Spin waves in a four-sublattice Heisenberg ferrimagnet or ferromagnet with different exchange constants ($J_{ab}=J_{cd}\neq J_{bc}=J_{da}$)

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The Hamiltonian for a four-sublattice Heisenberg ferrimagnet or ferromagnet with different exchange constants ($J_{ab}=J_{cd}\neq J_{bc}=J_{da}$) was established. An extended Bogoliubov transformation was developed by solving an equation group, consisting of 20 equations and 20 unknowns. The procedure for solving the equation group was carried out by introducing a simple way of reducing the numbers of the equations and the unknowns. The spin-wave spectra in the present system have been determined by performing the standard Holstein-Primakoff transformation and the Bogoliubov one. It has been found that the spin-wave spectra of the present system depend on the exchange constants and that the degeneracy of the spin-wave spectra remains. The results for a special case ($J_{ab}=J_{bc}$), i.e., an antiferromagnet, are discussed briefly. The spin-wave spectra of the four-sublattice Heisenberg antiferromagnet are found to be degenerative also and they are linear in k for small k.

I. INTRODUCTION

Spin waves have been investigated extensively in magnetic systems, since the early work of Bloch[1] and Holstein and Primakoff[2] on ferromagnets, and its extension to antiferromagnets by Anderson[3] and Kubo.[4] At first, spin waves have been considered as elementary excitations from which one can derive the thermodynamic properties of magnetic systems at low temperatures. Second, spin waves can be used to calculate various time-dependent properties of magnetic systems, such as dynamic response functions and correlation functions.[5,6]

Many results of the previous work have been obtained by using the boson formalism of Holstein and Primakoff[2] and expanding the Hamiltonian in powers of the occupation numbers, or by using the Dyson-Maleev formalism.[7,8] The former procedure can lead to incorrect results for small values of the spin S, but the difficulties are avoided if S^{-1} is formally treated as a small parameter and calculations are performed consistently to each order in S^{-1}. For diagonalizing the quadratic part of the Hamiltonian, the Bogoliubov transformation[9] is usually introduced. However, the attention of most work has been focused on the two-sublattice systems.

On the other hand, the rare-earth (R)–transition-metal (T) intermetallics have attracted great interest due to their outstanding permanent magnetic properties,[10–12] which can be usually explained by a two-sublattice model.[13,14] del Moral used the spin-wave theory to analyze the spin reorientations in the rare-earth–transition-metal intermetallics $(R'_xR_{1-x})_2Fe_{14}B$ and $(R'_xR_{1-x})Co_5$.[15] But in some compounds,[10,11,15] the existence of different rare-earth and transition-metal sites results in the necessity of a multisublattice model. These give an impetus to theoretical investigation on the spin waves of the multisublattice Heisenberg antiferromagnets, ferrimagnets and/or ferromagnets. The application of the spin-wave approximation to the multisublattice system is a very formidable problem.[15,16] To our knowledge the attempts to deal with the spin-wave excitations in the systems with multiple structurally ordered magnetic sublattices were done by various authors in the later 1950's and early 1960's.[16–25] Kaplan studied the wave functions and the energy spectrum for the spin-wave problem in a normal spinel.[16] Sáenz[17] and Wallace[18] paid attention to a lattice which has an arbitrary number of magnetic atoms, or spins, in each magnetic unit cell. Sáenz proved that there exists at least one "acoustic" branch among the $\leqslant n$ distinct branches of the spin-wave spectrum when the magnetic anisotropy and magnetic-field contributions vanish.[17] Wallace gave explicitly the transformations which diagonalize the Hamiltonian of the complex lattices in terms of harmonic oscillator formalism.[18] Meyer and Harris[19] as well as Douglass[20] used the approximations of Anderson in calculating the spin-wave spectrum of yttrium-iron-garnet (YIG). A common fact is that with the exception of completely ferromagnetic exchanges when the lattice is complicated, i.e., the cases when number of spins per magnetic unit cell is more than two, the problem cannot be solved explicitly in terms of the elements of the matrices for each **k** and it will generally be necessary to use numerical methods to solve the problem.[18–20] Another fact is that contradictory results were obtained by these authors by proceeding with different methods.[16,21–25]

The aim of the present paper is to study the spin-wave spectra at low temperatures of a four-sublattice Heisenberg ferrimagnet or ferromagnet with different exchange constants ($J_{ab}=J_{cd}\neq J_{bc}=J_{da}$), in terms of creation and annihilation operators. In the present work, to simplify, we shall only deal with the case with the same spin amplitudes for the different sublattices. Since the linear-spin-wave theory[3,4] (the leading term in the $1/S$ expansion) gives fairly good results for the quantum corrections to various physical quantities, we shall treat the spin fluctuations within the linear-spin-wave theory. In Sec. II, the Hamiltonian and formalism will be represented. For diagonalizing the Hamiltonian, the extended Bo-

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goliubov transformation will be developed by establishing an equation group consisting of 20 equations and 20 unknowns. The spin-wave spectra will be obtained in Sec. III by solving the equation group developed and consequently by performing the new transformation. In Sec. IV, the results for a special case ($J_{ab}=J_{bc}$), i.e., an antiferromagnet, will be discussed briefly. Section V is for concluding remarks. The results, obtained in this work, might be suitable for the four-sublattice systems with exchange constants ($J_{ab}\approx J_{cd}\neq J_{bc}\approx J_{da}$).

II. HAMILTONIAN AND FORMALISM

In the present work, we consider the four-sublattice Heisenberg ferrimagnet and/or ferromagnet with different isotropic exchange constants between spins in the different sublattices, modeled by the following Hamiltonian:

$$H=-\sum_{\langle l,i\rangle} J_{l,i;l,i+\delta}S_{l,i}\cdot S_{l,i+\delta}=-\sum_{i,\delta} J_{ab}S_{a,i}\cdot S_{b,i+\delta}-\sum_{j,\delta} J_{bc}S_{b,j}\cdot S_{c,j+\delta}-\sum_{m,\delta} J_{cd}S_{c,m}\cdot S_{d,m+\delta}-\sum_{n,\delta} J_{da}S_{d,n}\cdot S_{a,n+\delta} \quad (l=a,b,c,d), \tag{1}$$

where l denotes four sublattices a, b, c, and d. In the case of ferrimagnet the exchange constants are negative whereas in the case of the ferromagnet they are positive. δ represents that only the exchanges between the nearest neighbors are taken into account. The number of the nearest neighbors is Z. $S_i=\langle S_i^x,S_i^y,S_i^z\rangle$ are operators belonging to the spin-S representation, whose commutation relations are

$$[S_i^{\pm},S_j^z]=\mp\delta_{ij}S_i^{\pm};\quad [S_i^{+},S_j^{-}]=2\delta_{ij}S_i^z \tag{2}$$

with

$$S_i^{\pm}\equiv S_i^x\pm iS_i^y.$$

To simplify, in this work we shall study the case of $J_{ab}=J_{cd}\neq J_{bc}=J_{da}$ and we shall only deal with the case of the same spin amplitudes for the different sublattices. We restrict ourselves to the low-temperature region of $T\ll T_c$. The initial state is assumed to be of the completely ordered state, in which all spins couple antiparallel or parallel along the z axis for the ferrimagnet or the ferromagnet, respectively. By use of the Holstein-Primakoff transform[2] and the linear spin-wave approximation,[3,4] retaining terms up to the second order in the boson operators a_i^+, a_i; b_j^+, b_j; c_m^+, c_m; d_n^+ and d_n for the sublattices a, b, c, and d, respectively, we have

