Interpretation of the low damping of subthermal capillary waves (ripplons) on superfluid ^{4}He

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The problem of ripplon damping on superfluid ^{4}He is reexamined to show how imposing the boundary condition constraint at the perturbed rather than unperturbed surface position results in strong cancellation of the previous result for the three-ripplon interaction. This has orders of magnitude consequences for the relaxation rate of subthermal ripplons, which is shown to be dominated instead by ripplon-phonon scattering $\propto qT^4$, in excellent quantitative agreement with recent experiments.

I. INTRODUCTION

Atkins[1] first proposed the idea of treating the long-wavelength excitations at the free surface of superfluid helium as quantized capillary waves, or ripplons. It was very successful in explaining the temperature dependence of the surface tension and also accounts well for the frequencies.[2–4] The idea was further developed by Saam to examine the stability of ripplons to decay by ripplon-ripplon[5] and ripplon-phonon[6] scattering as a result of hydrodynamic nonlinearity. Saam's formalism was consequently used to estimate finite temperature lifetimes[7,8] where it was concluded that subthermal ripplon decay should be dominated by ripplon-ripplon scattering. Experiment however shows ripplons to be much longer lived than these predictions. The upper bound on the damping deduced from observed lifetimes of coupled surface charge-ripplon modes[3,4] was already 2 orders of magnitude lower, but was conceivably within the range of error introduced by approximations.[8] Now that measurements of ripplon damping[9] have revealed a six orders of magnitude discrepancy, a bolder explanation is called for. Either there is an error in the logic relating the calculated result to the model or the model itself is wrong.

We have therefore reexamined the formalism. The constraint imposed by the surface boundary condition relates the velocity potential to the surface deformation and thereby determines the form of the Lagrangian in terms of the surface variables. The correction to the ripplon-ripplon scattering Hamiltonian, introduced by applying the constraint at the perturbed rather than the unperturbed surface configuration, is of the same order as the term obtained by Saam using the unperturbed constraint. The sign, however, is opposite and the two terms almost exactly cancel in the limit of subthermal ripplons. Indeed, the cancellation is so strong ($\approx 10^{-12}$) that the mechanism of ripplon-phonon scattering, previously thought to be unimportant, becomes dominant. An explicit calculation shows that this process accounts well for the new experimental results.

In what follows, we dissert the problem in such a way as to try to understand why taking account of the curvature on applying the surface boundary condition—itself a well-known procedure[10]—leads to such a dramatic cancellation in the damping rate.

In a coordinate system defined by the three unit vectors $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$, let the liquid be located, at time t, in the half space $z \leqslant \zeta(x,y,t)$. In the absence of any excitation, $\zeta = 0$. The superfluid is described by an irrotational velocity field $\mathbf{v} = \nabla\phi(x,y,z,t)$ subject to the boundary condition that there be no particle flow though the free surface. For an element ds of the boundary surface with normal vector $\hat{\mathbf{n}}$, $\hat{\mathbf{z}}\cdot\hat{\mathbf{n}}\dot{\zeta}ds = \mathbf{v}\cdot\hat{\mathbf{n}}ds$. Since $\hat{\mathbf{n}}$ is proportional to $(\hat{\mathbf{z}} - \nabla_\parallel\zeta)$, $[\nabla_\parallel = (\partial/\partial x)\hat{\mathbf{x}} + (\partial/\partial y)\hat{\mathbf{y}}]$, the boundary condition becomes[10]

$$\dot{\zeta} = (\partial\phi/\partial z)\big|_{z=\zeta} - \nabla_\parallel\zeta\cdot\nabla_\parallel\phi\big|_{z=\zeta}. \tag{1}$$

Expanding Eq. (1) to second order in (ζ,ϕ),

$$\dot{\zeta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} + \zeta\left.\frac{\partial^2\phi}{\partial z^2}\right|_{z=0} - \nabla_\parallel\zeta\cdot\nabla_\parallel\phi\big|_{z=0}. \tag{2}$$

This is the constraint which was approximated to the first term in Ref. 5. We define Fourier transforms in the (x,y) plane $\mathbf{r}_\parallel = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$

$$\phi(\mathbf{r}_\parallel,z) = \sum_{\mathbf{q}} \phi_{\mathbf{q}}(z)e^{i\mathbf{q}\cdot\mathbf{r}_\parallel}, \quad \zeta(\mathbf{r}_\parallel) = \sum_{\mathbf{q}} \zeta_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}_\parallel}. \tag{3}$$

