Replica treatment of the effective elastic behavior of a composite

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We use the replica trick and a variational method to determine the effective elastic coefficients of a disordered composite. We obtain for them a self-consistent formula, which is satisfactory from the points of view of low disorder and low dilution expansions. When the bulk moduli K and the shear moduli μ are such that $K \gtrsim 2\mu$, it satisfies Hashin-Shtrikman bounds and is close to the usual effective-medium approximation. In the case $K \gtrsim 2\mu$, we observe a deviation of Hashin-Shtrikman bounds which can be understood by analogy with an equivalent one-dimensional problem. Finally, this calculation allows us to derive the rigidity threshold p_r for any dimension d.

INTRODUCTION

An effective medium is a quasihomogeneous description of an inhomogeneous system. Such a description is meaningful when the typical length of these inhomogeneities is regarded as small compared with the observation length. The determination of the effective behavior of a disordered system is not simple, and one usually uses the effective-medium theory (introduced by Bruggeman[1] in the dielectric case: for a review see Ref. 2), which is essentially a self-consistent one-site approximation.[3] Besides, there are exacts bounds (Hashin and Shtrikman[4]) that any approximation must satisfy.

This problem is of great importance because of its wide range of applications due to the development of composite materials. A quantity of interest is, in particular, the rigidity threshold. When one of the two materials is a quasivacuum with a proportion $1-p$, the effective behavior of the composite will differ from the vacuum one only above a finite value of p: the "rigidity threshold" which will be denoted by p_r. Our method allows its computation for any space dimension.

It was shown in Ref. 5 that one can relate the effective coefficients to the average of the inverse of a random operator. In order to calculate this average, we use the replica trick and a variational approximation used in Ref. 6. This method was applied to the propagation of electromagnetic waves in disordered dielectrics[7] and to the Hall effect.[8] We here adapt this point of view to the elastic case and derive the corresponding formulas for the effective static behavior of a binary mixture of two materials for which the usual linear isotropic Hooke's law is valid.

Section I introduces the model and explains how the problem reduces to the approximation of the average of a fluctuating operator. In Sec. II, we explain the method through the simple case of pure compressive materials. In Sec. III, we study the general case and we derive two self-consistent equations for the effective elastic coefficients of the mixture. In Sec. IV, we discuss the results, perform comparison with Hashin-Shtrikman bounds and Bruggeman's effective-medium approximation and study analytically the percolation problem.

I. MODEL AND BASIC EQUATIONS

We study a d-dimensional binary mixture of two microscopically isotropic materials. The elastic coefficients are distributed according to

$$P(\mu,K)=p\,\delta(\mu-\mu_1)\,\delta(K-K_1)+(1-p)\,\delta(\mu-\mu_2)\,\delta(K-K_2), \quad (1)$$

where μ (the shear modulus) and K (the bulk modulus) are defined as usual by Hooke's law:

$$\sigma_{ij}=2\mu\left(u_{ij}-\frac{1}{d}u_{ll}\delta_{ij}\right)+Ku_{ll}\delta_{ij}; \quad (2)$$

where σ_{ij} is the stress tensor and u_{ij} the strain tensor.[9] We use Einstein's convention for repeated indices. Moreover we define

$$\nu=2\mu+\lambda \quad \text{and} \quad \lambda=K-\frac{2}{d}u. \quad (3)$$

The effective coefficients, which will be superscripted by *, describe the response of the whole material to the stress, that is[2]

$$\langle\sigma_{ij}\rangle=2\mu^*\left\langle u_{ij}-\frac{1}{d}u_{ll}\delta_{ij}\right\rangle+K^*\langle u_{ll}\delta_{ij}\rangle, \quad (4)$$

where the brackets $\langle\ \rangle$ denote the average over the fluctuations of the elastic coefficients. The elastic equilibrium equation

$$\partial_i\sigma_{ij}=0$$

can be rewritten as

$$(L_0u)_j+\partial_i[\delta\mu(\partial_iu_j+\partial_ju_i)]+\partial_j(\delta\lambda\,\partial_iu_i)=0, \quad (5)$$

where

$$(L_0u)_j=\mu_0(\partial_i^2u_j+\partial_j\partial_iu_i)+\lambda_0\partial_j\partial_iu_i$$

with $\delta\mu(r)=\mu(r)-\mu_0, \delta\lambda(r)=\lambda(r)-\lambda_0$. The quantities μ_0 and λ_0 are arbitrary (strictly positive) parameters. The Fourier-transformed Green function of the differential operator L_0 is

$$G_{ij}=-\frac{\delta_{ij}}{k^2\mu_0}+\frac{\mu_0+\lambda_0}{\mu_0(2\mu_0+\lambda_0)}\frac{k_ik_j}{k^4}. \tag{6}$$

We denote by the same letter an operator in real or Fourier representation and this should be clear from the context. Equation (4) can be rewritten as the following integral equation:

$$\partial_j u_i=\partial_j u_i^0-[\partial_\rho\partial_j G_{ik}*(2\delta\mu u_{\rho k})+\partial_\rho\partial_j G_{ik}*(\delta\lambda u_{ll}\delta_{\rho k})], \tag{7}$$

where * is the convolution operator and where u^0 is the solution with no disorder: it is fixed by the boundary conditions and so it is not fluctuating. We symmetrize in (i,j) in order to obtain u_{ij} and also in (ρ,k). We can then rewrite u as a function of u^0

$$u_{ij}(r)=\int dr' M_{ijkl}(r,r')u_{kl}^0(r'), \tag{8}$$

$$M_{ijkl}^{-1}(r,r')=1-2G_{ijkl}(r-r')\delta\mu(r') - G_{ij\rho\rho}(r-r')\delta\lambda(r')\delta_{kl}, \tag{9}$$

$$G_{ijkl}=-\frac{1}{4\mu_0}\left(\delta_{jk}\frac{k_ik_l}{k^2}+\delta_{il}\frac{k_jk_k}{k^2}+\delta_{ik}\frac{k_jk_l}{k^2}+\delta_{jl}\frac{k_ik_k}{k^2}\right) + \frac{\mu_0+\lambda_0}{\mu_0(2\mu_0+\lambda_0)\ \frac{k_ik_jk_kk_l}{k^4}}. \tag{10}$$

The effective medium is homogeneous with coefficients m^* and λ^*. Its equilibrium equation

$$\partial_i\langle\sigma_{ij}\rangle=0$$

can be rewritten as (8) with (4) and the substitution

$$\delta\mu(r')\rightarrow\delta\mu^* \quad\text{and}\quad \delta\lambda(r')\rightarrow\delta\lambda^*.$$

