbf{G}}{\Lambda}\cdot\mathbf{x}\right),$$

$$\frac{4\pi J_y(\mathbf{x})}{c}=-i\Phi_0\frac{N}{A}\sum_{\mathbf{G}}\Omega'\frac{G_x}{\Lambda}\exp\left(i\frac{\mathbf{G}}{\Lambda}\cdot\mathbf{x}\right), \tag{92}$$

$$\frac{4\pi J_z(\mathbf{x})}{c}=0,$$

$$K_{xx}=B_z^2\sum_{\mathbf{G}}\Omega'^2G_y^2,\quad K_{yy}=B_z^2\sum_{\mathbf{G}}\Omega'^2G_x^2,$$

$$K_{zz}=K_{xz}=0, \tag{93}$$

$$f_{xx}=f_{yy}=0,\quad f_{zz}=B_z^2\sum_{\mathbf{G}}\Omega'^2. \tag{94}$$

The isotropic free energy is

$$F_{\rm field}=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}\Omega'^2,\quad F_{\rm kin}=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}(\Omega'-\Omega'^2),$$

$$F=\frac{B_z^2}{8\pi}\sum_{\mathbf{G}}\Omega'. \tag{95}$$

In the isotropic limit the local magnetic field is only along the direction of the vortex lines ($h_x=h_y=0$, $h_z\neq0$), and its intensity decays exponentially away from the core of each vortex located on the lattice.

The isotropic magnetic field $\mathbf{H}$, obtained either by thermodynamics, Eqs. (77) and (76), or by scaling, Eqs. (86) and (C9), is such that $H_z=(B_z/2)\Sigma_{\mathbf{G}}(\Omega'+\Omega'^2)$ and $H_x=0$. Thus $\mathbf{H}$ and $\mathbf{B}$ are aligned to each other in the isotropic limit, a fact that is not true for the anisotropic superconductor, where their misalignment produces an intrinsic magnetic torque[26,27] that has led to interesting experimental consequences.[28] There is no isotropic kinetic energy along the z axis, $K_{zz}=0$, since the current is constrained to the plane xy, orthogonal to the vortex lines. Thus the scaling identities in the B frame demand that the average kinetic energies, along the x and y directions be equal, $K_{xx}=K_{yy}$, in order to have the magnetic induction oriented along the z axis ($B_x=B_y=0$). These identities require the fulfillment of the first of the two lattice conditions, Eqs. (73) and (74), that extremize the free energy with respect to the unit cell parameters:

$$\frac{\partial F}{\partial\sigma}=\sum_{\mathbf{G}}\Omega'^2(G_x^2-G_y^2)=0, \tag{96}$$

$$\frac{\partial F}{\partial\omega}=\sum_{\mathbf{G}}\Omega'^2G_xG_y=0. \tag{97}$$

Among the many possible choices of unit cell for a given lattice, we pick the simplest ones for each of these two lattices. The corresponding reciprocal space vectors are obtained from Eq. (B9).

Square lattice. For the unit cell $L_1/L_2=1$, $\phi=90°$, or $\sigma=1$, $\omega=0$, one gets the reciprocal space vector components $G_x=(2\pi\Lambda/L)q_1$ and $G_y=(2\pi\Lambda/L)q_2$. Equations (96) and (97) become

$$\sum_{q_1,q_2}\frac{1}{1+\left(\frac{2\pi\Lambda}{L}\right)^2(q_1^2+q_2^2)}(q_1^2-q_2^2)=0, \tag{98}$$

$$\sum_{q_1,q_2}\frac{1}{1+\left(\frac{2\pi\Lambda}{L}\right)^2(q_1^2+q_2^2)}q_1q_2=0. \tag{99}$$