Incompressible fluid: the three-ripplon interaction. The equation of continuity in the incompressible liquid $\Delta\phi = 0$ leads to $\phi_{\mathbf{q}}(z) = \phi_{\mathbf{q}}e^{qz}$, with $q = |\mathbf{q}|$. The boundary condition (1) imposes

$$q\phi_{\mathbf{q}} = \dot{\zeta}_{\mathbf{q}} - \sum_{\mathbf{q}'} \frac{\mathbf{q}\cdot\mathbf{q}'}{q'}\dot{\zeta}_{\mathbf{q}'}\zeta_{\mathbf{q}-\mathbf{q}'}, \tag{4}$$

where the last term corresponds to the sum of the second-order terms in Eq. (2). The Lagrangian of a nonviscous incompressible fluid is $\mathscr{L} = T - V$, where

$$T = \frac{\rho_0}{2}\int_{-\infty}^{\infty} dxdy \int_{-\infty}^{\zeta(x,y)} \mathbf{v}^2(x,y,z)dz \tag{5}$$

represents the kinetic energy[11] and

$$V = \sigma_0\int_{-\infty}^{\infty} dxdy\{[1 + \nabla_\parallel^2\zeta]^{1/2} - 1\} \tag{6}$$

is the potential energy from surface-tension forces.[12] ρ_0 represents the bulk ^{4}He density and σ_0 the surface tension at

zero temperature. Expanding T and V to order three in (ζ,ϕ), one obtains the same expression as Saam:[5]

$$\mathscr{L}=\frac{\rho_0}{2}\sum_{\mathbf{q}} q|\phi_{\mathbf{q}}|^2-\frac{\sigma_0}{2}\sum_{\mathbf{q}} q^2|\zeta_{\mathbf{q}}|^2$$

$$-\frac{\rho_0}{2}\sum_{\mathbf{q},\mathbf{q}'}\{\mathbf{q}\cdot\mathbf{q}'-qq'\}\phi_{\mathbf{q}}\zeta_{-\mathbf{q}-\mathbf{q}'}\phi_{\mathbf{q}'}. \tag{7}$$

The variables $(\phi_{\mathbf{q}},\zeta_{\mathbf{q}})$ are coupled through Eq. (4), from which $\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1$ with

$$\mathscr{L}_0=\frac{\rho_0}{2}\sum_{\mathbf{q}}\frac{1}{q}|\dot{\zeta}_{\mathbf{q}}|^2-\frac{\sigma_0}{2}\sum_{\mathbf{q}} q^2|\zeta_{\mathbf{q}}|^2,$$

$$\mathscr{L}_1=\frac{\rho_0}{2}\sum_{\mathbf{q},\mathbf{q}'}\left\{\frac{\mathbf{q}\cdot\mathbf{q}'+qq'}{qq'}\right\}\dot{\zeta}_{\mathbf{q}}\zeta_{-\mathbf{q}-\mathbf{q}'}\dot{\zeta}_{\mathbf{q}'} \tag{8}$$

and the classical Hamiltonian, correct to order 3, is $\mathscr{H}=\mathscr{H}_0+\mathscr{H}_1$ with

$$\mathscr{H}_0=\frac{1}{2\rho_0}\sum_{\mathbf{q}} q|\pi_{\mathbf{q}}|^2+\frac{\sigma_0}{2}\sum_{\mathbf{q}} q^2|\zeta_{\mathbf{q}}|^2,$$

$$\mathscr{H}_1=-\frac{1}{2\rho_0}\sum_{\mathbf{q},\mathbf{q}'}\{\mathbf{q}\cdot\mathbf{q}'+qq'\}\pi_{\mathbf{q}}\zeta_{-\mathbf{q}-\mathbf{q}'}\pi_{\mathbf{q}'}, \tag{9}$$

where $\pi_{-\mathbf{q}}=\partial\mathscr{L}/\partial\dot{\zeta}_{\mathbf{q}}$ is the momentum conjugate to $\zeta_{\mathbf{q}}$. The quantum version is obtained through the usual transformation of canonical variables into Bose operators.[5]