$$H=-2NZS^2(|J_{ab}|+|J_{bc}|)+ZS(|J_{ab}|+|J_{bc}|)\left[\sum_i a_i^+a_i+\sum_j b_j^+b_j+\sum_m c_m^+c_m+\sum_n d_n^+d_n\right]+S\left[|J_{ab}|\sum_{i\delta}(a_ib_{i+\delta}+a_i^+b_{i+\delta}^+)+|J_{bc}|\sum_{j\delta}(b_jc_{j+\delta}+b_j^+c_{j+\delta}^+)+|J_{ab}|\sum_{m\delta}(c_md_{m+\delta}+c_m^+d_{m+\delta}^+)+|J_{bc}|\sum_{n\delta}(d_na_{n+\delta}+d_n^+a_{n+\delta}^+)\right] \tag{3a}$$

and

$$H=-2NZS^2(J_{ab}+J_{bc})+ZS(J_{ab}+J_{bc})\left[\sum_i a_i^+a_i+\sum_j b_j^+b_j+\sum_m c_m^+c_m+\sum_n d_n^+d_n\right]+S\left[J_{ab}\sum_{i\delta}(a_i^+b_{i+\delta}+a_ib_{i+\delta}^+)+J_{bc}\sum_{j\delta}(b_j^+c_{j+\delta}+b_jc_{j+\delta}^+)+J_{ab}\sum_{m\delta}(c_m^+d_{m+\delta}+c_md_{m+\delta}^+)+J_{bc}\sum_{n\delta}(d_n^+a_{n+\delta}+d_na_{n+\delta}^+)\right] \tag{3b}$$

for the ferrimagnet and the ferromagnet, respectively.

The Hamiltonians are rewritten by introducing the Fourier transforms of the boson operators in the reduced Brillouin zone:

$$H=-2NZS^2(|J_{ab}|+|J_{bc}|)+ZS(|J_{ab}|+|J_{bc}|)\sum_k [a_k^+a_k+b_k^+b_k+c_k^+c_k+d_k^+d_k]+ZS\sum_k \gamma_k[|J_{ab}|(a_k^+b_k^++a_kb_k)+|J_{bc}|(b_k^+c_k^++b_kc_k)+|J_{ab}|(c_k^+d_k^++c_kd_k)+|J_{bc}|(d_k^+a_k^++d_ka_k)] \tag{4a}$$

and

$$H=-2NZS^2(J_{ab}+J_{bc})+ZS(J_{ab}+J_{bc})\sum_k [a_k^+a_k+b_k^+b_k+c_k^+c_k+d_k^+d_k]+ZS\sum_k \gamma_k[J_{ab}(a_k^+b_k+a_kb_k^+)+J_{bc}(b_k^+c_k+b_kc_k^+)+J_{ab}(c_k^+d_k+c_kd_k^+)+J_{bc}(d_k^+a_k+d_ka_k^+)]. \tag{4b}$$

Here

$$\gamma_k=\frac{1}{Z}\sum_{\delta} e^{ik\cdot\delta}.$$

Usually, at this step, one needs to perform the Bogoliubov transformation[9] for eliminating the nondiagonal terms in the Hamiltonian. In the present case, the Bogoliubov transformation should be extended so that it is applicable for the four-sublattice systems. The new transformation may be written as the following matrixes:

$$\begin{bmatrix}\alpha_k^+\\ \beta_k\\ \xi_k^+\\ \eta_k\end{bmatrix}=\begin{bmatrix}a_{1k} & a_{2k} & a_{3k} & a_{4k}\\ a_{2k} & a_{5k} & a_{6k} & a_{7k}\\ a_{3k} & a_{6k} & a_{8k} & a_{9k}\\ a_{4k} & a_{7k} & a_{9k} & a_{10k}\end{bmatrix}\begin{bmatrix}a_k^+\\ b_k\\ c_k^+\\ d_k\end{bmatrix} \tag{5a}$$

and

$$\begin{bmatrix}\alpha_k^+\\ \beta_k^+\\ \xi_k^+\\ \eta_k^+\end{bmatrix}=\begin{bmatrix}a_{1k} & a_{2k} & a_{3k} & a_{4k}\\ a_{2k} & a_{5k} & a_{6k} & a_{7k}\\ a_{3k} & a_{6k} & a_{8k} & a_{9k}\\ a_{4k} & a_{7k} & a_{9k} & a_{10k}\end{bmatrix}\begin{bmatrix}a_k^+\\ b_k^+\\ c_k^+\\ d_k^+\end{bmatrix} \tag{5b}$$

for the ferrimagnet and the ferromagnet, respectively, and consequently their reversed transformation can be written as

$$\begin{bmatrix}a_k^+\\ b_k\\ c_k^+\\ d_k\end{bmatrix}=\begin{bmatrix}A_{1k} & A_{2k} & A_{3k} & A_{4k}\\ A_{2k} & A_{5k} & A_{6k} & A_{7k}\\ A_{3k} & A_{6k} & A_{8k} & A_{9k}\\ A_{4k} & A_{7k} & A_{9k} & A_{10k}\end{bmatrix}\begin{bmatrix}\alpha_k^+\\ \beta_k\\ \xi_k^+\\ \eta_k\end{bmatrix} \tag{6a}$$

and

$$\begin{bmatrix}a_k^+\\ b_k^+\\ c_k^+\\ d_k^+\end{bmatrix}=\begin{bmatrix}A_{1k} & A_{2k} & A_{3k} & A_{4k}\\ A_{2k} & A_{5k} & A_{6k} & A_{7k}\\ A_{3k} & A_{6k} & A_{8k} & A_{9k}\\ A_{4k} & A_{7k} & A_{9k} & A_{10k}\end{bmatrix}\begin{bmatrix}\alpha_k^+\\ \beta_k^+\\ \xi_k^+\\ \eta_k^+\end{bmatrix}. \tag{6b}$$

From the relationship between the parameters a_{ik} and A_{jk} ($i=1,2,...10$ and $j=1,2,...10$), one obtains ten equations (here we omit them to simplify). The commutation relations of the new operators,

$$[\alpha_k,\alpha_{k'}^+]=\delta_{kk'},$$

$$[\beta_k,\beta_{k'}^+]=\delta_{kk'},$$

$$[\xi_k,\xi_{k'}^+]=\delta_{kk'},$$

$$[\eta_k,\eta_{k'}^+]=\delta_{kk'},$$

lead to four equations

$$a_{1k}^2+a_{3k}^2-a_{2k}^2-a_{4k}^2=1, \tag{7a}$$

$$a_{5k}^2+a_{7k}^2-a_{2k}^2-a_{6k}^2=1, \tag{8a}$$

$$a_{3k}^2+a_{8k}^2-a_{6k}^2-a_{9k}^2=1, \tag{9a}$$

$$a_{7k}^2+a_{10k}^2-a_{4k}^2-a_{9k}^2=1 \tag{10a}$$

or

$$a_{1k}^2+a_{3k}^2+a_{2k}^2+a_{4k}^2=1, \tag{7b}$$

$$a_{5k}^2+a_{7k}^2+a_{2k}^2+a_{6k}^2=1, \tag{8b}$$

$$a_{3k}^2+a_{8k}^2+a_{6k}^2+a_{9k}^2=1, \tag{9b}$$

$$a_{7k}^2+a_{10k}^2+a_{4k}^2+a_{9k}^2=1. \tag{10b}$$

For eliminating the nondiagonal terms, i.e., $\alpha_k\beta_k+\alpha_k^+\beta_k^+$, $\alpha_k\xi_k^++\xi_k\alpha_k^+$, $\alpha_k\eta_k+\alpha_k^+\eta_k^+$, $\beta_k\xi_k+\beta_k^+\xi_k^+$, $\beta_k\eta_k^++\eta_k\beta_k^+$, and $\xi_k\eta_k+\xi_k^+\eta_k^+$ for the ferrimagnet or $\alpha_k^+\beta_k+\alpha_k\beta_k^+$, $\alpha_k^+\xi_k+\alpha_k\xi_k^+$, $\alpha_k^+\eta_k+\alpha_k\eta_k^+$, $\beta_k^+\xi_k+\beta_k\xi_k^+$, $\beta_k^+\eta_k+\beta_k\eta_k^+$, and $\xi_k^+\eta_k+\xi_k\eta_k^+$ for the ferromagnet, one needs to establish the following six equations, respectively:

