Quantum phonon optics: Coherent and squeezed atomic displacements

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We investigate coherent and squeezed quantum states of phonons. The latter allow the possibility of modulating the quantum fluctuations of atomic displacements below the zero-point quantum noise level of coherent states. The expectation values and quantum fluctuations of both the atomic displacement and the lattice amplitude operators are calculated in these states—in some cases analytically. We also study the possibility of squeezing quantum noise in the atomic displacement using a polariton-based approach.

I. INTRODUCTION

Classical phonon optics[1] has succeeded in producing many acoustic analogs of *classical optics*, such as phonon mirrors, phonon lenses, phonon filters, and even "phonon microscopes" that can generate acoustic pictures with a resolution comparable to that of visible light microscopy. Most phonon optics experiments use heat pulses or superconducting transducers to generate *incoherent* phonons, which propagate ballistically in the crystal. These ballistic incoherent phonons can then be manipulated by the above-mentioned devices, just as in geometric optics.

Phonons can also be excited *phase coherently*. For instance, coherent acoustic waves with frequencies of up to 10^{10} Hz can be generated by piezoelectric oscillators. Lasers have also been used to generate coherent acoustic and optical phonons through stimulated Brillouin and Raman scattering experiments. Furthermore, in recent years, it has been possible to track the phases of coherent optical phonons,[2] due to the availability of femtosecond-pulse ultrafast lasers (with a pulse duration shorter than a phonon period),[3] and techniques that can measure optical reflectivity with accuracy of one part in 10^6.

In most situations involving phonons, a *classical* description is adequate. However, at low enough temperatures, *quantum* fluctuations become dominant. For example, a recent study[4] shows that quantum fluctuations in the atomic positions can indeed influence observable quantities (e.g., the Raman line shape) even when temperatures are not very low. With these facts in mind, and prompted by the many exciting developments in *classical* phonon optics, coherent phonon experiments, and (on the other hand) squeezed states of light,[5] we would like to explore phonon analogs of *quantum* optics. In particular, we study the dynamical and quantum fluctuation properties of the atomic displacements, in analogy with the modulation of quantum noise in light. Specifically, we study single-mode and two-mode phonon coherent and squeezed states, and then focus on a polariton-based approach to achieve smaller quantum noise than the zero-point fluctuations of the atomic lattice.

The concepts of coherent and squeezed states were both proposed in the context of quantum optics. A coherent state is a phase-coherent sum of number states. In it, the quantum fluctuations in any pair of conjugate variables are at the lower limit of the Heisenberg uncertainty principle. In other words, a coherent state is as "quiet" as the vacuum state. Squeezed states[5] are interesting because they can have *smaller quantum noise than the vacuum state* in one of the conjugate variables, thus having a promising future in different applications ranging from gravitational wave detection to optical communications. In addition, squeezed states form an exciting group of states and can provide unique insight into quantum mechanical fluctuations. Indeed, squeezed states are now being explored in a variety of non-quantum-optics systems, including *classical* squeezed states.[6]

In Sec. II we introduce some quantities of interest and study the fluctuation properties of the phonon vacuum and number states. In Secs. III and IV we investigate phonon coherent and squeezed states. In Sec. V we propose a way of squeezing quantum noise in the atomic displacement operator using a polariton-based mechanism. The Appendix summarizes the derivation of the time evolution of the relevant operators in this polariton approach. Finally, Sec. VI presents some concluding remarks.

II. PHONON OPERATORS AND THE PHONON VACUUM AND NUMBER STATES

A phonon with quasimomentum $\mathbf{p}=\hbar\mathbf{q}$ and branch subscript λ has energy $\epsilon_{\mathbf{q}\lambda}=\hbar\omega_{\mathbf{q}\lambda}$; the corresponding creation and annihilation operators satisfy the boson commutation relations

$$[b_{\mathbf{q}'\lambda'},b^{\dagger}_{\mathbf{q}\lambda}]=\delta_{\mathbf{q}\mathbf{q}'}\delta_{\lambda\lambda'},\quad [b_{\mathbf{q}\lambda},b_{\mathbf{q}'\lambda'}]=0. \tag{1}$$

The atomic displacements $u_{i\alpha}$ of a crystal lattice are given by

$$u_{i\alpha}=\frac{1}{\sqrt{Nm}}\sum_{\mathbf{q}\lambda}^{N} U^{\lambda}_{\mathbf{q}\alpha}Q^{\lambda}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{R}_i}. \tag{2}$$

Here $\mathbf{R}_i$ refers to the equilibrium lattice positions, α to a particular direction, and $Q^{\lambda}_{\mathbf{q}}$ is the normal-mode amplitude operator

$$Q^{\lambda}_{\mathbf{q}}=\sqrt{\frac{\hbar}{2\omega_{\mathbf{q}\lambda}}}(b_{\mathbf{q}\lambda}+b^{\dagger}_{-\mathbf{q}\lambda}). \tag{3}$$