The interaction Hamiltonian $\mathscr{H}_1$ differs from Eq. (6) of Ref. 5 only in the sign of the qq' term. This seemingly innocent remark is crucial. First of all, the sign ensures the invariance of the physics to a z translation of the coordinate system. In a new set of coordinates defined by $(x'=x,\ y'=y,\ z'=z+\epsilon)$ the new Fourier components are related to the old by $\zeta'_{\mathbf{q}}=\zeta_{\mathbf{q}},\ \forall\mathbf{q}\neq 0$, and $\zeta_0=\zeta_0+\epsilon$ and the Hamiltonian is transformed into

$$\mathscr{H}'=\mathscr{H}(\zeta_{\mathbf{q}}=\zeta'_{\mathbf{q}},\pi_{\mathbf{q}}=\pi'_{\mathbf{q}}\forall\mathbf{q})$$

$$-\frac{\epsilon}{2\rho_0}\sum_{\mathbf{q}}\{\mathbf{q}\cdot\mathbf{q}'+qq'\}_{\mathbf{q}'=-\mathbf{q}}|\pi_{\mathbf{q}}|^2. \tag{10}$$

Only if the term in curly brackets is null are the form and the physical results invariant; otherwise an arbitrary effective mass term is added to $\mathscr{H}$ resulting in arbitrary ripplon frequencies depending on the choice of coordinate reference.

By the same token, the present form changes radically the previous result for the damping of subthermal ripplons. For ripplons of wave vector $\mathbf{Q}$, the dominant effect arises from terms involving $\zeta_{\mathbf{Q}}$. Taking account of the vector addition $\mathbf{Q}=-\mathbf{q}'-\mathbf{q}$, these now become

$$\frac{1}{2\rho_0}\sum_{\mathbf{q}}\frac{1}{2}\{(q-q')^2-Q^2\}\pi_{\mathbf{q}}\zeta_{\mathbf{Q}}\pi_{-\mathbf{q}-\mathbf{Q}} \tag{11}$$

instead of

$$\frac{1}{2\rho_0}\sum_{\mathbf{q}}\frac{1}{2}\{(q+q')^2-Q^2\}\pi_{\mathbf{q}}\zeta_{\mathbf{Q}}\pi_{-\mathbf{q}-\mathbf{Q}}. \tag{12}$$

The new coefficients are bounded by $Q^2/2$, whereas previously the only bound was imposed by the availability of thermally excited ripplons $q,q'\approx q_T$. Very roughly, the effect on the damping is to introduce a factor $(Q/q)^4\sim(\hbar\omega_Q/k_BT)^{8/3}\sim 10^{-13}$ in the typical conditions of the experiment of Ref. 9. In fact, in contrast with the previous result, the principal contribution now arises from $q\gtrsim Q$ and not from $q\sim q_T$, so that the decay rate becomes proportional to T. A good approximation to the numerically integrated result, accurate to 15% for $10^{-5}<\hbar\omega_Q/k_BT<10^{-3}$ is given by

$$(\omega_Q\tau_Q)^{-1}=\frac{1.2}{18\pi}(aQ)^2\frac{k_BT}{\sigma_0a^2}. \tag{13}$$

If a is taken to be the Wigner-Seitz radius (≈ 2.2 Å), $\sigma_0a^2/k_B\approx 1.4$ K. It is related to the previous estimate[7] by a factor $0.05(\hbar\omega_Q/k_BT)^{7/3}$ ($\approx 10^{-12}$ for 10 μm wavelength at 0.5 K). In view of the dramatic reduction factor for this mechanism, it becomes important to estimate other mechanisms and in particular the ripplon-phonon interaction.

Compressible fluid: the one-ripplon–two-phonon interaction. Saam[6] also set up a formalism for the compressible fluid using linearized continuity equations. We shall see however that the second-order terms in the continuity equations have only a small effect on the ripplon-phonon interaction for subthermal ripplons.