$$\begin{aligned}(J_{ab}+J_{bc})[&A_{1k}A_{2k}+A_{2k}A_{5k}+A_{3k}A_{6k}+A_{4k}A_{7k}]\\ &+\gamma_k[J_{ab}(A_{1k}A_{5k}+A_{2k}^2+A_{3k}A_{7k}+A_{6k}A_{4k})\\ &+J_{bc}(A_{2k}A_{6k}+A_{5k}A_{3k}+A_{4k}A_{2k}+A_{7k}A_{1k})]=0,\end{aligned} \tag{11}$$

$$\begin{aligned}(J_{ab}+J_{bc})[&A_{1k}A_{3k}+A_{2k}A_{6k}+A_{3k}A_{8k}+A_{4k}A_{9k}]\\ &+\gamma_k[J_{ab}(A_{1k}A_{6k}+A_{3k}A_{2k}+A_{3k}A_{9k}+A_{8k}A_{4k})\\ &+J_{bc}(A_{2k}A_{8k}+A_{6k}A_{3k}+A_{4k}A_{3k}+A_{9k}A_{1k})]=0,\end{aligned} \tag{12}$$

$$\begin{aligned}(J_{ab}+J_{bc})[&A_{1k}A_{4k}+A_{2k}A_{7k}+A_{3k}A_{9k}+A_{4k}A_{10k}]\\ &+\gamma_k[J_{ab}(A_{1k}A_{7k}+A_{4k}A_{2k}+A_{3k}A_{10k}+A_{9k}A_{4k})\\ &+J_{bc}(A_{2k}A_{9k}+A_{7k}A_{3k}+A_{4k}^2+A_{10k}A_{1k})]=0,\end{aligned} \tag{13}$$

$$\begin{aligned}(J_{ab}+J_{bc})[&A_{2k}A_{3k}+A_{5k}A_{6k}+A_{6k}A_{8k}+A_{7k}A_{9k}]\\ &+\gamma_k[J_{ab}(A_{2k}A_{6k}+A_{3k}A_{5k}+A_{6k}A_{9k}+A_{8k}A_{7k})\\ &+J_{bc}(A_{5k}A_{8k}+A_{6k}^2+A_{7k}A_{3k}+A_{9k}A_{2k})]=0,\end{aligned} \tag{14}$$

$$\begin{aligned}(J_{ab}+J_{bc})[&A_{2k}A_{4k}+A_{5k}A_{7k}+A_{6k}A_{9k}+A_{7k}A_{10k}]\\ &+\gamma_k[J_{ab}(A_{2k}A_{7k}+A_{4k}A_{5k}+A_{6k}A_{10k}+A_{9k}A_{7k})\\ &+J_{bc}(A_{5k}A_{9k}+A_{7k}A_{6k}+A_{7k}A_{4k}+A_{10k}A_{2k})]=0,\end{aligned} \tag{15}$$

$$\begin{aligned}(J_{ab}+J_{bc})[&A_{3k}A_{4k}+A_{6k}A_{7k}+A_{8k}A_{9k}+A_{9k}A_{10k}]\\ &+\gamma_k[J_{ab}(A_{3k}A_{7k}+A_{4k}A_{6k}+A_{8k}A_{10k}+A_{9k}^2)\\ &+J_{bc}(A_{6k}A_{9k}+A_{7k}A_{8k}+A_{4k}A_{9k}+A_{10k}A_{3k})]=0.\end{aligned} \tag{16}$$

Although Eqs. (11)–(16) are suitable for both ferrimagnet and ferromagnet, the different commutation relations shown in Eqs. (7)–(10) will result in that the solutions for the parameters a_{ik} and A_{jk} will be quite different for both cases.

Then the Hamiltonian can be written as follows:

$$H=H_0+H_0'+H_1, \tag{17}$$

where

$$H_0=-2NZS^2(|J_{ab}|+|J_{bc}|), \tag{18a}$$

$$H_0'=ZS\sum_k \{2(|J_{ab}|+|J_{bc}|)(A_{2k}^2+A_{4k}^2+A_{6k}^2+A_{9k}^2)+[|J_{ab}|(A_{1k}A_{2k}+A_{2k}A_{5k}+A_{3k}A_{6k}+A_{4k}A_{7k}+A_{3k}A_{4k}+A_{6k}A_{7k}+A_{8k}A_{9k}+A_{9k}A_{10k})+|J_{bc}|(A_{2k}A_{3k}+A_{5k}A_{6k}+A_{6k}A_{8k}+A_{7k}A_{9k}+A_{4k}A_{1k}+A_{7k}A_{2k}+A_{9k}A_{3k}+A_{10k}A_{4k})]\gamma_k\}, \tag{19a}$$

$$H_1=ZS\sum_k (\{(|J_{ab}|+|J_{bc}|)(A_{1k}^2+A_{2k}^2+A_{3k}^2+A_{4k}^2)+2[|J_{ab}|(A_{1k}A_{2k}+A_{3k}A_{4k})+|J_{bc}|(A_{2k}A_{3k}+A_{4k}A_{1k})]\gamma_k\}\alpha_k^+\alpha_k+\{(|J_{ab}|+|J_{bc}|)(A_{2k}^2+A_{5k}^2+A_{6k}^2+A_{7k}^2)+2[|J_{ab}|(A_{2k}A_{5k}+A_{6k}A_{7k})+|J_{bc}|(A_{5k}A_{6k}+A_{7k}A_{2k})]\gamma_k\}\beta_k^+\beta_k+\{(|J_{ab}|+|J_{bc}|)(A_{3k}^2+A_{6k}^2+A_{8k}^2+A_{9k}^2)+2[|J_{ab}|(A_{3k}A_{6k}+A_{8k}A_{9k})+|J_{bc}|(A_{6k}A_{8k}+A_{9k}A_{3k})]\gamma_k\}\xi_k^+\xi_k+\{(|J_{ab}|+|J_{bc}|)(A_{4k}^2+A_{7k}^2+A_{9k}^2+A_{10k}^2)+2[|J_{ab}|(A_{4k}A_{7k}+A_{9k}A_{10k})+|J_{bc}|(A_{7k}A_{9k}+A_{10k}A_{4k})]\gamma_k\}\eta_k^+\eta_k) \tag{20a}$$

for the ferrimagnet and

$$H_0=-2NZS^2(J_{ab}+J_{bc}), \tag{18b}$$

$$H_0'=ZS\sum_k \{[J_{ab}(A_{1k}A_{2k}+A_{2k}A_{5k}+A_{3k}A_{6k}+A_{4k}A_{7k}+A_{3k}A_{4k}+A_{6k}A_{7k}+A_{8k}A_{9k}+A_{9k}A_{10k})+J_{bc}(A_{2k}A_{3k}+A_{5k}A_{6k}+A_{6k}A_{8k}+A_{7k}A_{9k}+A_{4k}A_{1k}+A_{7k}A_{2k}+A_{9k}A_{3k}+A_{10k}A_{4k})]\gamma_k\}, \tag{19b}$$