We obtain $\langle u\rangle=M^*u^0$ with

$$M_{ijkl}^{*-1}=\delta_{ik}\delta_{jl}+\frac{\delta\mu^*}{2\mu_0}\left(\delta_{jk}\frac{k_ik_l}{k^2}+\delta_{il}\frac{k_jk_k}{k^2}+\delta_{ik}\frac{k_jk_l}{k^2} + \delta_{jl}\frac{k_ik_k}{k^2}\right)-2\delta\mu^*\frac{\mu_0+\lambda_0}{\mu_0(2\mu_0+\lambda_0)}\frac{k_ik_jk_kk_l}{k^4} + \frac{\delta\lambda^*}{2\mu_0+\lambda_0}\delta_{kl}\frac{k_ik_j}{k^2}. \tag{11}$$

Averaging (8) we find $\langle u\rangle=\langle M\rangle u^0$ since u^0 does not fluctuate. Since it is arbitrary, we immediately find

$$\langle M\rangle=M^*. \tag{12}$$

We thus have shown that the problem reduces to the evaluation of the average of the random operator M_{ijkl} knowing its inverse from Eq. (9).

II. CALCULATION IN A LIMITING CASE: PURE COMPRESSION

In order to introduce the method of calculation of the effective coefficients, we study the case of pure compression. This case is, of course, trivial but it illustrates in a pedagogical way the method and it shows its limitations. Moreover, this case is equivalent to the one-dimensional random resistor network (where the exact result is known) and was studied by use of the replica method in Ref. 5. In the pure compression case, the coefficients μ are set equal to zero. So we have

$$\sigma_{ij}=Ku_{ll}\delta_{ij}.$$

The material does not support shear any more so we must take

$$u_{ij}=u\,\delta_{ij}$$

a pure compressive strain with u a function of r, the radial coordinate (only in this section). The equilibrium equation then reads

$$\sigma=dKu \quad\text{and}\quad \partial_r\sigma=0. \tag{13}$$

We are thus left with the electrical problem studied in Ref. 5, with an effective dimension $d=1$. Here, as in the general case, K is a binary random variable. The equilibrium equation is

$$\partial_r[K(r)u(r)]=0. \tag{14}$$

Writing $K(r)=K_0+\delta K(r)$, we have

$$u(r)=u^0-\frac{\delta K(r)}{K_0}u(r) \tag{15}$$

with u^0 a constant. The solution of the equation can be rewritten under the form

$$u(r)=\int dr' M(r,r')u^0 \tag{16}$$

with

$$M^{-1}(r,r')=\delta(r-r')\left(1-\frac{\delta K(r)}{K_0}\right). \tag{17}$$

We know from the previous section that we have to evaluate $\langle M\rangle$. Let us note here that

$$M^*(r,r')=\left(1-\frac{\delta K^*}{K_0}\right)^{-1}\delta(r-r'). \tag{18}$$

We write M as a function of M^{-1} with the help of a Gaussian integral formula with Grassman variables[10]

$$M(r,r')=\frac{\int \mathcal{D}\bar{\xi}\mathcal{D}\xi\,\bar{\xi}(r')\xi(r)e^{\int dr dr'\bar{\xi}(r)(M^{-1})(r,r')\xi(r')}}{\int \mathcal{D}\bar{\xi}\mathcal{D}\xi\,e^{\int dr dr'\bar{\xi}(r)(M^{-1})(r,r')\xi(r')}}, \tag{19}$$

where ξ and $\bar{\xi}$ are Grassmann fields satisfying

$$\{\xi(r),\xi(r')\}=\{\xi(r),\bar{\xi}(r')\}=\{\bar{\xi}(r),\bar{\xi}(r')\}=0.$$

We use here Grassman's integration (Ref. 10) because the operator is not symmetric and we can not use the usual Gaussian one. In order to average (19), we use the replica trick (see for example the book[11]): the denominator is written

$(\int\cdots)^{n-1}$ and we will perform the limit $n\to 0$ at the end. Of course, we suppose that although this calculation is only valid for integer values of n, we can take the limit $n\to 0$. It is an usual assumption in the replica approach and we will not try to prove the limit exits. We introduce the vector

$$\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)$$

and similarly the vector $\overline{\boldsymbol{\xi}}$. Equation (19) can be written as

$$M(r,r')=\int \mathscr{D}\bar{\xi}\mathscr{D}\xi\frac{\overline{\boldsymbol{\xi}}(r')\cdot\boldsymbol{\xi}(r)}{n}\exp\left(\int dr dr' \sum_{a=1}^{n}\bar{\xi}^a(r)\,\delta(r-r')[1-\delta K(r')/K_0]\xi^a(r')\right). \tag{20}$$

In order to average (20) we denote that if $g(r,r')$ is a given function and $f(r')$ a fluctuating function of r' distributed according to a binary law without spatial correlations (we note f_1 and f_2 the two values of f), we have

$$\begin{aligned}\langle e^{\int\int g(r,r')f(r')dr\,dr'}\rangle &= e^{\int\int g(r,r')f_1 dr\,dr'}\prod_{r'} p(1+\eta e^{(1/\Lambda)\int g(r,r')(f_2-f_1)dr})\\ &= e^{\int\int g(r,r')f_1 dr\,dr'}\exp\left(\Lambda\int dr_0\ln(1+\eta\, e^{(1/\Lambda)\int g(r,r_0)(f_2-f_1)dr})+\Omega\,\ln p\right)\end{aligned} \tag{21}$$

with Λ a short distance cutoff (which has the dimension of the inverse of a volume) which appears in the transformation of the integral in a Riemann sum. The quantity Ω is the volume of the sample and $\eta=(1-p)/p$, where p is the probability of f_1.