The Lagrangian of the ideal compressible fluid is $\mathscr{L}=T-V-V'$, where

$$T=\frac{1}{2}\int_{-\infty}^{\infty}dxdy\int_{-\infty}^{\zeta}dz[\rho_0+\rho']\mathbf{v}^2 \tag{14}$$

represents the kinetic energy, V is given by Eq. (6) and

$$V'=\frac{s^2}{2\rho_0}\int_{-\infty}^{\infty}dxdy\int_{-\infty}^{\zeta}dz\rho'^2 \tag{15}$$

is the contribution to the energy due to the compression of the liquid. ρ' represents the variation of the density $\rho=\rho_0+\rho'$ from the equilibrium value ρ_0 and s is the first sound velocity. ζ is coupled to ϕ by Eq. (2) and ρ' is coupled to $\mathbf{v}=\nabla\phi$ through the equation of continuity in the bulk[12]

$$\dot{\rho}'+(\rho_0+\rho')\Delta\phi+\nabla\phi\cdot\nabla\rho'=0. \tag{16}$$

We write ϕ as the sum of two terms, $\phi=\phi^0+\phi^1$, where ϕ^0 satisfies the continuity equation to first order

$$\dot{\rho}'+\rho_0\Delta\phi^0=0 \tag{17}$$

and ϕ^1 is a second-order term satisfying

$$\rho_0\Delta\phi^1+\rho'\Delta\phi^0+\nabla\phi^0\cdot\nabla\rho'=0. \tag{18}$$

The quadratic part of the Lagrangian, $\mathscr{L}_0=T_0-V_0-V'_0$, is obtained by expanding T, V, and V' to second order,

$$T_0=\frac{\rho_0}{2}\int_{-\infty}^{\infty}dxdy\int_{-\infty}^{0}dz|\nabla\phi^0|^2,$$

$$V_0=\frac{\sigma_0}{2}\int_{-\infty}^{\infty}dxdy\nabla_{\parallel}\zeta^2,$$

$$V_0'=\frac{s^2}{2\rho_0}\int_{-\infty}^{\infty}dxdy\int_{-\infty}^{0}dz\rho'^2, \tag{19}$$

and restricting the continuity equations (2) and (16) to linear terms. The normal modes of $\mathscr{L}_0$ are expressed in Ref. 6. We shall however reformulate the problem somewhat differently. The key point is that it is only possible to eliminate variables through the constraints of Eqs. (2) and (16) taken to second order if we choose ρ' and ζ rather than ϕ and ζ (Ref. 6) as independent generalized coordinates. In addition to the Fourier transforms of Eq. (3), we introduce $\rho_{\mathbf{q},k}$ defined by

$$\rho'(x,y,z)=\sum_{\mathbf{q},k}\rho_{\mathbf{q},k}e^{i(\mathbf{q}\cdot\mathbf{r}_{\parallel}+kz)},\quad \rho_{\mathbf{q},k}=\rho^*_{-\mathbf{q},-k}. \tag{20}$$

ϕ^0 must satisfy Eq. (17), the general solution for which is a sum over

$$\phi^0_{\mathbf{q}}(z)=\phi^R_{\mathbf{q}}e^{qz}+\frac{1}{\rho_0}\sum_k\frac{\dot\rho_{\mathbf{q},k}}{q^2+k^2}e^{ikz}. \tag{21}$$

The coefficient $\phi^R_{\mathbf{q}}$ is fixed by the boundary condition (2) at the free surface. ϕ^1 is determined by Eq. (18) to be of order 2.

To first order in the surface constraint

$$q\phi^R_{\mathbf{q}}=\dot\eta_{\mathbf{q}}\ \text{ with }\ \eta_{\mathbf{q}}=\zeta_{\mathbf{q}}-\frac{1}{\rho_0}\sum_k\frac{ik}{q^2+k^2}\rho_{\mathbf{q},k}. \tag{22}$$

Setting $\rho_{\mathbf{q},-k}=-R_{q,k}\rho_{\mathbf{q},k}$, where

$$R_{q,k}=\frac{q^2+k^2-2ib_qk}{q^2+k^2+2ib_qk}\ \text{ with }\ b_q=\frac{\sigma_0q^2}{2\rho_0s^2} \tag{23}$$

represents the reflection coefficient for phonons impinging on the surface, the Lagrangian $\mathscr{L}_0$ takes the diagonal form of Ref. 6