$$H_1=ZS\sum_k (\{(J_{ab}+J_{bc})(A_{1k}^2+A_{2k}^2+A_{3k}^2+A_{4k}^2)+2[J_{ab}(A_{1k}A_{2k}+A_{3k}A_{4k})+J_{bc}(A_{2k}A_{3k}+A_{4k}A_{1k})]\gamma_k\}\alpha_k^+\alpha_k+\{(J_{ab}+J_{bc})\times(A_{2k}^2+A_{5k}^2+A_{6k}^2+A_{7k}^2)+2[J_{ab}(A_{2k}A_{5k}+A_{6k}A_{7k})+J_{bc}(A_{5k}A_{6k}+A_{7k}A_{2k})]\gamma_k\}\beta_k^+\beta_k+\{(J_{ab}+J_{bc})(A_{3k}^2+A_{6k}^2+A_{8k}^2+A_{9k}^2)+2[J_{ab}(A_{3k}A_{6k}+A_{8k}A_{9k})+J_{bc}(A_{6k}A_{8k}+A_{9k}A_{3k})]\gamma_k\}\xi_k^+\xi_k+\{(J_{ab}+J_{bc})(A_{4k}^2+A_{7k}^2+A_{9k}^2+A_{10k}^2)+2[J_{ab}(A_{4k}A_{7k}+A_{9k}A_{10k})+J_{bc}(A_{7k}A_{9k}+A_{10k}A_{4k})]\gamma_k\}\eta_k^+\eta_k) \tag{20b}$$

for the ferromagnet. They are the Hamiltonians for the initial state, the zero-point vibrating and the spin waves, respectively.

Now the problem becomes to solve the equation group, mentioned above, consisting of 20 equations and 20 unknowns (i.e., parameters a_{ik} and A_{jk}). As soon as the equation group is solved, the transformation developed above can be performed and consequently the spin-wave spectra of the present system can be obtained. In the next section, the procedure of solving the equation group will be carried out.

III. PROCEDURE FOR SOLVING THE EQUATION GROUP

Following the analysis in the last section, one obtains an equation group consisting of 20 equations and 20 unknowns. In this section, we will try to find the solution of the equation group so that the spin-wave spectra for the present system may be calculated.

The equation group, established in the last section, is very complex and hard to be solved. In order to solve this equation group, we suggest an easy way to find out the simplest form of the transformation matrixes and to reduce the number of unknowns and equations. For the present system, it can be proved that it is necessary to establish an equation group, including eight equations and eight unknowns.

In this case, the transformation matrixes are

$$\begin{bmatrix} \alpha_k^+ \\ \beta_k \\ \xi_k^+ \\ \eta_k \end{bmatrix} = \begin{bmatrix} a_{1k} & a_{2k} & a_{3k} & a_{4k} \\ a_{2k} & a_{1k} & a_{4k} & a_{3k} \\ a_{3k} & a_{4k} & a_{1k} & a_{2k} \\ a_{4k} & a_{3k} & a_{2k} & a_{1k} \end{bmatrix} \begin{bmatrix} a_k^+ \\ b_k \\ c_k^+ \\ d_k \end{bmatrix} \tag{21a}$$

and

$$\begin{bmatrix} a_k^+ \\ b_k \\ c_k^+ \\ d_k \end{bmatrix} = \begin{bmatrix} A_{1k} & A_{2k} & A_{3k} & A_{4k} \\ A_{2k} & A_{1k} & A_{4k} & A_{3k} \\ A_{3k} & A_{4k} & A_{1k} & A_{2k} \\ A_{4k} & A_{3k} & A_{2k} & A_{1k} \end{bmatrix} \begin{bmatrix} \alpha_k^+ \\ \beta_k \\ \xi_k^+ \\ \eta_k \end{bmatrix} \tag{22a}$$

for the ferrimagnet, and

$$\begin{bmatrix} \alpha_k^+ \\ \beta_k^+ \\ \xi_k^+ \\ \eta_k^+ \end{bmatrix} = \begin{bmatrix} a_{1k} & a_{2k} & a_{3k} & a_{4k} \\ a_{2k} & a_{1k} & a_{4k} & a_{3k} \\ a_{3k} & a_{4k} & a_{1k} & a_{2k} \\ a_{4k} & a_{3k} & a_{2k} & a_{1k} \end{bmatrix} \begin{bmatrix} a_k^+ \\ b_k^+ \\ c_k^+ \\ d_k^+ \end{bmatrix} \tag{21b}$$

and

$$\begin{bmatrix} a_k^+ \\ b_k^+ \\ c_k^+ \\ d_k^+ \end{bmatrix} = \begin{bmatrix} A_{1k} & A_{2k} & A_{3k} & A_{4k} \\ A_{2k} & A_{1k} & A_{4k} & A_{3k} \\ A_{3k} & A_{4k} & A_{1k} & A_{2k} \\ A_{4k} & A_{3k} & A_{2k} & A_{1k} \end{bmatrix} \begin{bmatrix} \alpha_k^+ \\ \beta_k^+ \\ \xi_k^+ \\ \eta_k^+ \end{bmatrix} \tag{22b}$$

for the ferromagnet, respectively. Comparing the matrixes (21) and (22) with the matrixes (5) and (6), one has the relations between the parameters:

$$(1)\quad a_{1k}=a_{5k}=a_{8k}=a_{10k}, \quad a_{2k}=a_{9k}, \quad a_{3k}=a_{7k},$$
$$a_{4k}=a_{6k};$$

and

$$(2)\quad A_{1k}=A_{5k}=A_{8k}=A_{10k}, \quad A_{2k}=A_{9k}, \quad A_{3k}=A_{7k},$$
$$A_{4k}=A_{6k}.$$

Then the commutation relations of the new operators become one equation,

$$a_{1k}^2+a_{3k}^2-a_{2k}^2-a_{4k}^2=1 \tag{23a}$$

for the ferrimagnet or

$$a_{1k}^2+a_{3k}^2+a_{2k}^2+a_{4k}^2=1 \tag{23b}$$

for the ferromagnet. Equations (11)–(16) are reduced to the following three equations:

$$2(J_{ab}+J_{bc})(A_{1k}A_{2k}+A_{3k}A_{4k})+\gamma_k[J_{ab}(A_{1k}^2+A_{2k}^2+A_{3k}^2+A_{4k}^2)+2J_{bc}(A_{1k}A_{3k}+A_{2k}A_{4k})]=0, \tag{24}$$

$$(J_{ab}+J_{bc})(A_{1k}A_{3k}+A_{2k}A_{4k})+\gamma_k[J_{ab}(A_{1k}A_{4k}+A_{2k}A_{3k})+J_{bc}(A_{1k}A_{2k}+A_{3k}A_{4k})]=0, \tag{25}$$

$$2(J_{ab}+J_{bc})(A_{1k}A_{4k}+A_{2k}A_{3k})+\gamma_k[2J_{ab}(A_{1k}A_{3k}+A_{2k}A_{4k})+J_{bc}(A_{1k}^2+A_{2k}^2+A_{3k}^2+A_{4k}^2)]=0, \tag{26}$$

and the relations between the parameters a_{ik} and A_{jk} are

$$A_{1k}=\frac{1}{Y}[a_{1k}(a_{1k}^2-a_{2k}^2-a_{3k}^2-a_{4k}^2)+2a_{2k}a_{3k}a_{4k}], \tag{27}$$

$$A_{2k}=\frac{1}{Y}[a_{2k}(a_{2k}^2-a_{1k}^2-a_{3k}^2-a_{4k}^2)+2a_{1k}a_{3k}a_{4k}], \tag{28}$$

$$A_{3k}=\frac{1}{Y}[a_{3k}(a_{3k}^2-a_{1k}^2-a_{2k}^2-a_{4k}^2)+2a_{1k}a_{2k}a_{4k}], \tag{29}$$

$$A_{4k}=\frac{1}{Y}[a_{4k}(a_{4k}^2-a_{1k}^2-a_{2k}^2-a_{3k}^2)+2a_{1k}a_{2k}a_{3k}]. \tag{30}$$