An experimentally observable quantity is the real part of the Fourier transform of the atomic displacement:

$$\mathrm{Re}[u_\alpha(\mathbf{q})]=\sum_\lambda \sqrt{\frac{\hbar}{8m\omega_{\mathbf{q}\lambda}}}\{U^\lambda_{\mathbf{q}\alpha}(b_{\mathbf{q}\lambda}+b^\dagger_{-q\lambda}) + U^{\lambda *}_{\mathbf{q}\alpha}(b_{-\mathbf{q}\lambda}+b^\dagger_{\mathbf{q}\lambda})\}. \tag{4}$$

For simplicity, hereafter we will drop the branch subscript λ, assume that $U_{\mathbf{q}\alpha}$ is real, and define a $\mathbf{q}$-mode dimensionless lattice amplitude operator:

$$u(\pm\mathbf{q})=b_\mathbf{q}+b^\dagger_{-\mathbf{q}}+b_{-\mathbf{q}}+b^\dagger_\mathbf{q}. \tag{5}$$

This operator contains essential information on the lattice dynamics, including quantum fluctuations. It is the phonon analog of the electric field in the photon case.

Let us first consider the phonon vacuum state. When no phonon is excited, the crystal lattice is in the phonon vacuum state $|0\rangle$. The expectation values of the atomic displacement and the lattice amplitude are zero, but the fluctuations will be finite:

$$\langle(\Delta u_{i\alpha})^2\rangle_{\mathrm{vac}}\equiv\langle(u_{i\alpha})^2\rangle_{\mathrm{vac}}-\langle u_{i\alpha}\rangle^2_{\mathrm{vac}} \tag{6}$$

$$=\sum_\mathbf{q}^N \frac{\hbar|U_{\mathbf{q}\alpha}|^2}{2Nm\omega_{\mathbf{q}\alpha}}, \tag{7}$$

$$\langle[\Delta u(\pm\mathbf{q})]^2\rangle_{\mathrm{vac}}=2. \tag{8}$$

Let us now consider the phonon number states. The eigenstates of the harmonic phonon Hamiltonian are number states which satisfy $b_\mathbf{q}|n_\mathbf{q}\rangle=\sqrt{n_\mathbf{q}}|n_\mathbf{q}-1\rangle$. The phonon number and the phase of atomic vibrations are conjugate variables. Thus, due to the uncertainty principle, the phase is arbitrary when the phonon number is certain, as is the case with any number state $|n_\mathbf{q}\rangle$. Thus, in a number state, the expectation values of the atomic displacement $\langle n_\mathbf{q}|u_{i\alpha}|n_\mathbf{q}\rangle$ and $\mathbf{q}$-mode lattice amplitude $\langle n_\mathbf{q}|u(\pm\mathbf{q})|n_\mathbf{q}\rangle$ vanish due to the randomness in the phase of the atomic displacements. The fluctuations in a number state $|n_\mathbf{q}\rangle$ are

$$\langle(\Delta u_{i\alpha})^2\rangle_{\mathrm{num}}=\frac{\hbar|U_{\mathbf{q}\alpha}|^2 n_\mathbf{q}}{Nm\omega_{\mathbf{q}\alpha}}+\sum_{\mathbf{q}'\neq\mathbf{q}}^N\frac{\hbar|U_{\mathbf{q}'\alpha}|^2}{2Nm\omega_{\mathbf{q}'\alpha}}, \tag{9}$$

$$\langle[\Delta u(\pm\mathbf{q})]^2\rangle_{\mathrm{num}}=2+2n_\mathbf{q}. \tag{10}$$

III. PHONON COHERENT STATES

A single-mode $(\mathbf{q})$ phonon coherent state is an eigenstate of a phonon annihilation operator:

$$b_\mathbf{q}|\beta_\mathbf{q}\rangle=\beta_\mathbf{q}|\beta_\mathbf{q}\rangle. \tag{11}$$

It can also be generated by applying a phonon displacement operator $D_\mathbf{q}(\beta_\mathbf{q})$ to the phonon vacuum state

$$|\beta_\mathbf{q}\rangle=D_\mathbf{q}(\beta_\mathbf{q})|0\rangle=\exp(\beta_\mathbf{q}b^\dagger_\mathbf{q}-\beta^*_\mathbf{q}b_\mathbf{q})|0\rangle \tag{12}$$

$$=\exp\left(-\frac{|\beta_\mathbf{q}|^2}{2}\right)\sum_{n_\mathbf{q}=0}^\infty\frac{\beta_\mathbf{q}^{n_\mathbf{q}}}{\sqrt{n_\mathbf{q}!}}|n_\mathbf{q}\rangle. \tag{13}$$

Thus it can be seen that a phonon coherent state is a phase-coherent superposition of number states. Moreover, coherent states are a set of minimum-uncertainty states which are as noiseless as the vacuum state. Coherent states are also the set of quantum states that best describe the classical harmonic oscillators.[7]