Triangular lattice. For the unit cell $L_1/L_2=1$, $\phi=60°$, or $\sigma=\sqrt{3/2}$, $\omega=1/2$, one gets the reciprocal space vector components $G_x=(2\pi\Lambda/\sqrt{6}L)3q_1$ and $G_y=(2\pi\Lambda/\sqrt{6}L)(2q_2-q_1)$. Equations (96) and (97) become

$$\sum_{q_1,q_2} \frac{1}{1+\left(\frac{2\pi\Lambda}{\sqrt{6}L}\right)^2 [(3q_1)^2+(2q_2-q_1)^2]} \times[(3q_1)^2-(2q_2-q_1)^2]=0, \quad (100)$$

$$\sum_{q_1,q_2} \frac{1}{1+\left(\frac{2\pi\Lambda}{\sqrt{6}L}\right)^2 [(3q_1)^2+(2q_2-q_1)^2]}(3q_1)(2q_2-q_1) =0. \quad (101)$$

The arguments showing that the two free-energy derivatives vanish are the same for both lattices. For the first set of identities, Eqs. (98) and (100), pick one particular term in the first sum associated to $q_1=m$, m any integer. Then there is always a choice in the second sum, $q_2=-m$, that renders the series terms equal to zero. For the second set of identities, Eqs. (99) and (101), the sum of two distinct terms of the series, associated to $q_1=m$, $q_2=m$ and $q_1=-m$, $q_2=m$, give that they also vanish.

In the C frame, one gets that all local fields are nicely expressed as simple rotations of their B frame counterparts: the local magnetic field $h_1(\mathbf{x})=\sin\theta h_z(\mathbf{x})$, $h_2(\mathbf{x})=0$, $h_3(\mathbf{x})=\cos\theta h_z(\mathbf{x})$, the supercurrent $J_1(\mathbf{x})=\cos\theta J_x(\mathbf{x})$, $J_2(\mathbf{x})=J_y(\mathbf{x})$, $J_3(\mathbf{x})=-\sin\theta J_x(\mathbf{x})$, the kinetic energy $K_{11}=\cos^2\theta K_{xx}$, $K_{22}=K_{yy}$, $K_{33}=\sin^2\theta K_{xx}$, and the field energy $f_{11}=\sin^2\theta f_{zz}$, $f_{22}=f_{yy}$, $f_{33}=\cos^2\theta f_{zz}$. The isotropic limit of Eqs. (C7) and (C8) give that $H_1B_1/\sin^2\theta= H_3B_3/\cos^2\theta= B_z^2\Sigma_{\mathbf{G}}\Omega'^2(G_y^2+1)$. Using the unit cell condition of Eq. (96), it follows [see also Eqs. (87)] that the scaling relations become $H_1B_1=\sin^2\theta H_zB_z$, $H_2B_2=0$, and $H_3B_3=\cos^2\theta H_zB_z$. One also gets that $H_3B_1-H_1B_3=0$ consistent with the fact of no transverse field in the isotropic limit $H_x=0$.

In summary in the isotropic limit both triangular and square lattices were shown to verify the scaling identities corresponding to $H_xB_x=H_yB_y=0$. According to the scaling relations, and also to the thermodynamic equation [Eq. (1)], for the same vortex density, B_z, triangular and square lattices result in distinct values of H_z. The square lattice is not interesting because it is an unstable configuration of the free energy. As pointed out before, the anisotropic problem cannot be mapped into the isotropic case except in special limits.

V. CONCLUSION

The virial theorem is a useful tool to determine $\mathbf{H}$ when numerical techniques must be introduced in the Ginzburg-Landau theory. While the thermodynamic relation, Eq. (1), requires the free energy at two values of $\mathbf{B}$ in order to compute a derivative, the virial relation demands the knowledge of the free energy only at a single value of $\mathbf{B}$. Another application of the scalar virial theorem, pointed out some time ago,[29] is in the determination of the elastic properties of the vortex state within the Ginzburg-Landau context. For isotropic superconductors, the tilt modulus in the limit of very large wavelength can be directly related to the sum of kinetic and field free-energy contributions, as described in the scalar virial theorem, since $c_{44}=2HB$.[30] The scalar virial theorem does not determine the magnetic field completely and, especially in case of anisotropy, its generalization to a vector virial theorem becomes quite useful. In fact, such generalization has been proposed, although not studied, by the authors [1] of the scalar virial theorem. The present scaling procedure is very similar to the one leading to the scalar virial theorem, the only difference being that each spatial coordinate is independently scaled here, while previously this was done simultaneously in all coordinates.