Using this formula with $f(r')=-\delta K(r')/K_0$ and $g(r,r')=\Sigma_{a=1}^{n}\bar{\xi}^a(r)\,\delta(r-r')\xi^a(r')$, we obtain from (20)

$$\langle M(r,r')\rangle=\int \mathscr{D}\bar{\xi}\mathscr{D}\xi\frac{\overline{\boldsymbol{\xi}}(r')\cdot\boldsymbol{\xi}(r)}{n}e^{\mathscr{H}_e(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})}, \tag{22}$$

where the effective Hamiltonian is

$$\mathscr{H}_e(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})=\int dr dr'\,\bar{\xi}^a(r)\,\delta(r-r')(1-\delta K_1/K_0)\xi^a(r')+\Lambda\int dr_0\ln\left[1+\eta\exp\left(-\frac{1}{\Lambda}\int dr\sum_a \bar{\xi}^a(r)\,\delta(r-r_0)\Delta_K/K_0\xi^a(r_0)\right)\right] \tag{23}$$

with

$$\delta_K=K_2-K_1,$$

$$\delta K_1=K_1-K_0,$$

$$\eta=\frac{1-p}{p}.$$

Let us recall that we want to compute the propagator of this effective Hamiltonian. Moreover we want a nonperturbative result and one way to do this is to use a variational method. This consists in approximating $\mathscr{H}_e$ by a Gaussian Hamiltonian $\mathscr{H}_0$ (diagonal in the replica space). We thus introduce

$$Z_0=\int \mathscr{D}\bar{\xi}\mathscr{D}\xi e^{\mathscr{H}_0}=\int \mathscr{D}\bar{\xi}\mathscr{D}\xi e^{\int dr\,dr'\Sigma_a\bar{\xi}^a(r)\mathscr{A}^{-1}(r-r')\xi^a(r')}. \tag{24}$$

The average using Z_0 will be denoted by $\langle\ \rangle_{\mathscr{H}_0}$. Once $\mathscr{H}_0$ is determined, the approximation reads

$$\mathscr{A}\approx\langle M\rangle, \tag{25}$$

so $\mathscr{A}=M^*$ [Eq. (12)] from which we can deduce K^*. We now have to find the best Gaussian Hamiltonian $\mathscr{H}_0$. In order to do this, one can easily show that for all $\mathscr{H}_0$ and all operator $\mathscr{O}$:

$$\langle\mathscr{O}_{\mathscr{H}_e}\rangle=\langle\mathscr{O}\rangle_{\mathscr{H}_0}+(\langle\mathscr{O}\mathscr{V}\rangle_{\mathscr{H}_0}-\langle\mathscr{O}\rangle_{\mathscr{H}_0}\langle\mathscr{V}\rangle_{\mathscr{H}_0})+o(\mathscr{V}) \tag{26}$$

with $\mathscr{V}=\mathscr{H}_e-\mathscr{H}_0$ small. Therefore the best $\mathscr{H}_0$ to second order in $\mathscr{V}$ must satisfy

$$\langle\mathscr{O}\mathscr{V}\rangle_{\mathscr{H}_0}-\langle\mathscr{O}_{\mathscr{H}_0}\rangle\langle\mathscr{V}\rangle_{\mathscr{H}_0}=0. \tag{27}$$

Note that the best $\mathscr{H}_0$ depends on $\mathscr{O}$ and that in our case, $\mathscr{O}=\bar{\xi}\xi$ is the propagator. One can express the result (27) in a simple form reminiscent of the usual Gibbs-Bogoliubov variational principle.[12] To this end, we introduce the variational free energy

$$\mathscr{F}[\mathscr{A}]=\langle\mathscr{H}_e-\mathscr{H}_0\rangle_{\mathscr{H}_0}+\mathscr{F}_0 \tag{28}$$

with $\mathscr{F}_0=\ln Z_0$. Using

$$\begin{aligned}\frac{\delta\mathscr{F}_0}{\delta\mathscr{A}^{-1}}&=\langle\bar{\xi}\xi\rangle_{\mathscr{H}_0},\\ \frac{\delta\langle\mathscr{V}\rangle_{\mathscr{H}_0}}{\delta\mathscr{A}^{-1}}&=\langle\bar{\xi}\xi\mathscr{V}\rangle_{\mathscr{H}_0}-\langle\bar{\xi}\xi\rangle_{\mathscr{H}_0}\langle\mathscr{V}\rangle_{\mathscr{H}_0}-\langle\bar{\xi}\xi\rangle_{\mathscr{H}_0},\end{aligned} \tag{29}$$

we transform (27) into

$$\frac{\delta\mathscr{F}}{\delta\mathscr{A}(k)}=0. \tag{30}$$

(Let us note that the usual convexity inequality of Gibbs-Bogoliubov does not hold here, since we are dealing with Grassman variables.)

We get for the variational free energy from (23) and (28)

$$\mathcal{F}=-n\Omega\int \frac{d^dk}{(2\pi)^d}\mathrm{tr}\ \ln\mathcal{A}(k)+n\Omega\left(1-\frac{\delta K_1}{K_0}\right)\int \frac{d^dk}{(2\pi)^d}\mathcal{A}(k)$$
$$+\Lambda\int dr_0\left\langle \ln\left[1+\eta\ \exp\left(-\frac{1}{\Lambda K_0}\int dr\sum_a \bar{\xi}^a(r)\,\delta(r-r_0)\Delta_K\xi^a(r_0)\right)\right]\right\rangle_{\mathcal{H}_0}. \tag{31}$$

The calculation of the last term is the main difficulty here and can be found in Ref. 5. In the general case (next section) we will give a more detailed calculation. After all calculations, the variational free energy per replica and unit volume is (since one expect $\mathcal{F}\sim n\Omega$ in the limits $n\to 0$ and $\Omega\to\infty$ we expand the free energy to the first order in n):

$$\frac{\mathcal{F}}{n\Omega}=-\int \frac{d^dk}{(2\pi)^d}\mathrm{tr}\ \ln\mathcal{A}(k)+\left(1-\frac{\delta K_1}{K_0}\right)\int \frac{d^dk}{(2\pi)^d}\mathcal{A}(k)$$
$$-\Lambda\sum_{h>0}(-1)^h\frac{\eta^h}{h}\ln\left[1-\frac{h\Delta_K}{\Lambda K_0}\int \frac{d^dk}{(2\pi)^d}\mathcal{A}(k)\right]. \tag{32}$$

The variational equation (30) leads to

$$\frac{1}{\mathcal{A}(k)}=1-\frac{\delta K_1}{K_0}+\frac{\Delta_K}{K_0}\sum_{h>0}(-1)^h\eta^h$$
$$\times\left[\frac{1}{1-(h\Delta_K/\Lambda K_0)\int[d^dk/(2\pi)^d]\mathcal{A}}\right]. \tag{33}$$

Using (18), and

$$\frac{1}{x}=\int_0^\infty du e^{-ux},$$

we find

$$K^*=\int_0^1 dz\frac{\langle Kz^{K/K^*}\rangle}{\langle z^{K/K^*}\rangle}. \tag{34}$$

This equation determines K^* in a self-consistent way. It is the result of Ref. 5 for $d=1$. It does not coincide with the exact result ($K^*=1/\langle 1/K\rangle$). We will discuss this interesting limiting case later.