A single-mode phonon coherent state can be generated by the Hamiltonian

$$H=\hbar\omega_\mathbf{q}\left(b^\dagger_\mathbf{q}b_\mathbf{q}+\frac{1}{2}\right)+\lambda^*_\mathbf{q}(t)b_\mathbf{q}+\lambda_\mathbf{q}(t)b^\dagger_\mathbf{q} \tag{14}$$

and an appropriate initial state. Here $\lambda_\mathbf{q}(t)$ represents the interaction strength between phonons and the external source. More specifically, if the initial state is a vacuum state, $|\psi(0)\rangle=|0\rangle$, then the state vector becomes a single-mode coherent state thereafter,

$$|\psi(t)\rangle=|\Lambda_\mathbf{q}(t)e^{-i\omega_\mathbf{q}t}\rangle, \tag{15}$$

where

$$\Lambda_\mathbf{q}(t)=-\frac{i}{\hbar}\int_{-\infty}^t\lambda_\mathbf{q}(\tau)e^{i\omega_\mathbf{q}\tau}d\tau \tag{16}$$

is the coherent amplitude of mode $\mathbf{q}$. If the initial state is a single-mode coherent state $|\psi(0)\rangle=|\alpha_\mathbf{q}\rangle$, then the state vector at time t takes the form

$$|\psi(t)\rangle=|\{\Lambda_\mathbf{q}(t)+\alpha_\mathbf{q}\}e^{-i\omega_\mathbf{q}t}\rangle, \tag{17}$$

which is still a coherent state.

In a single-mode $(\mathbf{q})$ coherent state $|\Lambda_\mathbf{q}(t)e^{-i\omega_\mathbf{q}t}\rangle$, $\langle u_{i\alpha}(t)\rangle_{\mathrm{coh}}$ and $\langle u(\pm\mathbf{q})\rangle_{\mathrm{coh}}$ are sinusoidal functions of time. The fluctuation in the atomic displacements is

$$\langle(\Delta u_{i\alpha})^2\rangle_{\mathrm{coh}}=\sum_\mathbf{q}^N\frac{\hbar|U_{\mathbf{q}\alpha}|^2}{2Nm\omega_{\mathbf{q}\alpha}}. \tag{18}$$

The unexcited modes are in the vacuum state and thus all contribute to the noise in the form of zero-point fluctuations. Furthermore,

$$\langle[\Delta u(\pm\mathbf{q})]^2\rangle_{\mathrm{coh}}=2. \tag{19}$$

From the expressions of the noise $\langle(\Delta u_{i\alpha})^2\rangle_{\mathrm{coh}}$ and $\langle[\Delta u(\pm\mathbf{q})]^2\rangle_{\mathrm{coh}}$, it is impossible to know which state (if any) has been excited, while this information is clearly present in the expression of the expectation value of the lattice amplitude $\langle u(\pm\mathbf{q})\rangle_{\mathrm{coh}}$. These results can be straightforwardly generalized to multimode coherent states.

IV. PHONON SQUEEZED STATES

In order to reduce quantum noise to a level below the zero-point fluctuation level, we need to consider phonon squeezed states. Quadrature squeezed states are generalized coherent states.[8] Here ''quadrature'' refers to the dimensionless coordinate and momentum. Compared to coherent states, squeezed ones can achieve smaller variances for one of the quadratures during certain time intervals and are therefore helpful for decreasing quantum noise. Figures 1 and 2 schematically illustrate several types of phonon states, including vacuum, number, coherent, and squeezed states. These figures are the phonon analogs of the illuminating schematic diagrams used for photons.[8]