Anisotropy introduces some interesting theoretical remarks because scaling along the crystal's axes (C frame) and along the orthogonal axes, defined by magnetic induction (B frame), result into distinct scaling relations. In the B frame the scaling relations only determine H_z, the projection of $\mathbf{H}$ along $\mathbf{B}=B_z\hat{\mathbf{z}}$. Nevertheless in the B frame two identities are obtained, related to the fact that $H_xB_x=0$, Eq. (16), and $H_yB_y=0$, Eq. (17). Such identities give information on the collective properties of the vortex state that extremizes the Helmholtz free energy.

We have compared the scaling to the thermodynamic relations in a special limit of the AGL theory, where $\mathbf{H}$ is obtained analytically. This is the London limit and we have treated the problem of straight parallel vortices tilted with respect to the c axis. We have found that elastic properties of the vortex lattice can be related to the thermodynamic fields according to $H_xB_x-H_yB_y=8\pi\partial F/\partial\sigma$ [see Eq. (85)]. Hence two identities, related to $H_xB_x=0$ and $H_yB_y=0$, are fulfilled once a property of the unit cell that extremizes the free energy is included, $\partial F/\partial\sigma=0$. Thus we have obtained, through the scaling relation, identities that provide information on the properties of the collective state of the vortices, as previously discussed. In case the collective vortex state is driven by disorder, like in a spin glass,[16] we believe that such B frame identities should be helpful to elucidate the properties of such state. Notice that we were only able to obtain the magnetic field component perpendicular to the magnetic induction through the help of the vector identity $H_xB_z=H_1B_3-H_3B_1$. This shows that the B frame scaling relations are not able to determine all components of $\mathbf{H}$, although they supply information on the collective state not given by the C frame scaling relations. It seems to us that in the case of anisotropic superconductors, the two sets of scaling relations are both useful because they contain some complementary information about the superconductor's state.

In the present AGL context disorder can be introduced through the assumption that the parameters acquire a spatial dependence, $\alpha_o(\mathbf{x})$, $T_c(\mathbf{x})$, $\beta_o(\mathbf{x})$, $\mu(\mathbf{x})$, and $m(\mathbf{x})$. Recently there have been suggestions of unconventional pairing symmetry for the high temperature superconductors.[31] Extensions of the Ginzburg-Landau theory containing order parameters of higher symmetry have been studied before.[32] Although we have not derived expressions for $\mathbf{H}$ in these cases, it is obvious that the present scaling procedure easily applies to them. We believe that, for both cases of disorder and of an order parameter of higher symmetry, the present vector virial theorem provides a useful tool to unveil the magnetic properties of the underlying GL theories.

In summary we have obtained in this paper, by scaling arguments, new relations that determine $\mathbf{H}$ for the AGL theory. Perfect agreement between the scaling relations and the thermodynamic method has been found in the London limit. We propose that the **H** relations should be valid throughout the mixed state.