Here

$$Y=a_{1k}^4+a_{2k}^4+a_{3k}^4+a_{4k}^4+8a_{1k}a_{2k}a_{3k}a_{4k}-2a_{1k}^2a_{2k}^2-2a_{1k}^2a_{3k}^2-2a_{1k}^2a_{4k}^2-2a_{2k}^2a_{3k}^2-2a_{2k}^2a_{4k}^2-2a_{3k}^2a_{4k}^2. \tag{31}$$

Using the relation of Eq. (23a) or (23b), one may rewrite Eqs. (27)–(30) as

$$A_{1k}=\frac{1}{Y}[a_{1k}+2a_{3k}(a_{2k}a_{4k}-a_{1k}a_{3k})], \tag{32a}$$

$$A_{2k}=-\frac{1}{Y}[a_{2k}+2a_{4k}(a_{2k}a_{4k}-a_{1k}a_{3k})], \tag{33a}$$

$$A_{3k}=\frac{1}{Y}[a_{3k}+2a_{1k}(a_{2k}a_{4k}-a_{1k}a_{3k})], \tag{34a}$$

$$A_{4k}=-\frac{1}{Y}[a_{4k}+2a_{2k}(a_{2k}a_{4k}-a_{1k}a_{3k})], \tag{35a}$$

or

$$A_{1k}=\frac{1}{Y}[a_{1k}(2a_{1k}^2-1)+2a_{2k}a_{3k}a_{4k}], \tag{32b}$$

$$A_{2k}=-\frac{1}{Y}[a_{2k}(2a_{2k}^2-1)+2a_{1k}a_{3k}a_{4k}], \tag{33b}$$

$$A_{3k}=\frac{1}{Y}[a_{3k}(2a_{3k}-1)+2a_{1k}a_{2k}a_{4k}], \tag{34b}$$

$$A_{4k}=-\frac{1}{Y}[a_{4k}(2a_{4k}-1)+2a_{1k}a_{2k}a_{3k}]. \tag{35b}$$

To simplify, one defines the following parameters:

$$p_{1k}=a_{1k}a_{2k}+a_{3k}a_{4k}, \tag{36}$$

$$p_{2k}=a_{1k}a_{4k}+a_{2k}a_{3k}, \tag{37}$$

$$p_{3k}=a_{1k}a_{3k}+a_{2k}a_{4k}, \tag{38}$$

$$p_k=a_{2k}a_{4k}-a_{1k}a_{3k}, \tag{39}$$

$$q_k=a_{1k}^2+a_{2k}^2+a_{3k}^2+a_{4k}^2. \tag{40}$$

Considering Eqs. (32)–(35) above, Eqs. (24) minus Eq. (26) results in

$$2p_k=1 \tag{41a}$$

for the ferrimagnet and

$$2a_{1k}^2+2a_{3k}^2+2a_{1k}a_{3k}-2a_{2k}a_{4k}-1=0 \tag{41b}$$

for the ferromagnet, respectively, and/or

$$(J_{ab}-J_{bc})\gamma_k(a_{1k}-a_{3k})^2-2(J_{ab}+J_{bc})(a_{1k}-a_{3k})(a_{2k}-a_{4k})+(J_{ab}-J_{bc})\gamma_k(a_{2k}-a_{4k})^2=0. \tag{42}$$

Inserting Eq. (41a) into Eqs. (24)–(26) leads to

$$p_{1k}+p_{2k}=\frac{\gamma_k(q_k+2p_{3k})}{2} \tag{43}$$

and

$$p_{1k}+p_{2k}=\frac{q_k+2p_{3k}}{2\gamma_k}, \tag{44}$$

which corresponds to $\gamma_k^2\equiv 1$ and is not meaningful.

When $J_{ab}\neq J_{bc}$, Eq. (42) equals

$$a_{1k}-a_{3k}=K(a_{2k}-a_{4k}). \tag{45}$$

Here

$$K=\frac{1\pm\sqrt{1-X^2}}{X} \tag{46}$$

with

$$X=\frac{(J_{ab}-J_{bc})\gamma_k}{J_{ab}+J_{bc}}. \tag{47}$$

On the other hand, from Eq. (42), one may obtain

$$q_k=2\left[p_{3k}+\frac{p_{1k}-p_{2k}}{X}\right]. \tag{48}$$

Inserting Eq. (48) into Eq. (24) and/or (26), one has

$$2p_k=-1 \tag{49}$$

or

$$J_{ab}p_{2k}-J_{bc}p_{1k}-\gamma_k(J_{ab}-J_{bc})p_{3k}=0. \tag{50}$$

Combining Eq. (49) with Eq. (25) results in

$$q_k=2[p_{3k}+X(p_{1k}-p_{2k})]. \tag{51}$$

Equations (48) and (51) lead to the relation of $X^2\equiv 1$, namely,

$$\gamma_k^2\equiv\frac{(J_{ab}+J_{bc})^2}{(J_{ab}-J_{bc})^2}\geqslant 1,$$

which is not meaningful. Thus Eq. (49) should be omitted, also.

It can be proved that Eq. (41b) is not meaningful either.

When $J_{ab}\neq J_{bc}$, Eq. (50) becomes

$$p_{3k}=\frac{J_{ab}p_{2k}-J_{bc}p_{1k}}{\gamma_k(J_{ab}-J_{bc})}. \tag{52}$$

Due to Eq. (52), Eq. (48) can be rewritten as

$$q_k=2\,\frac{J_{ab}p_{1k}-J_{bc}p_{2k}}{\gamma_k(J_{ab}-J_{bc})}. \tag{53}$$

Inserting Eqs. (52) and (53) into Eq. (25), one has

$$(1+\gamma_k^2)J_{ab}J_{bc}(p_{1k}-p_{2k})(1-2p_k)^2+(1-\gamma_k^2)[(J_{bc}^2p_{1k}-J_{ab}^2p_{2k})(1+4p_k^2)+(J_{bc}^2p_{2k}-J_{ab}^2p_{1k})4p_k]=0. \tag{54}$$

Now Eqs. (24)–(26) become Eqs. (52), (53), [or Eq. (45)], and (54). The equation group consists of these three equations together with Eq. (23) and Eqs. (27)–(30). Since the latter four equations are relations between the parameters a_{ik} and A_{jk}, at this step, they are not important to be dealt with for solving the equation group. Therefore, one is only concerned with a smaller equation group, consisting of Eqs. (23), (52), (53) [or Eq. (45)], and (54).

From Eqs. (23) and (45), one has

$$a_{1k}=\frac{1}{2}K(a_{2k}-a_{4k})\pm\frac{\sqrt{\Delta}}{4}, \tag{55}$$

$$a_{3k}=-\frac{1}{2}K(a_{2k}-a_{4k})\pm\frac{\sqrt{\Delta}}{4}. \tag{56}$$

Here

$$\Delta=8a_{2k}^2+8a_{4k}^2+8-4K^2(a_{2k}-a_{4k})^2 \tag{57a}$$

for the ferrimagnet and

$$\Delta=8-8a_{2k}^2-8a_{4k}^2-4K^2(a_{2k}-a_{4k})^2 \tag{57b}$$

for the ferromagnet, respectively.