$$\mathscr{L}_0=\frac{\rho_0}{2}\sum_{\mathbf{q}}\frac{|\dot\eta_{\mathbf{q}}|^2}{(q^2+b_q^2)^{1/2}}-\frac{\sigma_0}{2}\sum_{\mathbf{q}}\frac{q^2\kappa_q|\eta_{\mathbf{q}}|^2}{(q^2+b_q^2)^{1/2}}+\frac{1}{4\rho_0}\sum_{\mathbf{q},k}\frac{|\dot{\tilde\rho}_{\mathbf{q},k}|^2}{q^2+k^2}-\frac{s^2}{4\rho_0}\sum_{\mathbf{q},k}|\tilde\rho_{\mathbf{q},k}|^2, \tag{24}$$

where $\kappa_q=-b_q+(q^2+b_q^2)^{1/2}$ and

$$\tilde\rho_{\mathbf{q},k}=\rho_{\mathbf{q},k}-\frac{4ib_qk\rho_0}{q^2+k^2-2ib_qk}\eta_{\mathbf{q}}. \tag{25}$$

The eigenfrequencies for ripplons are $\omega_q=(\sigma_0\kappa_qq^2/\rho_0)^{1/2}$ and for phonons $\varpi_{q,k}=s(q^2+k^2)^{1/2}$.[6]

The next higher order introduces the interaction terms which give rise to damping. We shall limit ourselves here[13] to the terms of interest for damping of subthermal ripplons ($\hbar\omega_q\ll k_BT$). Only three-ripplon and one-ripplon–two-phonon scattering is allowed by the kinematics. Futhermore we retain only the lowest order in b_q/q (≈0.015 for thermal phonons at 1 K). Within this approximation, $R_{q,k}\approx1$, $\rho'|_{z=0}\approx0$, and $\rho_{\mathbf{q},k}\approx\tilde\rho_{\mathbf{q},k}$. Hence, we neglect the third-order term [order $(b_q/q)^2$] in the expansion of V'. After an integration by parts with the help of Eq. (18), the kinetic energy may be written $T=T_0+T_1$ where

$$T_1=\int_{-\infty}^{\infty}dxdy\left[\frac{\rho_0\zeta}{2}|\nabla\phi^0|^2_{z=0}-\int_{-\infty}^{0}dz\frac{\rho'}{2}|\nabla\phi^0|^2\right]+\int_{-\infty}^{\infty}dxdy\left[\rho_0\phi^0\frac{\partial\phi^1}{\partial z}+\rho'\phi^0\frac{\partial\phi^0}{\partial z}\right]_{z=0}. \tag{26}$$

$(\partial\phi^1/\partial z)|_{z=0}$ can be expressed in terms of its Fourier transform in the (x,y) plane, after integration of Eq. (18)

$$\left.\frac{\partial\phi^1_{\mathbf{q}}}{\partial z}\right|_{z=0}=i\sum_{\mathbf{q}',k}\frac{(k^2+2q^2)\mathbf{q}'(\mathbf{q}'-\mathbf{q})-qq'k^2}{\rho_0kq'(k^2+4q^2)}\dot\eta_{\mathbf{q}'}\tilde\rho_{\mathbf{q}-\mathbf{q}',k}.$$

As T_1 is already of order 3, the variable $\phi^0_{\mathbf{q}}$ may be eliminated from it by using the first-order surface boundary condition of Eq. (22) and Eq. (21). The resulting contribution to the three-ripplon interaction is

$$\mathscr{L}_1^a=\frac{\rho_0}{2}\sum_{\mathbf{q},\mathbf{q}'}\left\{\frac{-\mathbf{q}\cdot\mathbf{q}'+qq'}{qq'}\right\}\dot\eta_{\mathbf{q}}\eta_{-\mathbf{q}-\mathbf{q}'}\dot\eta_{\mathbf{q}'}. \tag{27}$$

Neglecting terms of order $\dot\eta\tilde\rho\tilde\rho$ compared with $\eta\dot{\tilde\rho}\tilde\rho$, the lowest order one-ripplon–two-phonon interaction from T_1 is described by

$$\mathscr{L}_1'=-\frac{1}{2\rho_0}\sum_{\substack{\mathbf{q},k\\ \mathbf{q}',k'}}\frac{kk'}{(q^2+k^2)(q'^2+k'^2)}\dot{\tilde\rho}_{\mathbf{q},k}\eta_{-\mathbf{q}-\mathbf{q}'}\dot{\tilde\rho}_{\mathbf{q}',k'}. \tag{28}$$