APPENDIX A: MATRIX PROPERTIES

Under the rotation $R(\theta)$ given in Eq. (12) the covariant derivative transforms, as any other vector, according to $\mathbf{D}\psi_\theta = R(\theta)\cdot\mathbf{D}\psi_{\,\mathrm{crystal}}$. Knowledge of the invariance $[\mathbf{D}\psi]_\theta^\dagger\cdot\mathbf{m}'(\theta)\cdot[\mathbf{D}\psi]_\theta = [\mathbf{D}\psi]_{\mathrm{crystal}}^\dagger\cdot\mathbf{m}'\cdot[\mathbf{D}\psi]_{\mathrm{crystal}}$ gives that $\mathbf{m}'(\theta) = \mathbf{R}^\top(\theta)\cdot\mathbf{m}'\cdot\mathbf{R}(\theta)$. Consider the dimensionless mass matrices, $\mathbf{m}$ and $\mathbf{m}'$, upon rotation. The orthogonal transformation gives that $\det\mathbf{m}'(\theta) = \det\mathbf{m}'(0)$ and so

$$\mu_{xx}\mu_{zz} - \mu_{xz}^2 = \mu_a\mu_c\,, \quad m_{xx}m_{zz} - m_{xz}^2 = m_a m_c\,. \quad \text{(A1)}$$

Since we have

$$\cos\theta m_{xz} + \sin\theta m_{zz} = \sin\theta m_a\,, \quad \text{(A2)}$$

$$\cos\theta m_{zz} - \sin\theta m_{xz} = \sin\theta m_c\,, \quad \text{(A3)}$$

the explicit θ dependence of these matrix elements leads to the following relations

$$\frac{m_{xx}}{m_a m_c} = \mu_{zz}\,, \quad \frac{m_{xz}}{m_a m_c} = -\mu_{xz}\,, \quad \frac{m_{zz}}{m_a m_c} = \mu_{xx}\,. \quad \text{(A4)}$$

The dimensionless matrices are the inverse of each other, $\mathbf{m}'\cdot\mathbf{m} = I$, and in terms of components one gets

$$m_{xx}\mu_{xx} + m_{xz}\mu_{xz} = m_{zz}\mu_{zz} + m_{xz}\mu_{xz} = 1,$$

$$m_{xz}\mu_{xx} + m_{zz}\mu_{xz} = m_{xz}\mu_{zz} + m_{xx}\mu_{xz} = 0. \quad \text{(A5)}$$

The scaling relations can be expressed more compactly, if the kinetic term, proportional to $[\mathbf{D}\psi]^\dagger\cdot\mathbf{V_j}\cdot[\mathbf{D}\psi]$, is expressed in terms of the matrices

C frame:

$$\mathbf{V_1} = \begin{pmatrix} -\mu_a & 0 & 0 \\ 0 & \mu_a & 0 \\ 0 & 0 & \mu_c \end{pmatrix}, \quad \mathbf{V_2} = \begin{pmatrix} \mu_a & 0 & 0 \\ 0 & -\mu_a & 0 \\ 0 & 0 & \mu_c \end{pmatrix},$$

$$\mathbf{V_3} = \begin{pmatrix} \mu_a & 0 & 0 \\ 0 & \mu_a & 0 \\ 0 & 0 & -\mu_c \end{pmatrix}, \quad \text{(A6)}$$

B frame:

$$\mathbf{V_x} = \begin{pmatrix} -\mu_{xx} & 0 & 0 \\ 0 & \mu_a & 0 \\ 0 & 0 & \mu_{zz} \end{pmatrix}, \quad \mathbf{V_y} = \begin{pmatrix} \mu_{xx} & 0 & \mu_{xz} \\ 0 & -\mu_a & 0 \\ \mu_{xz} & 0 & \mu_{zz} \end{pmatrix},$$

$$\mathbf{V_z} = \begin{pmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_a & 0 \\ 0 & 0 & -\mu_{zz} \end{pmatrix}. \quad \text{(A7)}$$

In order to express the kinetic energy in terms of ρ and $\mathbf{J}$, we must calculate

$$\mathbf{W_j} \equiv \mathbf{m}\cdot\mathbf{V_j}\cdot\mathbf{m}, \quad \text{(A8)}$$

where $\mathbf{m}$ is given by Eq. (13). With the help of Eqs. (A1), (A4), (A5) one obtains