III. THE GENERAL CASE

After this pedagogical introduction, we will now study the general case defined in Sec. I. The key ideas of the method have already been explained, and we will focus on technical difficulties.

In order to compute $\langle M_{ijkl}\rangle$, we write M as a function of M^{-1}:

$$M_{ijkl}(r,r')=\frac{\int \mathcal{D}\bar{\xi}\mathcal{D}\xi\bar{\xi}_{kl}(r')\xi_{ij}(r)\exp\left(\int drdr'\sum_{ijkl}\bar{\xi}_{ij}(r)(M^{-1})_{ijkl}(r,r')\xi_{kl}(r')\right)}{\int \mathcal{D}\bar{\xi}\mathcal{D}\xi\exp\left(\int drdr'\sum_{ijkl}\bar{\xi}_{ij}(r)(M^{-1})_{ijkl}(r,r')\xi_{kl}(r')\right)}, \tag{35}$$

where ξ and $\bar{\xi}$ are Grassmann fields.

Equation (35) can be rewritten as

$$M_{ijkl}(r,r')=\int \mathcal{D}\bar{\xi}\mathcal{D}\xi\frac{\overline{\boldsymbol{\xi}}_{kl}(r')\cdot\boldsymbol{\xi}_{ij}(r)}{n}\exp\left(\int drdr'\sum_{i,j,k,l=1}^{d}\sum_{a=1}^{n}\bar{\xi}^a_{ij}(r)[\delta_{ik}\delta_{jl}\delta(r-r')-2\delta\mu(r')G_{ijkl}(r-r')\right.$$
$$\left.-\delta\lambda(r')G_{ij\rho\rho}(r-r')\delta_{kl}]\xi^a_{kl}(r')\right). \tag{36}$$

Using (21), we obtain from averaging (36)

$$\langle M_{ijkl}(r,r')\rangle=\int \mathscr{D}\bar{\xi}\mathscr{D}\xi\frac{\overline{\boldsymbol{\xi}_{kl}}(r')\cdot\boldsymbol{\xi}_{ij}(r)}{n}e^{\mathscr{H}_e(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})},$$

where the effective Hamiltonian is

$$\mathscr{H}_e(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})=\int dr\sum_{i,j,a}\bar{\xi}^a_{ij}(r)\xi^a_{ij}(r)-\int dr dr'\sum_{i,j,k,l,a}\bar{\xi}^a_{ij}(r)[2\,\delta\mu_1 G_{ijkl}(r-r')+\delta\lambda_1 G_{ij\rho\rho}(r-r')\,\delta_{kl}]\xi^a_{kl}(r')$$
$$+\Lambda\int dr_0\ln\left[1+\eta\exp\left(-\frac{1}{\Lambda}\int dr\sum_{i,j,k,l,a}\bar{\xi}^a_{ij}(r)[2\Delta_\mu G_{ijkl}(r-r_0)+\Delta_\lambda G_{ij\rho\rho}(r-r_0)\,\delta_{kl}]\xi^a_{kl}(r_0)\right)\right] \tag{37}$$

with

$$\Delta_\mu=\mu_2-\mu_1,$$

$$\Delta_\lambda=\lambda_2-\lambda_1,$$

$$\eta=\frac{1-p}{p}.$$

The variational free energy [Eq. (28)] is

$$\mathscr{F}=-n\Omega\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\ \ln\mathscr{A}(k)+n\Omega\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\mathscr{A}(k)-n\Omega\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}(\mathscr{G}^1\mathscr{A})(k)$$
$$+\Lambda\int dr_0\left\langle\ln\left[1+\eta\ \exp\left(-\frac{1}{\Lambda}\int dr\sum_{i,j,k,l,a}\bar{\xi}^a_{ij}(r)\mathscr{G}(r-r_0)\xi^a_{kl}(r_0)\right)\right]\right\rangle_{\mathscr{H}_0}, \tag{38}$$

where

$$\mathscr{G}=2\Delta_\mu G_{ijkl}+\Delta_\lambda G_{ij\rho\rho}\delta_{kl},$$
$$\mathscr{G}^1=2\,\delta\mu_1 G_{ijkl}+\delta\lambda_1 G_{ij\rho\rho}\delta_{kl}. \tag{39}$$

In order to compute the last term of (38), we expand the logarithm. This leads us to compute terms of the form

$$\left\langle\lambda\int dr_0\exp\left(-\frac{h}{\Lambda}\int dr\sum_{i,j,k,l,a}\bar{\xi}^a_{ij}(r)\mathscr{G}(r-r_0)\xi^a_{kl}(r_0)\right)\right\rangle_{\mathscr{H}_0}, \tag{40}$$

(with h an integer) which is equal to

$$\Lambda\int dr_0\exp\left(n\mathrm{Tr}\ \ln\left[\delta_{ik}\delta_{jl}\delta(r-r')-\frac{h}{\Lambda}\sum_{mn}\mathscr{G}_{ijmn}(r-r_0)\mathscr{A}_{mnkl}(r_0-r')\right]\right). \tag{41}$$

Expanding this last expression to first order in n, we obtain

$$\Lambda\Omega+n\Lambda\int dr_0\mathrm{Tr}\ \ln\left[\delta_{ik}\delta_{jl}\delta(r-r')-\frac{h}{\Lambda}\sum_{mn}\mathscr{G}_{ijmn}(r-r_0)\mathscr{A}_{mnkl}(r_0-r')\right]+O(n^2)$$
$$=\Lambda\Omega+n\Omega\Lambda\ \mathrm{tr}\ \ln\left(\delta_{ik}\delta_{jl}-\frac{h}{\Lambda}\int\frac{d^dk}{(2\pi)^d}\sum_{mn}\mathscr{A}_{ijmn}(k)\mathscr{G}_{mnkl}(k)\right)+O(n^2). \tag{42}$$

Here, tr denotes the usual matrical trace and Tr the operator trace (including spatial indices). We finally find (up to a constant)

$$\frac{\mathcal{F}}{n\Omega}=-\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\,\ln\mathcal{A}(k)+\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\mathcal{A}(k)-\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}(\mathcal{G}^1\mathcal{A})(k)$$
$$-\Lambda\sum_{h>0}(-1)^h\frac{\eta^h}{h}\mathrm{tr}\,\ln\left(\delta_{ik}\delta_{jl}-\frac{h}{\Lambda}\int\frac{d^dk}{(2\pi)^d}(\mathcal{A}\mathcal{G})_{ijkl}(k)\right).$$