A single-mode quadrature phonon squeezed state is gen-

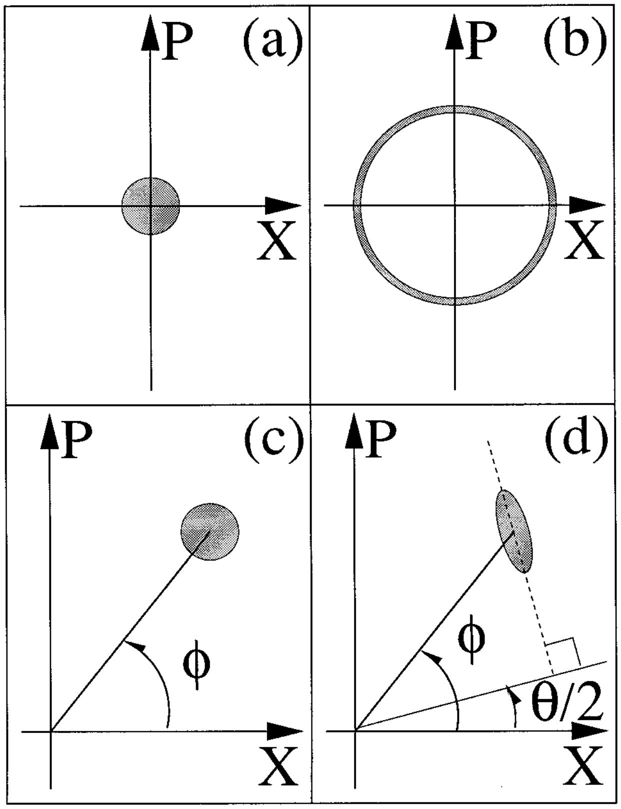

FIG. 1. Schematic diagram of the uncertainty areas (shaded) in the generalized coordinate and momentum $(X(\mathbf{q},-\mathbf{q}),\ P(\mathbf{q},-\mathbf{q}))$ phase space of (a) the phonon vacuum state, (b) a phonon number state, (c) a phonon coherent state, and (d) a phonon squeezed state. Here $X(\mathbf{q},-\mathbf{q})$ and $P(\mathbf{q},-\mathbf{q})$ are the two-mode $(\pm\mathbf{q})$ coordinate and momentum operators, which are the direct generalizations of their corresponding single-mode operators. Notice that the phonon coherent state has the same uncertainty area as the vacuum state, and that both areas are circular, while the squeezed state has an elliptical uncertainty area. Therefore, in the direction parallel to the $\theta/2$ line, the squeezed state has a smaller noise than both the vacuum and coherent states.

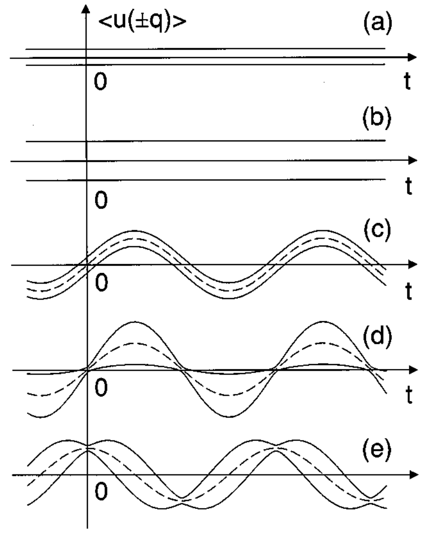

FIG. 2. Schematic diagram of the time evolution of the expectation value and the fluctuation of the lattice amplitude operator $u(\pm\mathbf{q})$ in different states. Dashed lines represent $\langle u(\pm\mathbf{q})\rangle$, while the solid lines represent the envelopes $\langle u(\pm\mathbf{q})\rangle \pm \sqrt{\langle[\Delta u(\pm\mathbf{q})]^2\rangle}$. (a) The phonon vacuum state $|0\rangle$, where $\langle u(\pm\mathbf{q})\rangle=0$ and $\langle[\Delta u(\pm\mathbf{q})]^2\rangle=2$. (b) A phonon number state $|n_{\mathbf{q}},n_{-\mathbf{q}}\rangle$, where $\langle u(\pm\mathbf{q})\rangle=0$ and $\langle[\Delta u(\pm\mathbf{q})]^2\rangle=2(n_{\mathbf{q}}+n_{-\mathbf{q}})+2$. (c) A single-mode phonon coherent state $|\alpha_{\mathbf{q}}\rangle$, where $\langle u(\pm\mathbf{q})\rangle=2|\alpha_{\mathbf{q}}|\cos\omega_{\mathbf{q}}t$ (i.e., $\alpha_{\mathbf{q}}$ is real), and $\langle[\Delta u(\pm\mathbf{q})]^2\rangle=2$. (d) A single-mode phonon squeezed state $|\alpha_{\mathbf{q}}e^{-i\omega_{\mathbf{q}}t},\xi(t)\rangle$, with the squeezing factor $\xi(t)=re^{-2i\omega_{\mathbf{q}}t}$ and $r=1$. Here, $\langle u(\pm\mathbf{q})\rangle=2|\alpha_{\mathbf{q}}|\cos\omega_{\mathbf{q}}t$, and $\langle[\Delta u(\pm\mathbf{q})]^2\rangle=2(e^{-2r}\cos^2\omega_{\mathbf{q}}t+e^{2r}\sin^2\omega_{\mathbf{q}}t)$. (e) A single-mode phonon squeezed state, as in (d); now the expectation value of u is $\langle u(\pm\mathbf{q})\rangle=2|\alpha_{\mathbf{q}}|\sin\omega_{\mathbf{q}}t$ (i.e., $\alpha_{\mathbf{q}}$ is purely imaginary), and the fluctuation $\langle[\Delta u(\pm\mathbf{q})]^2\rangle$ has the same time dependence as in (d). Notice that the squeezing effect now appears at the times when the lattice amplitude $\langle u(\pm\mathbf{q})\rangle$ reaches its maxima, while in (d) the squeezing effect is present at the times when $\langle u(\pm\mathbf{q})\rangle$ is close to zero.

erated from a vacuum state as

$$|\alpha_{\mathbf{q}},\xi\rangle=D_{\mathbf{q}}(\alpha_{\mathbf{q}})S_{\mathbf{q}}(\xi)|0\rangle; \tag{20}$$

a two-mode quadrature phonon squeezed state is generated as

$$|\alpha_{\mathbf{q}_1},\alpha_{\mathbf{q}_2},\xi\rangle=D_{\mathbf{q}_1}(\alpha_{\mathbf{q}_1})D_{\mathbf{q}_2}(\alpha_{\mathbf{q}_2})S_{\mathbf{q}_1,\mathbf{q}_2}(\xi)|0\rangle. \tag{21}$$