C frame:

$$\mathbf{W_1} = \begin{pmatrix} -m_a & 0 & 0 \\ 0 & m_a & 0 \\ 0 & 0 & m_c \end{pmatrix}, \quad \mathbf{W_2} = \begin{pmatrix} m_a & 0 & 0 \\ 0 & -m_a & 0 \\ 0 & 0 & m_c \end{pmatrix},$$

$$\mathbf{W_3} = \begin{pmatrix} m_a & 0 & 0 \\ 0 & m_a & 0 \\ 0 & 0 & -m_c \end{pmatrix}, \quad \text{(A9)}$$

B frame:

$$\mathbf{W_x} = \begin{pmatrix} -m_{xx} & 0 & 0 \\ 0 & m_a & 0 \\ 0 & 0 & m_{zz} \end{pmatrix},$$

$$\mathbf{W_y} = \begin{pmatrix} m_{xx} & 0 & m_{xz} \\ 0 & -m_a & 0 \\ m_{xz} & 0 & m_{zz} \end{pmatrix},$$

$$\mathbf{W_z} = \begin{pmatrix} m_{xx} & 0 & 0 \\ 0 & m_a & 0 \\ 0 & 0 & -m_{zz} \end{pmatrix}. \quad \text{(A10)}$$

APPENDIX B: RECIPROCAL SPACE

Let the local magnetic field created by a single vortex in real space be $\mathbf{h}(\mathbf{x})$, where $\mathbf{x}$ is the coordinate on the plane xy, orthogonal to the vortex line. Take that its Fourier transform is

$$\mathbf{h}(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\mathbf{h}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}). \quad \text{(B1)}$$

Now consider a periodic array of vortices whose positions in real space are determined by

$$\mathbf{L}(\mathbf{n}) = n_1\mathbf{L_1} + n_2\mathbf{L_2}, \quad \mathbf{L_1} = L_1\hat{\mathbf{e}}_\mathbf{1}, \quad \mathbf{L_2} = L_2\hat{\mathbf{e}}_\mathbf{2}, \quad \text{(B2)}$$

where (n_1, n_2) are a set of integers, and the unit vectors are chosen such that

$$\hat{\mathbf{e}}_\mathbf{1} \equiv \hat{\mathbf{x}}, \quad \hat{\mathbf{e}}_\mathbf{2} \equiv \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}. \quad \text{(B3)}$$

The vector $\hat{\mathbf{x}}$ is on the plane defined by the c-axis and the magnetic induction, and $\hat{\mathbf{y}}$ is orthogonal to this plane (see Fig. 2).

The local magnetic field produced by a periodic array of vortices takes the contribution of all vortices, and so must be given by

$$\tilde{\mathbf{h}}(\mathbf{x}) = \sum_{\mathbf{n}} \mathbf{h}[\mathbf{x} + \mathbf{L}(\mathbf{n})]. \quad \text{(B4)}$$

This is a periodic function, since discrete translations multiple of the basic lengths, L_1 and L_2, just shift the field to a position identical to the starting one: $\tilde{\mathbf{h}}[\mathbf{x}+\mathbf{L}(\mathbf{m})]=\tilde{\mathbf{h}}(\mathbf{x})$. Such a periodic function is better described in the so-called reciprocal space,

$$\tilde{\mathbf{h}}(\mathbf{x})=\frac{N}{A}\sum_{\mathbf{g}} \mathbf{h}(\mathbf{g})\exp(i\mathbf{g}\cdot\mathbf{x}), \tag{B5}$$

where N/A is the ratio between the total number of vortices, N, and the total area of the xy plane, A. An exponential condition, $\exp[i\mathbf{g}\cdot\mathbf{L}(\mathbf{n})]=1$, valid for all (n_1,n_2), assures the translational invariant property of the function $\tilde{\mathbf{h}}(\mathbf{x})$. The function $\mathbf{h}(\mathbf{g})$ is just the original Fourier transform, $\mathbf{h}(\mathbf{k})$. The reciprocal space is just a subset of the momentum space, such that $\mathbf{k}$ is restricted to the set of vectors $\mathbf{g}$, that satisfy the exponential condition, just described.