The variational equation

$$\frac{\delta\mathcal{F}}{\delta\mathcal{A}_{klij}(k)}=0 \tag{43}$$

reads

$$(\mathcal{A}^{-1})_{ijkl}=\delta_{ik}\delta_{jl}-\mathcal{G}^1_{ijkl}(k)+\sum_{h>0}(-1)^h\eta^h\mathcal{G}_{ijmn}\times\left[\frac{1}{1-(h/\Lambda)\int[d^dk/(2\pi)^d](\mathcal{A}\mathcal{G})}\right]_{mnkl}. \tag{44}$$

Using (25) and (11) in (44), we get after a straightforward but tedious calculation the following self-consistent equations for the coefficients of isotropic elasticity (see the Appendix for some details),

$$\mu^*=\int_0^1 dz\frac{\langle\mu z^{[2\mu/d(d+2)](d/\mu^*+2/\nu^*)}\rangle}{\langle z^{[2\mu/d(d+2)](d/\mu^*+2/\nu^*)}\rangle},$$
$$K^*=\int_0^1 dz\frac{\langle Kz^{K/\nu^*}\rangle}{\langle z^{K/\nu^*}\rangle}, \tag{45}$$

where

$$\nu^*=2\mu^*+\lambda^* \quad\text{and}\quad \lambda^*=K^*-\frac{2}{d}\mu^*.$$

Let us note that in the last term of (44), the integral has to be regularized by the cutoff Λ because of short distance divergence, since we have to integrate a homogeneous function of degree zero. One can show that the integral is proportional to Λ (see the Appendix for a complete expression of the useful integrals). Note that the cutoff and the arbitrary parameters μ_0 and λ_0 disappear at the end of the calculation as expected. Equations (45) are the main result of our paper and we now discuss them. Let us note that when $\mu_i=0$, we retrieve the result of Sec. II.

IV. DISCUSSION

A. The low disorder expansion

First, we extract the low disorder expansion (that is when the variance of μ and K are small) from our formulas and compare it with the straightforward low disorder expansion of (8).

First, if we iterate twice the exact integral equation (8) and average, we find

$$\langle u_{ij}\rangle=u^0_{ij}+\Bigg\{-4\langle\delta\mu^2\rangle\frac{1}{d(d+2)}\left(\frac{d}{\mu_0}+\frac{2}{\nu_0}\right)\Bigg[-\frac{1}{4\mu_0}\left(\delta_{jk}\frac{k_ik_l}{k^2}+\delta_{il}\frac{k_jk_k}{k^2}+\delta_{il}\frac{k_jk_k}{k^2}+\delta_{il}\frac{k_jk_l}{k^2}+\delta_{jk}\frac{k_ik_k}{k^2}\right)$$
$$+\left(\frac{1}{\mu_0}-\frac{1}{\nu_0}\right)\frac{k_ik_jk_kk_l}{k^4}-\frac{2}{d(d+2)\nu_0}\left(\frac{1}{\mu_0}-\frac{1}{\nu_0}\right)\delta_{kl}\frac{k_ik_j}{k^2}\Bigg]+4\langle\delta\mu\,\delta\lambda\rangle\frac{1}{d\nu_0^2}\delta_{kl}\frac{k_ik_j}{k^2}+\langle\delta\lambda^2\rangle\frac{1}{\nu_0^2}\delta_{kl}\frac{k_ik_j}{k^2}\Bigg\}_{ijkl}u^0_{kl}(k), \tag{46}$$

and we can extract the perturbative expansion of μ^* and K^*

$$\mu^*\approx\mu_0-\frac{2\langle\delta\mu^2\rangle}{d(d+2)}\left(\frac{d}{\mu_0}+\frac{2}{\nu_0}\right),$$
$$K^*\approx K_0-\frac{\langle\delta K^2\rangle}{\nu_0}. \tag{47}$$

Let us then note that for a general formula of the form

$$C^*=\int_0^1 dz\frac{\langle Cz^{B(C^*)C}\rangle}{\langle z^{B(C^*)C}\rangle}, \tag{48}$$

where C is some coefficient with $\delta C=C-C_0\ll C_0$ and $B(C^*)$ a given function of C^*, we find the perturbative expansion

$$C^*=C_0-B(C_0)\langle\delta C^2\rangle. \tag{49}$$

Applying this result to (45) leads to the perturbative expansion (47) for μ^* and K^*. The formula (45) has thus the correct low disorder expansion, and it is the only one with the form of (48).

B. Comparison with the Hashin-Shtrikman bounds

A more difficult test for (45) are Hashin-Shtrikman bounds, because we could not prove *a priori* that they are satisfied. These bounds are given by[4]

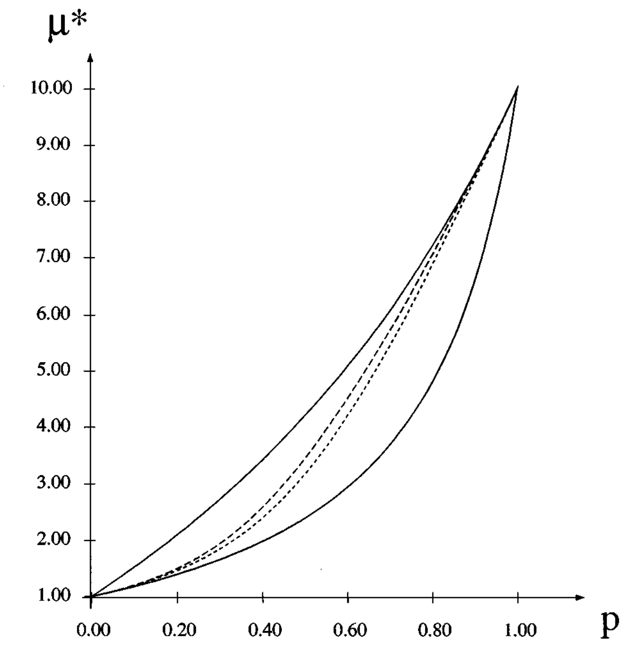

FIG. 1. Effective shear modulus μ^* as a function of concentration p for $\mu_2=10$, $\mu_1=1$, $K_2=15$, $K_1=1.5$, and $d=3$, given by our approximation (45) (long dash), effective-medium theory (short dash). The Hashin-Shtrikman bounds are also given (continuous curves). The respective positions of the curves for K^* are the same in this case.