Here $D_{\mathbf{q}}(\alpha_{\mathbf{q}})$ is the coherent state displacement operator with $\alpha_{\mathbf{q}}=|\alpha_{\mathbf{q}}|e^{i\phi}$,

$$S_{\mathbf{q}}(\xi)=\exp\left(\frac{\xi^*}{2}b_{\mathbf{q}}^2-\frac{\xi}{2}b_{\mathbf{q}}^{\dagger 2}\right), \tag{22}$$

$$S_{\mathbf{q}_1,\mathbf{q}_2}(\xi)=\exp(\xi^* b_{\mathbf{q}_1}b_{\mathbf{q}_2}-\xi b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_2}^\dagger) \tag{23}$$

are the single- and two-mode squeezing operators, and $\xi=re^{i\theta}$ is the complex squeezing factor with $r\geqslant 0$ and $0\leqslant\theta<2\pi$. The squeezing operator $S_{\mathbf{q}_1,\mathbf{q}_2}(\xi)$ can be produced by the following Hamiltonian:

$$H_{\mathbf{q}_1,\mathbf{q}_2}=\hbar\omega_{\mathbf{q}_1}b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_1}+\hbar\omega_{\mathbf{q}_2}b_{\mathbf{q}_2}^\dagger b_{\mathbf{q}_2}+\zeta(t)b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_2}^\dagger + \zeta^*(t)b_{\mathbf{q}_1}b_{\mathbf{q}_2}. \tag{24}$$

The time-evolution operator has the form

$$U(t)=\exp\left(-\frac{i}{\hbar}H_0 t\right)\exp[\xi^*(t)b_{\mathbf{q}_1}b_{\mathbf{q}_2}-\xi(t)b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_2}^\dagger], \tag{25}$$

where

$$H_0=\hbar\omega_{\mathbf{q}_1}b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_1}+\hbar\omega_{\mathbf{q}_2}b_{\mathbf{q}_2}^\dagger b_{\mathbf{q}_2} \tag{26}$$

and

$$\xi(t)=\frac{i}{\hbar}\int_{-\infty}^{t}\zeta(\tau)e^{i(\omega_{\mathbf{q}_1}+\omega_{\mathbf{q}_2})\tau}d\tau. \tag{27}$$

Here $\xi(t)$ is the squeezing factor and $\zeta(t)$ is the strength of the interaction between the phonon system and the external source; this interaction allows the generation and absorption of two phonons at a time. The two-mode phonon quadrature operators have the form

$$X(\mathbf{q},-\mathbf{q})=2^{-3/2}(b_{\mathbf{q}}+b_{\mathbf{q}}^{\dagger}+b_{-\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}) \tag{28}$$

$$=2^{-3/2}u(\pm\mathbf{q}), \tag{29}$$

$$P(\mathbf{q},-\mathbf{q})=-i2^{-3/2}(b_{\mathbf{q}}-b_{\mathbf{q}}^{\dagger}+b_{-\mathbf{q}}-b_{-\mathbf{q}}^{\dagger}). \tag{30}$$

We have considered two cases where squeezed states were involved in modes $\pm\mathbf{q}$. In the first case, the system is in a two-mode ($\pm\mathbf{q}$) squeezed state $|\alpha_{\mathbf{q}},\alpha_{-\mathbf{q}},\xi\rangle$, ($\xi=re^{i\theta}$), and its fluctuation is

$$\langle[\Delta u(\pm\mathbf{q})]^2\rangle=2\left(e^{-2r}\cos^2\frac{\theta}{2}+e^{2r}\sin^2\frac{\theta}{2}\right). \tag{31}$$

In the second case, the system is in a single-mode squeezed state $|\alpha_{\mathbf{q}},\xi\rangle$ ($\alpha_{\mathbf{q}}=|\alpha_{\mathbf{q}}|e^{i\phi}$) in the first mode and an arbitrary coherent state $|\beta_{-\mathbf{q}}\rangle$ in the second mode. The fluctuation is now

$$\langle[\Delta u(\pm\mathbf{q})]^2\rangle=1+e^{2r}\sin^2\left(\phi+\frac{\theta}{2}\right)+e^{-2r}\cos^2\left(\phi+\frac{\theta}{2}\right). \tag{32}$$

In both of these cases, $\langle[\Delta u(\pm\mathbf{q})]^2\rangle$ can be smaller than in coherent states (see Fig. 2).

V. POLARITON APPROACH

Phonon squeezed states can be generated through phonon-phonon interactions. This will be discussed elsewhere.[9] Here we focus on how to squeeze quantum noise in the atomic displacements through phonon-photon interactions. When an ionic crystal is illuminated by light, there can be a strong coupling between photons and the local polarization of the crystal in the form of phonons. Photons and phonons with the same wave vector can thus form polaritons.[10] Although now phonons and photons are not separable in a polariton, we can still study the quantum noise in the atomic displacements. Let us consider the simplest Hamiltonian[10] describing the above scenario:

$$H_{\text{polariton}}=\sum_{\mathbf{k}}\{E_{1\mathbf{k}}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}+E_{2\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+E_{3\mathbf{k}}(a_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-a_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-a_{\mathbf{k}}b_{-\mathbf{k}}+a_{-\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger})\}, \tag{33}$$

where

$$E_{1\mathbf{k}}=\hbar ck, \tag{34}$$

$$E_{2\mathbf{k}}=\hbar\omega_0\sqrt{1+\chi}, \tag{35}$$

$$E_{3\mathbf{k}}=i\left(\frac{\hbar^2 ck\omega_0\chi}{4\sqrt{1+\chi}}\right)^{1/2}. \tag{36}$$