Inserting Eqs. (55) and (56) into Eqs. (52) and (54), one, respectively, obtains

$$\pm\sqrt{\Delta}(a_{2k}+a_{4k})=2\gamma_k(a_{2k}+a_{4k})^2+M(a_{2k}-a_{4k})^2+2\gamma_k, \tag{58}$$

$$\pm\sqrt{\Delta}(a_{2k}+a_{4k})=-\frac{1}{N}\left[\frac{2-(K^2-1)(a_{2k}-a_{4k})^2}{a_{2k}-a_{4k}}\right]^2 \tag{59}$$

with

$$M=\mp 2\gamma_k K\sqrt{1-X^2}, \tag{60}$$

$$N=\frac{(1-\gamma_k^2)(K^2-1)^2(J_{bc}^2-J_{ab}^2)}{2K[(J_{ab}^2+J_{bc}^2)(1-\gamma_k^2)+2J_{ab}J_{bc}(1+\gamma_k^2)]}. \tag{61}$$

For the convenience, setting $a_{2k}=E+F$ and $a_{4k}=-E+F$, one has

$$\pm\sqrt{\Delta}F=4\gamma_kF^2+2ME^2+\gamma_k, \tag{62}$$

$$\pm\sqrt{\Delta}F=-\frac{[1-2(K^2-1)E^2]^2}{2NE^2}. \tag{63}$$

After defining $u=E^2$ and $v=F^2$, from Eqs. (62) and (63), one immediately obtains

$$16(1-\gamma_k^2)v^2+16(1-\gamma_kM-K^2)uv+8(1-\gamma_k^2)v-4M^2u^2-4\gamma_kMu-\gamma_k^2=0 \tag{64a}$$

for the ferrimagnet or

$$16(1+\gamma_k^2)v^2+16(1+\gamma_kM+K^2)uv-8(1-\gamma_k^2)v+4M^2u^2+4\gamma_kMu+\gamma_k^2=0 \tag{64b}$$

for the ferromagnet and

$$4[MN+(K^2-1)^2]u^2+2[\gamma_kN-2(K^2-1)]u+8\gamma_kNuv+1=0 \tag{65}$$

for both cases. Equation (64) may be reduced to a quartic equation by using the relation of Eq. (65):

$$u^4+pu^3+qu^2+ru+s=0, \tag{66}$$

where for the ferrimagnet one has

$$p=\frac{4(K^2-1)}{T}\,[\gamma_k^2N^2-2\gamma_kN(K^2-1)-2MN-2(1-\gamma_k^2)(K^2-1)^2], \tag{67a}$$

$$q=\frac{1}{T}\,[-\gamma_k^2N^2+2\gamma_kN(K^2-1)+2MN+6(1-\gamma_k^2)(K^2-1)^2], \tag{68a}$$

$$r=-\frac{2(1-\gamma_k^2)(K^2-1)}{T}, \tag{69a}$$

$$s=\frac{1-\gamma_k^2}{4T} \tag{70a}$$

with

$$T=4[M^2N^2+2\gamma_kMN^2(K^2-1)+2MN(K^2-1)^2+2\gamma_kN(K^2-1)^3+(1-\gamma_k^2)(K^2-1)^4], \tag{71a}$$

and for the ferromagnet one has

$$p=\frac{4}{T}\,[2\gamma_kMN^2+4\gamma_kNK^2(K^2-1)-2MN(K^2-1)-\gamma_k^2N^2(K^2+1)-2(1+\gamma_k^2)(K^2-1)^3], \tag{67b}$$

$$q=\frac{1}{T}\,[3\gamma_k^2N^2-2\gamma_kN(5K^2-3)+2MN+6(1+\gamma_k^2)(K^2-1)^2], \tag{68b}$$

$$r=\frac{2}{T}\,[\gamma_kN-(1+\gamma_k^2)(K^2-1)], \tag{69b}$$

$$s=\frac{1+\gamma_k^2}{4T} \tag{70b}$$

with

$$T=4[M^2N^2-2\gamma_kMN^2(K^2+1)+2MN(K^2-1)^2-2\gamma_kN(K^4-1)+(1+\gamma_k^2)(K^2-1)^4]. \tag{71b}$$

The quartic equation may be reduced to the form[26]

$$x^4+ax^2+bx+c=0, \tag{72}$$

by the substitution $u=x-p/4$. Here

$$a=q-\frac{3}{8}\,p^2, \tag{73}$$

$$b=r-\frac{1}{2}\,pq+\frac{1}{8}\,p^3, \tag{74}$$

$$c=s-\frac{1}{4}\,pr+\frac{1}{16}\,p^2q-\frac{3}{256}\,p^4. \tag{75}$$

Let l, m, and n denote the roots of the resolvent cubic:

$$t^3+p't^2+q't+r'=0, \tag{76}$$

where

$$p'=\frac{a}{2}=\frac{1}{2}\left(q-\frac{3}{8}\,p^2\right), \tag{77}$$

$$q'=\frac{1}{16}\,(a^2-4c)=\frac{1}{16}\left(q^2-qp^2+\frac{3}{16}\,p^4-4s+pr\right), \tag{78}$$

$$r'=-\frac{1}{64}\,b^2=-\frac{1}{64}\left(r-\frac{1}{2}\,pq+\frac{1}{8}\,p^3\right)^2. \tag{79}$$

The required roots of the reduced quartic are

$$x_1=+\sqrt{l}+\sqrt{m}+\sqrt{n}, \tag{80}$$

$$x_2=+\sqrt{l}-\sqrt{m}-\sqrt{n}, \tag{81}$$

$$x_3=-\sqrt{l}+\sqrt{m}-\sqrt{n}, \tag{82}$$

$$x_4=-\sqrt{l}-\sqrt{m}+\sqrt{n}, \tag{83}$$

where the selection of the square root to be attached satisfy

$$\sqrt{l}\sqrt{m}\sqrt{n}=-\frac{b}{8}. \tag{84}$$

The cubic equation of Eq. (76) may be reduced by the substitution of $t=y-p'/3$ to the normal form[26]

$$y^3+a'y+b'=0, \tag{85}$$

where

$$a'=\frac{1}{3}\,(3q'-p'^2)=-\frac{1}{48}\,q^2-\frac{1}{4}\,s+\frac{1}{16}\,pr, \tag{86}$$

$$b'=\frac{1}{27}\,(2p'^3-9p'q'+27r')=-\frac{1}{864}\,q^3-\frac{1}{64}\,r^2+\frac{1}{192}\,pqr+\frac{1}{24}\,sq-\frac{1}{64}\,sp^2, \tag{87}$$

which has the solutions y_1, y_2, and y_3:

$$y_1=A+B, \tag{88}$$

$$y_2=-\frac{1}{2}\,(A+B)+\frac{i\sqrt{3}}{2}\,(A-B), \tag{89}$$

$$y_3=-\frac{1}{2}\,(A+B)-\frac{i\sqrt{3}}{2}\,(A-B), \tag{90}$$

where $i^2=-1$ and,

$$A=\left[-\frac{b'}{2}+\left(\frac{b'^2}{4}+\frac{a'^3}{27}\right)^{(1/2)}\right]^{(1/3)}, \tag{91}$$

$$B=\left[-\frac{b'}{2}-\left(\frac{b'^2}{4}+\frac{a'^3}{27}\right)^{(1/2)}\right]^{(1/3)}. \tag{92}$$

Then one has solutions $t_g=y_g-p'/3$ ($g=1$, 2, and 3) which correspond to the roots of the cubic equation (76), namely, l, m, and n. Noticing the substitution relation, one also finds the roots of the quartic equation (66) are $u_h=x_h-p/4$ ($h=1$, 2, 3, and 4). From Eq. (65), one obtains the values for the solutions of v:

$$v=-\frac{1}{8\gamma_k Nu}\,(4[MN+(K^2-1)^2]u^2 + 2[\gamma_k N-2(K^2-1)]u+1). \tag{93}$$

According to the definitions above, finally, one obtains

$$a_{2k}=\pm\sqrt{u}\pm\sqrt{v}, \tag{94}$$

$$a_{4k}=\mp\sqrt{u}\pm\sqrt{v}, \tag{95}$$

or

$$a_{2k}=\pm\sqrt{u}\mp\sqrt{v}, \tag{96}$$

$$a_{4k}=\mp\sqrt{u}\mp\sqrt{v}. \tag{97}$$

Then a_{1k} and a_{3k} are determined by Eqs. (55) and (56) and A_{jk} ($j=1,2,3,4$) are given by Eqs. (27)–(30), respectively. Corresponding to the roots u_h ($h=1$, 2, 3, and 4), there are four groups of solutions for the parameters a_{ik} and A_{jk} ($i=1,2,3,4$; $j=1,2,3,4$). It can be proved that the four groups of the solutions, obtained above, are equal for performing the transformation and calculating the spin-wave spectra.