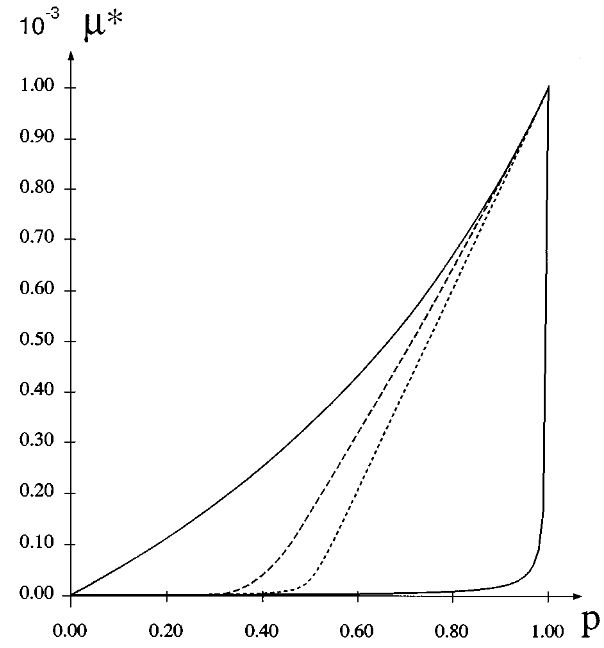

FIG. 2. Effective shear modulus μ^* as a function of concentration p for $\mu_2=1000$, $\mu_1=1$, $K_2=1500$, and $K_1=1.5$. The contrast between the two materials is high and the percolation thresholds are clearly displayed.

$$K_1^* \leqslant K^* \leqslant K_2^*,$$

$$\mu_1^* \leqslant \mu^* \leqslant \mu_2^*,$$

with

$$K_1^* = K_1 + \frac{1-p}{1/(K_2-K_1)+3p/(3K_1+4\mu_1)},$$

$$K_2^* = K_2 + \frac{p}{1/(K_1-K_2)+3(1-p)/(3K_2+4\mu_2)},$$

$$\mu_1^* = \mu_1 + \frac{1-p}{1/(\mu_2-\mu_1)+6(K_1+2\mu_1)p/5\mu_1(3K_1+4\mu_1)},$$

$$\mu_2^* = \mu_2 + \frac{p}{1/(\mu_1-\mu_2)+6(K_2+2\mu_2)(1-p)/5\mu_2(3K_2+4\mu_2)}. \tag{50}$$

Another interesting expansion is then the low dilution limit, when $p\sim 0$ (the case $p\sim 1$ is similar). Expanding (45), the Hashin-Shtrikman bounds, and the standard effective-medium approximation (see next section) in power of p, we find they are equal at first order in p: (45) is correct for low dilution.

Numerically, we observe that μ^* always satisfies Hashin-Shtrikman bounds. For K^*, the deviation from Hashin-Shtrikman bounds is negligible if the coefficients K_i ($i=1,2$) are of the same magnitude as the μ_j, or lower, even if the contrast between the two materials is high. But, when the ratios K_i/μ_j are high (more precisely >2 that we will denote by $K \gtrsim 2\mu$), K^* is out of Hashin-Shtrikman bounds (see Figs. 1–4). This can be easily understood. As it was shown in Sec. II, the self-consistent equation for K^* is (for $\mu_i=0$)

$$K^* = \int_0^1 dz \frac{\langle K z^{K/K^*}\rangle}{\langle z^{K/K^*}\rangle}. \tag{51}$$

In this case, the Hashin-Shtrikman bounds simply reduce to the exact result

$$\frac{1}{K^*} = \left\langle \frac{1}{K} \right\rangle. \tag{52}$$

The result of Ref. 5 is false for $d=1$ [Eq. (51)]: it verifies Hashin-Shtrikman bounds only for $d>d_c$ with $d_c\approx 1.5$ numerically. This is not important for the physics in the electric case for $d=3$, but here it is, because the wide range of possible elastic coefficients allows to be very close to this limiting situation.

Let us note that we face the same problem with the limit $d\rightarrow +\infty$. We have $\mu^* = \langle \mu \rangle$ but the formula for K^* with $\mu\approx 0$ is wrong. Even if d is high, the computation of K^* (for $\mu^*\approx 0$) is still a one-dimensional problem.

We tried to solve the difficulty in the following way. First, in the electric case, if we calculate the effective resistivity ρ^* by the same method as for the effective conductivity g^*,[5] we find

$$\rho^* = \int_0^1 dz \frac{\langle \rho z^{(\rho/\rho^*)(1-1/d)} \rangle}{\langle z^{(\rho/\rho^*)(1-1/d)} \rangle}. \tag{53}$$

This formula is exact for $d=1$, but false for $d=+\infty$. One inconsistency of the replica method is that $\rho^* g^* > 1$.[13,14] If we compare the two results with the electrical Hashin-Shtrikman bounds we conclude that ρ^* must be rejected for $d > d_c$, and g^* for $d < d_c$. One can think that in the elasticity problem, the formula involving the inverse coefficient can be correct when $K \gtrsim 2\mu$. If we define the inverse coefficients by

$$\alpha = \frac{1}{2\mu} \text{ and } \beta = \frac{1}{d^2 K}, \tag{54}$$

$$u_{ij} = \alpha \left(\sigma_{ij} - \frac{1}{d} \sigma_{ll} \delta_{ij} \right) + \beta \sigma_{ll} \delta_{ij}. \tag{55}$$

We can rewrite the integral equation (8) as

$$\sigma_{ij} = \sigma_{ij}^0 + \int dr' \{ [\mathscr{G}_\alpha(r-r') \delta\alpha(r') + \mathscr{G}_\beta(r-r') \delta\beta(r')] \sigma(r') \}_{ij} \tag{56}$$

with

$$\mathscr{G}_\alpha(k) = -\frac{1}{\alpha_0} \delta_{ij}\delta_{jl} + \frac{1}{2\alpha_0} \left(\delta_{jk} \frac{k_i k_l}{k^2} + \delta_{il} \frac{k_j k_k}{k^2} \delta_{ik} \frac{k_j k_l}{k^2} + \delta_{jl} \frac{k_i k_k}{k^2} \right) + \frac{[-2\alpha_0 + \beta_0(2d-d^2)]}{\alpha_0[\alpha_0 + \beta_0 d(d-1)]} \frac{k_i k_j k_k k_l}{k^4}$$
$$- \frac{d\beta_0}{\alpha_0[\alpha_0 + \beta_0 d(d-1)]} \left(\delta_{kl} \frac{k_i k_j}{k^2} - \delta_{ij}\delta_{kl} \right) + \frac{(\alpha_0 - d\beta_0)}{\alpha_0[\alpha_0 + \beta_0 d(d-1)]} \delta_{ij} \frac{k_k k_l}{k^2},$$
$$\mathscr{G}_\beta(k) = \frac{d\alpha_0}{\alpha_0[\alpha_0 + \beta_0 d(d-1)]} \left(\delta_{kl} \frac{k_i k_j}{k^2} - \delta_{ij}\delta_{kl} \right). \tag{57}$$