Here $\mathbf{k}$ is the wave vector for both photons and phonons and ω_0 is the bare phonon frequency. χ is the dimensionless dielectric susceptibility of the crystal (the strength of the phonon-photon interaction) defined by

$$\chi\omega_0^2\varepsilon_0\mathbf{E}=\ddot{\mathbf{P}}+\omega_0^2\mathbf{P}, \tag{37}$$

where $\mathbf{E}$ is the electric field of the incoming light and $\mathbf{P}$ is the polarization generated by optical phonons in the crystal. In $H_{\text{polariton}}$, the two free oscillator sums correspond to free photons and free phonons, while the mixing terms come from the interaction $\mathbf{E}\cdot\mathbf{P}$ between photons and phonons. The phonon energy $E_{2\mathbf{k}}$ has been corrected as ω_0 is substituted by $\omega_0\sqrt{1+\chi}$, so that we have "dressed" phonons.

TABLE I. Different combinations of $t=0$ initial states (modes $\pm\mathbf{k}$) for the polariton approach to lattice amplitude squeezing and the corresponding effects in the fluctuations of the lattice amplitude operator $u(\pm\mathbf{k})$. Here CS($\mathbf{k}$), VS($\mathbf{k}$), TS($\mathbf{k}$), SMST($\mathbf{k}$), and TMST ($\pm\mathbf{k}$) refer, respectively, to coherent, vacuum, thermal, single-mode, and two-mode squeezed states in the mode inside the parentheses, $\mathbf{k}$ or $\pm\mathbf{k}$. $T_s(\chi)$ is the temperature below which squeezing is obtained (see Fig. 4). By squeezing we mean that the quantum noise of the relevant variable is below its corresponding vacuum state value.

$t=0$ photons	$t=0$ phonons	Squeezed $u(\pm\mathbf{k})$?
CS($\pm\mathbf{k}$)	CS($\pm\mathbf{k}$)	yes (no) if $\chi>(\leqslant)$ 0.1
SMST($\mathbf{k}$), VS($-\mathbf{k}$)	VS($\pm\mathbf{k}$)	yes (no) if $\chi>(\leqslant)$ 0.1
SMST($\mathbf{k}$), VS($-\mathbf{k}$)	TS($\pm\mathbf{k}$)	yes if $T<T_s(\chi)$
TMST($\pm\mathbf{k}$)	VS($\pm\mathbf{k}$)	weak (no) if $\chi>(\leqslant)$ 0.1

Our goal is to compute the fluctuations of the lattice amplitude operator $u(\pm\mathbf{k},t)=b_{\mathbf{k}}(t)+b_{-\mathbf{k}}^{\dagger}(t)+b_{-\mathbf{k}}(t)+b_{\mathbf{k}}^{\dagger}(t)$. In a two-mode ($\pm\mathbf{k}$) coherent state $|\alpha_{\mathbf{q}},\alpha_{-\mathbf{q}}\rangle$, its variance is $\langle[\Delta u(\pm\mathbf{q})]^2\rangle_{\text{coh}}=2$. Therefore, if at any given time we obtain a value less than 2, the lattice amplitude of the relevant mode is squeezed. In our calculation, we diagonalize the polariton Hamiltonian and find the time dependence of $u(\pm\mathbf{q})$. The Appendix presents in more detail the derivation of the time evolution of $u(\pm\mathbf{q})$.

Our results show that the fluctuation property of $u(\pm\mathbf{q})$ sensitively depends on the $t=0$ initial state $|\psi(0)\rangle$ of both phonons and photons. Our results are summarized in Table I, and some numerical examples are shown in Fig. 3. These calculations focus on the case where ck is close to ω_0 (the bare phonon frequency, which is typically $\sim$10 THz for optical phonons) and thus our typical time is $\sim$0.1 ps. More specifically, squeezing effects in $u(\pm\mathbf{k})$ are relatively strong for either one of the following two sets of $t=0$ initial states: (i) photon and phonon coherent states, or (ii) single-mode photon squeezed state and phonon vacuum state. For instance, the maximum squeezing exponent r is 0.015 when the incident photon state has a squeezing factor $\xi=0.1e^{2ickt}$ (where ck is the photon frequency). On the other hand, with an initial two-mode photon ($\pm\mathbf{k}$) squeezed state and two-mode ($\pm\mathbf{k}$) phonon vacuum state, the squeezing effect in $u(\pm\mathbf{k})$ is weak. We have also used initial conditions with a single-mode photon squeezed state and thermal states in the two phonon modes.