Additional interaction terms, those omitted in Ref. 6, arise from T_0 on expressing $\phi^R_{\mathbf{q}}$ in terms of the generalized coordinates by means of the surface boundary condition of Eq. (2) taken to second order. Equation (22) is then replaced by

$$q\phi^R_{\mathbf{q}}=\dot\eta_{\mathbf{q}}-\sum_{\mathbf{q}'}\frac{\mathbf{q}\cdot\mathbf{q}'}{q'}\dot\eta_{\mathbf{q}'}\eta_{\mathbf{q}-\mathbf{q}'}-\left.\frac{\partial\phi^1_{\mathbf{q}}}{\partial z}\right|_{z=0}-\frac{1}{\rho_0}\sum_{\mathbf{q}',k}\frac{ik\mathbf{q}\cdot\mathbf{q}'\dot\eta_{\mathbf{q}'}\tilde\rho_{\mathbf{q}-\mathbf{q}',k}}{q'[(\mathbf{q}-\mathbf{q}')^2+k^2]}. \tag{29}$$

The first second-order term in Eq. (29) contributes to the three-ripplon interaction

$$\mathscr{L}_1^b=\rho_0\sum_{\mathbf{q},\mathbf{q}'}\left\{\frac{\mathbf{q}\cdot\mathbf{q}'}{qq'}\right\}\dot\eta_{\mathbf{q}}\eta_{-\mathbf{q}-\mathbf{q}'}\dot\eta_{\mathbf{q}'}. \tag{30}$$

The sum $\mathscr{L}_1=\mathscr{L}_1^a+\mathscr{L}_1^b$ is identical to the three-ripplon term of Eq. (8) for the incompressible fluid. The other second-order terms induce two-ripplon–one-phonon interactions that are kinematically forbidden in the regime that we consider here. The relevant Lagrangian is then $\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1+\mathscr{L}_1'$. In terms of the conjugate momenta $\pi_{-\mathbf{q}}=\partial\mathscr{L}/\partial\dot\eta_{\mathbf{q}}$ and $\psi_{-\mathbf{q},-k}=\partial\mathscr{L}/\partial\dot{\tilde\rho}_{\mathbf{q},k}$, the Hamiltonian is $\mathscr{H}=\mathscr{H}_0+\mathscr{H}_1+\mathscr{H}_1'$ with

$$\mathscr{H}_0=\sum_{\mathbf{q}}\left[\frac{(q^2+b_q^2)^{1/2}}{2\rho_0}|\pi_{\mathbf{q}}|^2+\frac{\sigma_0 q^2\kappa_q}{2(q^2+b_q^2)^{1/2}}|\eta_{\mathbf{q}}|^2\right]$$
$$+\sum_{\mathbf{q},k}\left[\rho_0(q^2+k^2)|\psi_{\mathbf{q},k}|^2+\frac{s^2}{4\rho_0}|\tilde{\rho}_{\mathbf{q},k}|^2\right],$$

$$\mathscr{H}_1=-\frac{1}{2\rho_0}\sum_{\mathbf{q},\mathbf{q}'}\{\mathbf{q}\cdot\mathbf{q}'+qq'\}\pi_{\mathbf{q}}\eta_{-\mathbf{q}-\mathbf{q}'}\pi_{\mathbf{q}'},$$

$$\mathscr{H}_1'=2\rho_0\sum_{\mathbf{q},k}\sum_{\mathbf{q}',k'}kk'\psi_{\mathbf{q},k}\eta_{-\mathbf{q}-\mathbf{q}'}\psi_{\mathbf{q}',k'}. \qquad (31)$$

The three-ripplon term $\mathscr{H}_1$ is identical to Eq. (9) and, as already pointed out, differs from the previously accepted result. Our leading term $\mathscr{H}_1'$ for the one-ripplon–two-phonon interaction, on the other hand, is identical to that derived in Ref 6; the higher order terms in the continuity equations make no significant contribution here. The ripplon damping factor associated with $\mathscr{H}_1'$ for $\hbar\omega_Q \ll k_BT$ is

$$\frac{2}{3}(\omega_Q\tau_Q)^{-1}\approx\left(\frac{2}{3}\right)\frac{\pi^2}{60}\frac{\hbar Q}{\rho_0\omega_Q}\left(\frac{k_BT}{\hbar s}\right)^4$$
$$\approx 7.7\left(\frac{W_0}{\sigma_0 a^2}\right)^{\frac{1}{2}}\left(\frac{T}{\theta_D}\right)^4(Qa)^{-1/2}. \qquad (32)$$