By a computation analogous to the previous one, we find

$$\alpha^* = \int_0^1 dz \frac{\langle \alpha z^{(\alpha/\alpha^*)[d/(d+2)][\alpha^* + \beta^*(d^2-d-2)]/[\alpha^* + \beta^* d(d-1)]} \rangle}{\langle z^{(\alpha/\alpha^*)[d/(d+2)][\alpha^* + \beta^*(d^2-d-2)]/[\alpha^{*+} \beta^* d(d-1)]}},$$
$$\beta^* = \int_0^1 dz \frac{\langle \beta z^{(\alpha/\alpha^*)[-2\alpha^* + \beta^* d(2-d)]/(d+2)[\alpha^* + \beta^* d(d-1)] + \beta d(d-1)/[\alpha^* + \beta^* d(d-1)]} \rangle}{\langle z^{(\alpha/\alpha^*)[-2\alpha^* + \beta^* d(2-d)]/(d+2)[\alpha^* + \beta^* d(d-1)] + \beta d(d-1)/[\alpha^* + \beta^* d(d-1)]} \rangle}. \tag{58}$$

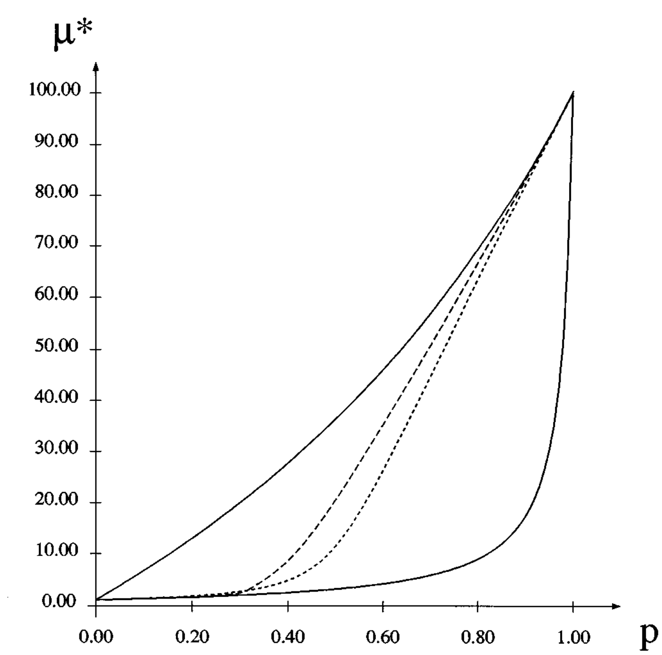

FIG. 3. Effective shear modulus μ^* as a function of concentration p for $\mu_2=100$, μ_1-1, $K_2=400$, $K_1=4$, and $d=3$ with the same legend as for Fig. 1. Hashin-Shtrikman bounds are satisfied.

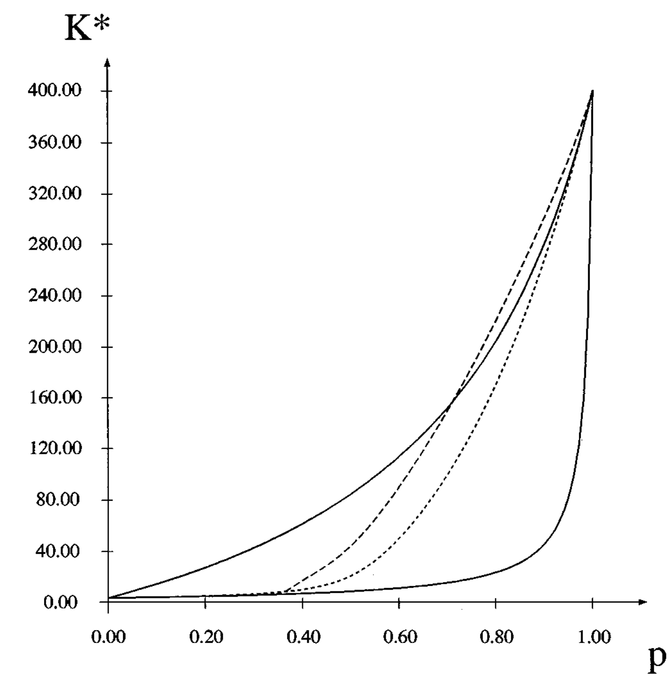

FIG. 4. Effective bulk modulus K^* as a function of concentration p for $\mu_2=100$, $\mu_1=1$, $K_2=400$, $K_1=4$, and $d=3$ with the same legend as for Fig. 1. We note a deviation from Hashin-Shtrikman bounds.

β behaves as expected in the limit $\alpha\to+\infty, d\to+\infty$, that is $\beta^*=\langle\beta\rangle$. But, due to the first term in the exponent of z in the formula for β^*, when $\alpha\to+\infty$ (with d fixed), we have not $\beta^*=\langle\beta\rangle$ except if $\alpha_1=\alpha_2$. As a result, it seems to be impossible to find in this way a formula analogous to (45), which would verify Hashin-Shtrikman bounds for $K\gtrsim 2\mu$ with a finite contrast for μ.

As a conclusion, our approximation (45) can be applied to materials whose Poisson coefficient is less than 0.3. This restriction has physical sense since there exist materials whose Poisson coefficient is greater or lower than this value. In the following we suppose that $K\lesssim 2\mu$.

C. The percolation threshold

When one of the two materials (say 2) is quasivacuum ($\mu_2\approx 0$ and $K_2\approx 0$), there is a percolation effect: the elastic coefficients are nonzero only above a strictly positive value of the concentration of matter p. We will call this value the *rigidity threshold* and denote it by p_r. We obtain the value of p_r in the following way. First, we have a general formula: if C is a quantity which is determined by

$$C^*=\int_0^1 dz\,\frac{pCz^{XC}}{(pz^{XC}+1-p)},$$

we find the expansion

$$C^*=\frac{1}{X}\int_0^1 dt\,t^{1/XC}\frac{1}{t+(1-p)/p}\approx-\frac{\ln(1-p)}{X}$$

when $X\to+\infty$. Applying this to (45) next to the percolation threshold with μ and K in place of C and with A given as a function of μ^* and K^* by (45), we find that μ^* and K^* percolate at the same value of the proportion p and that

$$\lim_{p\to p_r+}\frac{\mu^*}{\nu^*}=\frac{d}{2(d+1)},\tag{59}$$

$$p_r=1-e^{-2/(d+1)}.\tag{60}$$

In the electric case, there is a percolation effect too (see Ref. 15). If we denote by p_c the threshold in this case the same method leads to[5]

$$p_c=1-e^{-(1/d)}.\tag{61}$$

We have $p_c\leqslant p_r$: it needs more material to have a rigid percolation than a simple topological percolation like in the electric case. In other words the existence of an infinite cluster is not sufficient to ensure the rigidity of the whole sample (see Ref. 16 and references therein[17,18] for the calculation of p_r in a network model).