Figure 4 shows the temperature dependence of the squeezing effect for several values of the dielectric susceptibility χ of the crystal. Our numerical results show that squeezing effects are quickly overshadowed by the thermal noise for small χ, while for large χ (e.g., $\chi=0.5$) the squeezing effect can exist up to $T\approx250$ K, as illustrated in Fig. 4.

VI. CONCLUSIONS

In conclusion, we have investigated the dynamics and quantum fluctuation properties of phonon coherent and

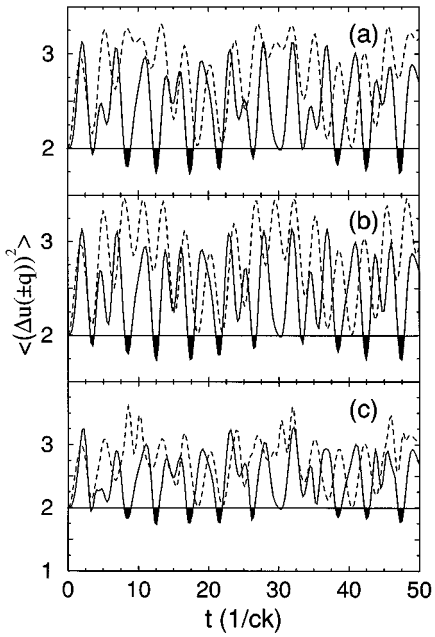

FIG. 3. Calculated $\langle[\Delta u(\pm\mathbf{k})]^2\rangle$ versus time for different combinations of photon and phonon initial states using a polariton mechanism for lattice amplitude squeezing. Dashed (solid) lines correspond to a susceptibility $\chi=0.1(0.4)$. Time is measured in units of $1/ck$, where ck is the free photon frequency. These calculations focus on the case where ck is close to ω_0 (the bare phonon frequency, which is typically $\sim$10 THz for optical phonons) and thus our typical time is $\sim$0.1 ps. The horizontal lines at $\langle[\Delta u(\pm\mathbf{k})]^2\rangle=2$ correspond to the noise level of coherent states. Thus, any time the fluctuation satisfies $\langle[\Delta u(\pm\mathbf{k})]^2\rangle<2$ (highlighted), the state is squeezed. Different combinations of initial states were considered. (a) Photon and phonon coherent states. (b) Single-mode squeezed state in photon mode $\mathbf{k}$ with squeezing factor $\xi=0.1$ and a vacuum state in the photon mode $-\mathbf{k}$; both phonon modes are in the vacuum state. (c) Same combination of states as in (b), but here $\xi=0.1e^{2it}$.

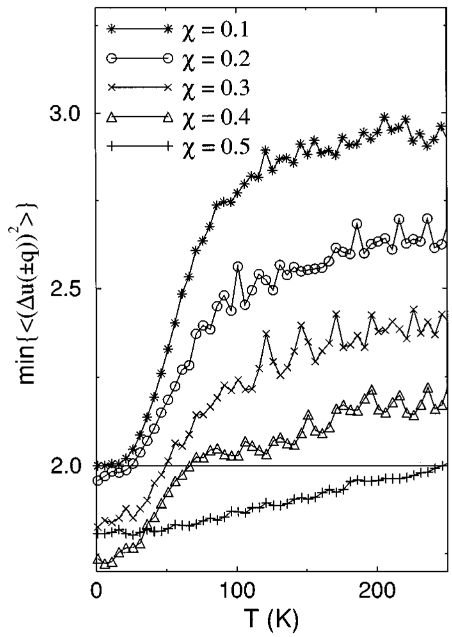

FIG. 4. Temperature dependence of the *minimum* fluctuation $\min\{\langle[\Delta u(\pm\mathbf{k})]^2\rangle\}$ in $u(\pm\mathbf{k})$ using a polariton mechanism for squeezing. The phonon frequency is 10 THz. The initial states are a single-mode squeezed state in photon mode $\mathbf{k}$, vacuum state in photon mode $-\mathbf{k}$, and thermal state in both $(\pm\mathbf{k})$ phonon modes. The squeezing factor is $\xi=0.1e^{2it}$. Squeezing can exist up to a temperature $T_s(\chi)$. For example, when $\chi=0.2$, squeezing effects vanish when $T\gtrsim25$ K. On the other hand, for stronger photon-phonon interaction (e.g., $\chi=0.5$), the squeezing effects can be obtained up to $T\approx250$ K.

squeezed states. In particular, we calculate the experimentally observable time evolution and fluctuation of the lattice amplitude operator $u(\pm\mathbf{q})$. We show that the $\langle u(\pm\mathbf{q})\rangle$ are sinusoidal functions of time in both coherent and squeezed states, but the fluctuation $\langle[\Delta u(\pm\mathbf{q})]^2\rangle$ in a squeezed phonon state is periodically smaller than its vacuum or coherent state value 2. Therefore phonon squeezed states are periodically quieter than the vacuum state. In the polariton approach to squeezing, we calculate the atomic displacement part of a polariton, and prove that the fluctuations of the associated lattice amplitude operator can be squeezed for different combinations of initial photon and phonon states and large enough $(\chi>0.1)$ interaction strength.