After performing the transformation, one obtains the final form of the Hamiltonian for the present system:

$$H=H_0+H_0'+H_1, \tag{98}$$

where

$$H_0=-2NZS^2(|J_{ab}|+|J_{bc}|), \tag{99}$$

$$H_0'=4ZS\sum_k \{(|J_{ab}|+|J_{bc}|)(A_{2k}^2+A_{4k}^2)+[|J_{ab}|(A_{1k}A_{2k} + A_{3k}A_{4k})+|J_{bc}|(A_{1k}A_{4k}+A_{2k}A_{3k})]\gamma_k\} \tag{100a}$$

or

$$H_0'=4ZS\sum_k \{[J_{ab}(A_{1k}A_{2k}+A_{3k}A_{4k}) + J_{bc}(A_{1k}A_{4k}+A_{2k}A_{3k})]\gamma_k\} \tag{100b}$$

and

$$H_1=ZS\sum_k \{(|J_{ab}|+|J_{bc}|)(A_{1k}^2+A_{2k}^2+A_{3k}^2+A_{4k}^2) + 2[|J_{ab}|(A_{1k}A_{2k}+A_{3k}A_{4k})+|J_{bc}|(A_{2k}A_{3k} + A_{4k}A_{1k})]\gamma_k\}[\alpha_k^+\alpha_k+\beta_k^+\beta_k+\xi_k^+\xi_k+\eta_k^+\eta_k]. \tag{101}$$

They are the energies for the initial state, the zero-point vibrating, and the spin waves, respectively.

The spin-wave spectrum of the ferromagnet has the same form as that of the ferrimagnet, but the different commutation relations of the new operators in the two systems result in the different parameters p, q, r, and s. It should be noticed that the zero-point vibrating energies in the ferrimagnetic and ferromagnetic systems are also different. It is obvious that the spin-wave spectra depend on the strength of the exchange constants J_{ab} and J_{bc}. It can be seen from Eq. (101) that, in the present four-sublattice system with the exchange constants being $J_{ab}=J_{cd}\neq J_{bc}=J_{da}$, the degeneracy of the spin-wave spectra exists and that the number of the degeneracy of the spin-wave spectra is four. There occurs the splitting of the energy level in the present system. It can be seen from Eqs. (98)–(101) that there are four different energy levels. The different values for the energy levels are ascribed to the fact that there exist positive and negative signs in the formula of K [see Eq. (46)], a_{ik} [see Eqs. (94) and (95)] and thus A_{jk}.

IV. A SPECIAL CASE ($J_{ab}=J_{bc}$): ANTIFERROMAGNET

Following the analysis in the last section, finally, one obtains the spin-wave spectra for the present system, which are very complicated. In this section, a special and simple case ($J_{ab}=J_{bc}$), i.e., an antiferromagnet, will be discussed briefly.

In this case, it can be proved that it is sufficient and necessary to establish an equation group, including six equations and six unknowns and that the transformation matrixes can be taken as

$$\begin{pmatrix}\alpha_k^+\\ \beta_k\\ \xi_k^+\\ \eta_k\end{pmatrix}=\begin{pmatrix}a_{1k} & a_{2k} & a_{3k} & a_{2k}\\ a_{2k} & a_{1k} & a_{2k} & a_{3k}\\ a_{3k} & a_{2k} & a_{1k} & a_{2k}\\ a_{2k} & a_{3k} & a_{2k} & a_{1k}\end{pmatrix}\begin{pmatrix}a_k^+\\ b_k\\ c_k^+\\ d_k\end{pmatrix} \tag{102}$$

and

$$\begin{pmatrix}a_k^+\\ b_k\\ c_k^+\\ d_k\end{pmatrix}=\begin{pmatrix}A_{1k} & A_{2k} & A_{3k} & A_{2k}\\ A_{2k} & A_{1k} & A_{2k} & A_{3k}\\ A_{3k} & A_{2k} & A_{1k} & A_{2k}\\ A_{2k} & A_{3k} & A_{2k} & A_{1k}\end{pmatrix}\begin{pmatrix}\alpha_k^+\\ \beta_k\\ \xi_k^+\\ \eta_k\end{pmatrix}. \tag{103}$$

Equations (23)–(26) become

$$a_{1k}^2+a_{3k}^2-2a_{2k}^2=1, \tag{104}$$

$$4(A_{1k}+A_{3k})A_{2k}+\gamma_k[(A_{1k}+A_{3k})^2+4A_{2k}^2]=0, \tag{105}$$

$$(A_{1k}A_{3k}+A_{2k}^2)+\gamma_k(A_{1k}+A_{3k})A_{2k}=0, \tag{106}$$

and the relations between the parameters a_{ik} and A_{jk} are

$$A_{1k}=\frac{1}{Y}\,(a_{1k}-a_{3k})[-2a_{2k}^2+a_{1k}(a_{1k}+a_{3k})], \tag{107}$$

$$A_{2k} = -\frac{1}{Y} a_{2k}(a_{1k} - a_{3k})^2, \tag{108}$$

$$A_{3k} = \frac{1}{Y}(a_{1k} - a_{3k})[2a_{2k}^2 - a_{3k}(a_{1k} + a_{3k})]. \tag{109}$$

Here

$$Y = (a_{1k} - a_{3k})^2[(a_{1k} + a_{3k})^2 - 4a_{2k}^2]. \tag{110}$$

Now the equation group consists of the six equations listed above, i.e., Eqs. (104)–(109). Solving the above equation group, one immediately obtains the two groups of solutions:

$$\begin{aligned}
a_{1k} &= \frac{1}{2}[1 \pm \sqrt{1+4W^2}],\\
a_{2k} &= W\\
a_{3k} &= \frac{1}{2}[1 \mp \sqrt{1+4W^2}],\\
A_{1k} &= \frac{1}{2}\left[\frac{1}{1-4W^2} \pm \frac{1}{\sqrt{1+4W^2}}\right],\\
A_{2k} &= \frac{W}{4W^2-1},\\
A_{3k} &= \frac{1}{2}\left[\frac{1}{1-4W^2} \mp \frac{1}{\sqrt{1+4W^2}}\right],
\end{aligned} \tag{111}$$

and

$$\begin{aligned}
a_{1k} &= -\frac{1}{2}[1 \mp \sqrt{1+4W^2}],\\
a_{2k} &= -W\\
a_{3k} &= -\frac{1}{2}[1 \pm \sqrt{1+4W^2}],\\
A_{1k} &= -\frac{1}{2}\left[\frac{1}{1-4W^2} \mp \frac{1}{\sqrt{1+4W^2}}\right],\\
A_{2k} &= \frac{W}{1-4W^2},\\
A_{3k} &= -\frac{1}{2}\left[\frac{1}{1-4W^2} \pm \frac{1}{\sqrt{1+4W^2}}\right]
\end{aligned} \tag{112}$$

with

$$W = \frac{1 \pm \sqrt{1-\gamma_k^2}}{2\gamma_k}. \tag{113}$$

It can be proved that the two groups of the solutions, listed above, are equal for performing the transformation and calculating the spin-wave spectra. After performing the transformation, one obtains the final form of the Hamiltonian for the present system:

$$\begin{aligned}
H = H_0 + H_0' + H_1 = &-4NZS^2|J| \mp Z|J|S\sum_k \frac{\gamma_k^2}{\sqrt{1-\gamma_k^2}}\\
&+2Z|J|S\sum_k (1 \pm \sqrt{1-\gamma_k^2})(\alpha_k^+\alpha_k + \beta_k^+\beta_k + \xi_k^+\xi_k\\
&+\eta_k^+\eta_k).
\end{aligned} \tag{114}$$