D. Comparison with the standard effective-medium approximation

We now compare our approximation with the standard effective-medium approximation which verifies Hashin-Shtrikman bounds. The result can be found for example in Ref. 19, where it is derived from the general Eshelby's calculation for ellipsoidal inhomogeneities. We generalized it in the d-dimensional case in order to compare it with our present result. We used an elementary method: a sphere of one material is regarded as embedded in the effective medium, we calculate the stress associated with a general boundary stress applied far from the sphere, and we deduce self-consistent equations for the effective coefficients:

$$\left\langle\frac{K'-K^*}{2(d-1)\mu^*+dK'}\right\rangle=0,$$

$$\left\langle\frac{\mu'-\mu^*}{3d^2K^*\mu^*-12\mu^{*2}+4d^2\mu^{*2}+6dK^*\mu'+4(2d+3)\mu^*\mu'}\right\rangle=0.\tag{62}$$

In order to compare with our results, we compute p_r in the effective-medium approximation

$$p_r^{\text{em}}=\frac{2d^2-\sqrt{d^4+6d^3+15d^2-18d}}{(d-3)(d+2)}.\tag{63}$$

In the electric case, $p_c^{\text{em}}=1/d$, so $p_c^{\text{em}}\leqslant p_r^{\text{em}}$ and we have the same qualitative conclusion as that in our approximation. The two approximations are the same in the limit $d\to+\infty$. More precisely, we see that in the elastic case (as in the electric one):

$$p_r-p_r^{\text{em}}=O(1/d^2).\tag{64}$$

We studied numerically the two approximations (45) and (62) for various contrasts between the materials and various dimensions (see, e.g., Figs. 1 and 2). We note that the two approximations are very close to each other, in particular when the contrast is small (0.5 for example) or when the dimension is high. Moreover, we clearly see the percolation values p_r, when the contrast is high (10 000 in Fig. 2).

CONCLUSION

We computed the elastic effective coefficients of a composite using the replica trick and together with a variational approximation. The result is in good agreement with Hashin-Shtrikman bounds and with the effective-medium approximation for $K\lesssim 2\mu$ and gives the rigidity threshold p_r in all dimensions easily. We found $p_r=1-\exp(-2/(d+1))$. Therigidity threshold is greater than p_c: the existence of an infinite cluster is not sufficient to ensure the rigidity of the whole sample. We note that the method can be generalized in the case of nonisotropic elasticity (for example when the compliance tensor has only one axis of symmetry): the variational equation is quite general and the derivation of the effective coefficients is then an automatic computation, although it can be tedious. Moreover this method may be useful in studying the propagation of waves in a composite. However, the problem of the failure of the electrical formula in dimension one can be a general limitation of the method. Even for a three-dimensional disorder, the microscopic parameters can be such that our equations degenerate to a one-dimensional problem.

APPENDIX: CALCULATION OF EFFECTIVE COEFFICIENTS

First, let us note that the following calculations are easily done with a diagrammatic representation of the fourth-rank tensor which are a combination of the tensor product of unity and a projector. We calculate $\mathscr{A}$ from (11) and we obtain for $\mathscr{A}\mathscr{G}$

$$(\mathscr{A}\mathscr{G})_{ijkl}=-\frac{\Delta_\mu}{2\mu^*}\left(\delta_{jk}\frac{k_ik_l}{k^2}+\delta_{il}\frac{k_jk_k}{k^2}+\delta_{ik}\frac{k_jk_l}{k^2}+\delta_{jl}\frac{k_ik_k}{k^2}\right)+2\Delta_\mu\left(\frac{1}{\mu^*}-\frac{1}{\nu^*}\right)\frac{k_ik_jk_kk_l}{k^4}-\frac{\Delta_\lambda}{\nu^*}\delta_{kl}\frac{k_ik_j}{k^2}. \quad \text{(A1)}$$

(In fact one can show directly that $\mathscr{A}\mathscr{G}=\mathscr{G}^*$, where $\mathscr{G}^*$ is $\mathscr{G}$ calculated in the effective medium.) To integrate (A1), we use the two formulas:

$$\int\frac{d^dk}{(2\pi)^d}\frac{k_ik_j}{k^2}=\Lambda\frac{\delta_{ij}}{d},$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{k_ik_jk_kk_l}{k^4}=\frac{\Lambda}{d(d+2)}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}+\delta_{ij}\delta_{kl}). \quad \text{(A2)}$$

We find

$$\left(1-h\int\mathscr{A}\mathscr{G}\right)_{ijkl}=\alpha\delta_{ik}\delta_{jl}+\beta\delta_{il}\delta_{jk}+\gamma\delta_{ij}\delta_{kl}$$

with

$$\alpha+\beta=1+\frac{2h\Delta_\mu}{d(d+2)}\left(\frac{d}{\mu^*}+\frac{2}{\nu^*}\right),$$

$$\alpha+\beta+d\gamma=1+h\frac{\Delta_K}{\nu^*}.$$

The linear independence of $\delta_{jk}k_ik_l/k^2+\delta_{il}k_jk_k/k^2+\delta_{ik}k_jk_l/k^2+\delta_{jl}k_ik_k/k^2, k_ik_jk_kk_l/k^4$, and $\delta_{kl}k_ik_j/k^2$ as functions of k allows us to identify term by term in the variational equation. We obtain

$$\mu^*=\mu_1+\sum_{h>0}(-1)^{h+1}\eta^h\Delta_\mu\frac{1}{\alpha+\beta},$$

$$K^*=K_1+\sum_{h>0}(-1)^{h+1}\eta^h\Delta_K\frac{1}{\alpha+\beta+d\gamma}.$$

Using

$$\frac{1}{x}=\int_0^\infty du e^{-ux},$$

we find the formulas (45) of the text.

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