It is difficult to generate squeezed states because they have noise levels which are even lower than the one for the vacuum state. Indeed, the experimental and theoretical development of photon coherent and squeezed states took decades. Likewise, the experimental realization of phonon squeezed states might require years of further theoretical and experimental work. Nevertheless, we believe that theoretical results in quantum phonon optics can help the development of the corresponding experiments. We hope that our effort[9] into this very rich problem will lead to more theoretical and experimental developments in the still unexplored area of quantum phonon optics and the manipulation of phonon quantum fluctuations.

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APPENDIX: TIME EVOLUTION OF THE PHONON OPERATORS IN A POLARITON

To derive the time evolution of the phonon operators, we need to first diagonalize the polariton Hamiltonian $H_{\text{polariton}}$. For this purpose, we introduce the polariton operators $\alpha_{i\mathbf{k}}$ in terms of the phonon and photon operators $b_{\mathbf{k}}$ and $a_{\mathbf{k}}$:

$$\alpha_{i\mathbf{k}}=w_i a_{\mathbf{k}}+x_i b_{\mathbf{k}}+y_i a^{\dagger}_{-\mathbf{k}}+z_i b^{\dagger}_{-\mathbf{k}}, \quad i=1,2. \tag{A1}$$

If we write

$$\alpha=(\alpha_{1\mathbf{k}},\alpha_{2\mathbf{k}},\alpha^{\dagger}_{1,-\mathbf{k}},\alpha^{\dagger}_{2,-\mathbf{k}})^T, \tag{A2}$$

$$a=(a_{\mathbf{k}},b_{\mathbf{k}},a^{\dagger}_{-\mathbf{k}},b^{\dagger}_{-\mathbf{k}})^T, \tag{A3}$$

the above relation can be written in a matrix form

$$\alpha=\mathbf{A}\cdot\mathbf{a}, \tag{A4}$$

with its inverse $\mathbf{a}=\mathbf{A}^{-1}\cdot\alpha$. Here $\mathbf{A}$ is a matrix given by

$$\mathbf{A}=\begin{pmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ y_1^* & z_1^* & w_1^* & x_1^* \\ y_2^* & z_2^* & w_2^* & x_2^* \end{pmatrix}. \tag{A5}$$

In the polariton representation, the Hamiltonian has the diagonal form

$$H'_{\text{polariton}}=\sum_{\mathbf{k}}\left[E_{\mathbf{k}}^{(1)}\left(\alpha_{1\mathbf{k}}^{\dagger}\alpha_{1\mathbf{k}}+\frac{1}{2}\right)+E_{\mathbf{k}}^{(2)}\left(\alpha_{2\mathbf{k}}^{\dagger}\alpha_{2\mathbf{k}}+\frac{1}{2}\right)\right]. \tag{A6}$$

The subindices $i=1,2$ specify the two polariton branches, with different dispersion relations $E_{\mathbf{k}}^{(1)}$ and $E_{\mathbf{k}}^{(2)}$. The transformation matrix elements w_i, x_i, y_i, and z_i are determined by requiring that the $\alpha_{i\mathbf{k}}$'s satisfy boson commutation relations

$$[\alpha_{i\mathbf{k}},\alpha_{j\mathbf{k}'}^{\dagger}]=\delta_{ij}\delta_{\mathbf{k}\mathbf{k}'}, \quad [\alpha_{i\mathbf{k}},\alpha_{j\mathbf{k}'}]=0, \tag{A7}$$

so that

$$[\alpha_{i\mathbf{k}},H]=E_{\mathbf{k}}^{(i)}\alpha_{i\mathbf{k}}, \tag{A8}$$

which is true if the two different polariton branches are independent of each other.

In the polariton representation, the Hamiltonian $H'_{\text{polariton}}$ describes two independent harmonic oscillators. From the Heisenberg equation

$$i\hbar\frac{d\hat{O}}{dt}=[\hat{O},H], \tag{A9}$$

we obtain

$$\alpha_{1\mathbf{k}}(t)=\alpha_{1\mathbf{k}}(0)e^{-iE_{\mathbf{k}}^{(1)}t/\hbar}, \tag{A10a}$$

$$\alpha_{2\mathbf{k}}(t)=\alpha_{2\mathbf{k}}(0)e^{-iE_{\mathbf{k}}^{(2)}t/\hbar}, \tag{A10b}$$

or in a more compact form

$$\alpha(t)=\mathbf{U}_{\alpha}(t)\alpha(0). \tag{A11}$$

Recall that the matrix form of the canonical transformation from the photon and phonon operators ($a_{\mathbf{k}}$ and $b_{\mathbf{k}}$) to the polariton operators ($\alpha_{\mathbf{k}}$) is $\boldsymbol{\alpha}=\mathbf{A}\cdot\mathbf{a}$. Thus at time t the photon and phonon operators can be expressed as

$$\mathbf{a}(t)=\mathbf{A}^{-1}\alpha(t)=\mathbf{A}^{-1}\mathbf{U}_{\alpha}(t)\mathbf{A}\mathbf{a}(0), \tag{A12}$$

which provides the time evolution of the photon and phonon operators.

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