It can be seen from Eq. (114) that in the four-sublattice system, with the same exchange constants and the same spin amplitudes for different sublattices, there exists the degeneracy of the spin-wave spectra. It has been well known that the value for the number of the degeneracy of the spin-wave spectra is two in a two-sublattice Heisenberg antiferromagnet. Thus it is easy to understand that the value for the number of such degeneracy is four in the present four-sublattice system. However, on the other hand, it is quite surprising that there occurs the splitting of the energy level. It can be seen, from Eq. (114), that there are two different energy levels. The two energy levels are ascribed to the coexistence of the signs $\pm$ in Eq. (113). This differs from the situation in the two sublattice antiferromagnet in which one of the signs $\pm$ is not meaningful and must be omitted during the diagonalizing process. The sublattice magnetization at low temperature is an interesting subject of many investigations. At first, one likes to discuss the ground state at 0 K of the present system. At $T=0$ K, the energy for the ground state is

$$E_0 = -4NZS^2|J| - Z|J|S\sum_k \frac{\gamma_k^2}{\sqrt{1-\gamma_k^2}}, \tag{115}$$

which is lower than that of the initial state and is not the same as that of the ground state in the two-sublattice system. The spin-wave spectrum, corresponding to the ground state, is

$$\hbar\omega_k^+ = (1 + \sqrt{1-\gamma_k^2}). \tag{116}$$

Another energy level in Eq. (114) is for the first exciton level whose effects can be neglected at very low temperatures.

Thus the energy at low temperatures of the four-sublattice Heisenberg antiferromagnet is

$$\begin{aligned}
H = H_0 + H_0' + H_1 = &-4NZS^2|J| - Z|J|S\sum_k \frac{\gamma_k^2}{\sqrt{1-\gamma_k^2}}\\
&+2Z|J|S\sum_k (1 + \sqrt{1-\gamma_k^2})(\alpha_k^+\alpha_k + \beta_k^+\beta_k + \xi_k^+\xi_k\\
&+\eta_k^+\eta_k).
\end{aligned} \tag{117}$$

It is easy to see that the spin-wave spectrum is linear in k for small k. The sublattice magnetization at low temperature of the present system is similar to that in the system of the two sublattice antiferromagnet. The temperature dependences of the magnetization and the specific heat of the present system behave as the T^3 laws of an antiferromagnet. This is the direct result of the assumption of the same exchange constants and the same spin amplitudes for different sublattices.

V. CONCLUDING REMARKS

In conclusion, the spin-wave spectra at low temperatures of the four-sublattice Heisenberg ferrimagnet and ferromagnet with exchange constants $J_{ab}=J_{cd}\neq J_{bc}=J_{da}$ have been determined by performing the standard Holstein-Primakoff and an extended Bogoliubov transformation. In order to perform the transformation, an equation group consisting of 20 equations and 20 unknowns has been established. The equation group was solved by reducing the numbers of the equations and the unknowns. It has been found that in the present four-sublattice ferrimagnetic or ferromagnetic system the degeneracy of the spin-wave spectra exists. On the other hand, there occurs the splitting of the energy level so that at $T=0$ K there are four energy levels. One of which is the ground state and others are exciton levels. The sublattice magnetization at low temperature of the present ferrimagnetic or ferromagnetic system behaves as that of the two-sublattice ferrimagnet or ferromagnet.

The spin-wave spectrum of the ferromagnet has the same form as that of the ferrimagnet, but the different commutation relations of the new operators in the two systems result in the different parameters p, q, r, and s. The zero-point vibrating energies in the two systems are also different.

A special case ($J_{ab}=J_{bc}$), i.e., a four-sublattice Heisenberg antiferromagnet, was discussed briefly. In the four-sublattice system, within the assumption of the same exchange constants and the same spin amplitudes for different sublattices, there also exists the degeneracy of the spin-wave spectra. There occurs the splitting of the energy level, and at $T=0$ K there are two energy levels for a ground state and an exciton level, respectively. The sublattice magnetization at low temperature of the antiferromagnet is similar to that in the two-sublattice antiferromagnet. The temperature dependences of the magnetization and the specific heat of the four-sublattice antiferromagnet satisfy the T^3 laws as those in the two-sublattice antiferromagnet.

The results obtained in this work for the four-sublattice ferrimagnet and/or ferromagnet with exchange constants $J_{ab}=J_{cd}\neq J_{bc}=J_{da}$, might be suitable for the four-sublattice systems with different exchange constants ($J_{ab}\approx J_{cd}\neq J_{bc}\approx J_{da}$). The method as well as the transformation can be applied to other four-sublattice systems or other multisublattice systems.

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[1] F. Bloch, Z. Phys. **61**, 206 (1930).
[2] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[3] P. W. Anderson, Phys. Rev. **86**, 694 (1952).
[4] R. Kubo, Phys. Rev. **87**, 568 (1952).
[5] J. van Kranendonk and J. H. van Vleck, Rev. Mod. Phys. **30**, 1 (1958).
[6] A. B. Harris, D. Kumar, B. I. Halperin, and P. C. Hohenberg, Phys. Rev. B **3**, 961 (1971).
[7] F. J. Dyson, Phys. Rev. **102**, 1217 (1956); **102**, 1230 (1956).
[8] S. V. Maleev, Sov. Phys. JETP **6**, 776 (1958).
[9] N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).
[10] K. H. J. Buschow, in *Handbooks on Ferromagnetic Materials*, edited by E. P. Wohlfarth (North-Holland, Amsterdam, 1983), Vol. 1, p. 297 and Vol. 4, p. 1.
[11] J. F. Herbst, Rev. Mod. Phys. **63**, 819 (1991).
[12] S. H. Li and J. M. D. Coey, in *Handbooks of Magnetic Materials*, edited by K. H. J. Buschow (North-Holland, Amsterdam, 1991), Vol. 6, p. 1.
[13] Zhi-dong Zhang, Tong Zhao, X. K. Sun, and Y. C. Chuang, J. Appl. Phys. **71**, 3434 (1992).
[14] Zhi-dong Zhang, Tong Zhao, P. F. de Châtel, and F. R. de Boer, J. Magn. Magn. Mater. **147**, 74 (1995).
[15] A. del Moral, J. Phys. Condens. Matter **4**, 4687 (1992).
[16] T. A. Kaplan, Phys. Rev. **119**, 1460 (1960); **109**, 782 (1958).
[17] A. W. Sáenz, Phys. Rev. **125**, 1940 (1962).
[18] D. C. Wallace, Phys. Rev. **128**, 1614 (1962).
[19] H. Meyer and A. B. Harris, J. Appl. Phys. **31**, 49S (1960).
[20] R. L. Douglass, Phys. Rev. **120**, 1612 (1960).
[21] H. Cofta, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **VI**, 519 (1958).
[22] J. M. Ziman, Proc. Phys. Soc. London Ser. A **65**, 540 (1952); **65**, 548 (1952); **66**, 89 (1953).
[23] L. Kowalewski, Acta Phys. Pol. **21**, 121 (1962); **20**, 545 (1961); **20**, 675 (1961); **19**, 80 (1960).
[24] J. Szaniecki, Acta Phys. Pol. **20**, 983 (1961); **20**, 995 (1961); **21**, 3 (1962).
[25] S. V. Vonsovski and Y. M. Seidov, Izv. Acad. Nauk SSSR **18**, 319 (1954).
[26] R. S. Burington, *Handbook of Mathematical Tables and Formulas*, 5th ed. (McGraw-Hill, New York, 1972), pp. 12–14.