Flux quantization demands that a fluxon Φ_0 be associated to the area A/N of each vortex unit cell:

$$\frac{A}{N}=L_1L_2\sin\phi=\frac{\Phi_0}{B_z}. \tag{B6}$$

Thus the vortex density is completely determined by the magnetic induction.

The reciprocal space vectors $\mathbf{g}$ also form an array characterized by two integers, (q_1,q_2), such that $\mathbf{g}(\mathbf{q})=q_1\mathbf{g_1}+q_2\mathbf{g_2}$. Choosing $\mathbf{g_1}\cdot\mathbf{L_2}=\mathbf{g_2}\cdot\mathbf{L_1}=0$ and $\mathbf{g_1}\cdot\mathbf{L_1}=\mathbf{g_2}\cdot\mathbf{L_2}=2\pi$ (see Fig. 2), then $\mathbf{g}\cdot\mathbf{L}(\mathbf{n})=2\pi I$, I any integer, and we find that

$$\mathbf{g}(\mathbf{q})=\frac{2\pi}{\sin\phi}\left(\frac{q_1}{L_1}\hat{\mathbf{v}}_\mathbf{1}+\frac{q_2}{L_2}\hat{\mathbf{v}}_\mathbf{2}\right),\quad \hat{\mathbf{v}}_\mathbf{1}=\sin\phi\hat{\mathbf{x}}-\cos\phi\hat{\mathbf{y}},$$

$$\hat{\mathbf{v}}_\mathbf{2}=\hat{\mathbf{x}}. \tag{B7}$$

We define new parameters L, σ and ω describing the real space unit cell,

$$L\equiv\sqrt{\frac{\Phi_0}{B_z}}=\sqrt{L_1L_2\sin\phi},\quad \sigma\equiv\sqrt{\frac{L_2\sin\phi}{L_1}}=\frac{L}{L_1},$$

$$\omega\equiv\frac{L_2\cos\phi}{L_1}. \tag{B8}$$

The optimal lattice is determined by minimizing the free energy with respect to the variables σ and ω which represent L_1/L_2 and ϕ. We define the dimensionless reciprocal space vector as $\mathbf{G}=\Lambda\mathbf{g}$ that becomes, in terms of the the previously defined variables,

$$\mathbf{G}=G_x\hat{\mathbf{x}}+G_y\hat{\mathbf{y}},\quad G_x=\frac{2\pi\Lambda}{L}q_1\sigma,$$

$$G_y=\frac{2\pi\Lambda}{L}[-q_1\omega+q_2]\frac{1}{\sigma}. \tag{B9}$$

The following derivatives are useful for the present purposes:

$$\frac{\partial G_x}{\partial\sigma}=\frac{G_x}{\sigma},\quad \frac{\partial G_y}{\partial\sigma}=-\frac{G_y}{\sigma},\quad \frac{\partial G_x}{\partial\omega}=0,\quad \frac{\partial G_y}{\partial\omega}=-\frac{G_y}{\sigma^2}. \tag{B10}$$

Then the derivatives of the G_x, G_y polynomials, defined by Eqs. (51)–(53), are easily obtained:

$$\frac{\partial P}{\partial\sigma}=\frac{2}{\sigma}m_{zz}(G_x^2-G_y^2),\quad \frac{\partial Q}{\partial\sigma}=\frac{2}{\sigma}m_a(G_x^2-G_y^2),$$

$$\frac{\partial R}{\partial\sigma}=\frac{2}{\sigma}(m_{zz}G_x^2-m_cG_y^2), \tag{B11}$$

$$\frac{\partial P}{\partial\omega}=-\frac{2}{\sigma^2}m_{zz}G_xG_y,\quad \frac{\partial Q}{\partial\omega}=-\frac{2}{\sigma}m_aG_xG_y,$$

$$\frac{\partial R}{\partial\omega}=-\frac{2}{\sigma}m_cG_xG_y. \tag{B12}$$

APPENDIX C: MATHEMATICAL RELATIONS

In this appendix we provide to the interested reader some further details on how some mathematical relations are established in Sec. IV B.