The Stefan-Boltzmann like part of this result arises from the energy (momentum) loss associated with the Doppler shift of phonons reflected from the moving surface and the Q/ρ_0 part from the ripplon effective mass. Introducing the Wigner-Seitz radius, a, highlights the more evidently dimensionless second form where $W_0=\hbar^2/ma^2\approx 2.5K$ (m is the ^{4}He atom mass) is a measure of the zero point energy and the Debye temperature θ_D is defined in the usual way.

The result is compared in Fig. 1 with recent measurements of the damping factor for ripplons with wavelength in the range 3–20 μm.[9] The agreement with this *ab initio* calculation is remarkable.

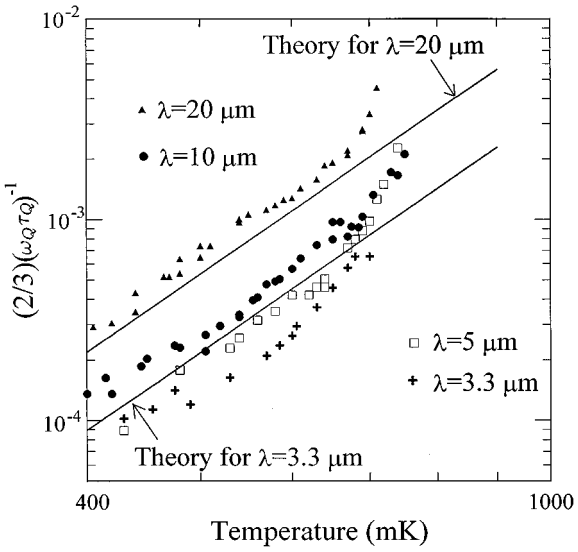

FIG. 1. Theoretical damping factor (straight lines) for wavelength 3.3 and 20 μm compared to experimental data (Ref. 9).

We conclude that the basic physics of the quantum hydrodynamical model of Atkins and Saam is not put into question by the new experiments. The problem stems rather from an inconsistency in the formalism. Once this is put right, the model, which contains no adjustable parameters, gives an extremely good quantitative account of the laboratory results on ripplon lifetimes.

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[1] K.R. Atkins, Can. J. Phys. **31**, 1165 (1953).
[2] K.A. Pickar and K.R. Atkins, Phys. Rev. **178**, 399 (1969).
[3] C.C. Grimes and G. Adams, Phys. Rev. Lett. **42**, 745 (1979).
[4] G. Deville, J. Low. Temp. Phys. **72**, 135 (1988).
[5] W.F. Saam, Phys. Rev. A **8**, 1918 (1973).
[6] W.F. Saam, Phys. Rev. B **12**, 163 (1975).
[7] H. Gould and V.K. Wong, Phys. Rev. B **18**, 2124 (1978).
[8] R.C. Jones, A.J.E. Williams, W.F. Vinen, and P.A. Ewbank, J. Low Temp. Phys. **92**, 239 (1993).
[9] P. Roche, G. Deville, K.O. Keshishev, N.J. Appleyard, and F.I.B. Williams, Phys. Rev. Lett. **75**, 3316 (1995).
[10] H. Lamb, *Hydrodynamics*, 6th ed. (Cambridge University Press, Cambridge, 1932), Chap. I.
[11] Applying Green's theorem with $\Delta\phi=0$, the kinetic energy can be represented as an integral over the free surface $\mathscr{S}$: $T=\int_{\mathscr{S}}\phi\mathbf{v}\cdot\hat{\mathbf{n}}ds$. Using the exact boundary condition, $T=\int dxdy\dot{\zeta}\phi|_{z=\zeta}$. Although this leads to a more elegant formalism, we have preferred to follow Saam.
[12] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959), Chap. VII.
[13] The other terms appear in P. Roche, Ph.D. thesis, University of Paris VI, France, 1995, and will be published elsewhere.