First we outline a few intermediate steps showing that Eq. (84) and Eq. (73) are indeed the same condition. We start adding and taking a term $m_{zz}G_y^2PQ$ to Eq. (73),

$$\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{(m_c-m_{zz})PQG_y^2$$
$$+[m_{zz}Q(P-R)+m_aPR](G_y^2-G_x^2)\}=0. \tag{C1}$$

Equation (55) gives

$$\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{(m_c-m_{zz})PQG_y^2+[m_aP^2+m_{zz}Q(P-R)$$
$$+m_a(m_c-m_{zz})G_y^2](G_y^2-G_x^2)]\}=0. \tag{C2}$$

Finally one obtains from the above equation (C2) that the LHS of Eq. (84) vanishes. We find that Eq. (56) is helpful when verifying the above relation. In summary the RHS of Eq. (84) vanishes due to the actual extremization of the free energy with respect to lattice parameters.

Second we present some of the intermediate steps necessary to show that H_x from scaling is the same thermodynamic relation of Eq. (76). Using the auxiliary identities of Eq. (57) and of Eq. (58) one obtains

$$H_1B_1=B_z^2\sum_{\mathbf{G}}\frac{1}{(QR)^2}[m_c\sin^2\theta Q^2G_y^2$$
$$+(m_{xz}^2\mathbf{G}^2G_y^2+P^2)-R^2\cos^2\theta], \tag{C3}$$

$$H_3B_3=B_z^2\sum_{\mathbf{G}}\frac{1}{(QR)^2}[m_a\cos^2\theta Q_c^2G_y^2$$
$$+(m_{xz}^2\mathbf{G}^2G_y^2+P^2)-R_a^2\cos^2\theta]. \tag{C4}$$

Then applying another auxiliary relation, Eq. (59), gives

$$H_1B_1=B_z^2\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{\sin^2\theta(m_cQ^2G_y^2+R^2)+[-R^2+P(Q+R)-QR]\},\tag{C5}$$

$$H_3B_3=B_z^2\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{\cos^2\theta(m_aQ_c^2G_y^2+R_a^2)+[-R_a^2+P(Q+R)-QR]\}.\tag{C6}$$

Finally the use of auxiliary relations of Eq. (60) and of Eq. (61) yields

$$H_1B_1=B_z^2\sin^2\theta\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{m_cQ^2G_y^2+R^2+(m_a-m_c)(Q+R)G_y^2\},\tag{C7}$$

$$H_3B_3=B_z^2\cos^2\theta\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{m_aQ_c^2G_y^2+R_a^2-(m_a-m_c)(R+R_{ac})G_y^2\}.\tag{C8}$$

Now one is ready to obtain the pseudovector component using the above formula:

$$\begin{aligned}-H_xB_z=H_3B_1-H_1B_3&=B_z^2\sin\theta\,\cos\theta,\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{(m_aQ_c^2-m_cQ^2)G_y^2+(R_a^2-R^2)-(m_a-m_c)(2R+R_{ca}+Q)G_y^2\}\\&=B_z^2\sin\theta\,\cos\theta(m_a-m_c)\sum_{\mathbf{G}}\frac{1}{(QR)^2}\{1-m_am_c\mathbf{G}^4+(m_a-m_c)G_y^2-Q-R_{ca}\}G_y^2\\&=B_z^2\sin\theta\,\cos\theta(m_c-m_a)\sum_{\mathbf{G}}\frac{Q_c}{QR^2}G_y^2.\end{aligned}\tag{C9}$$

Comparison of the above equation to Eq. (76) shows that they are identical.

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