Topological phase transition in the $\nu=2/3$ quantum Hall effect

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The double-layer $\nu=2/3$ fractional quantum Hall system is studied using the edge-state formalism and finite-size diagonalization subject to periodic boundary conditions. Transitions between three different ground states are observed as the separation as well as the tunneling between the two layers is varied. Experimental consequences are discussed. [S0163-1829(96)02423-X]

I. INTRODUCTION

Advances in semiconductor fabrication have made it possible to produce multilayer two-dimensional electron systems that allow exploration of the effects of interlayer as well as intralayer correlations. The possibilities raised by the introduction of an extra degree of freedom into the standard picture of the fractional quantum Hall effect were first examined by Halperin[1] in the context of spin-unpolarized ground states, and later by Haldane and Rezayi[2] who proposed applying Halperin's wave functions to the case where the electrons possess a double valued index indicating the quantum state of the electron in the third direction parallel to the magnetic field. These states have been analyzed in their spin as well as layer index form.[3] Recent experiments with multilayer electron systems[4,5] seem to suggest the existence of incompressible states that belong to the universality class described by Halperin's wave functions. These systems allow the manipulation of two parameters that greatly influence the character of the fluids, the distance between the effective layers of electrons, given by d, and the tunneling between the two layers, denoted by the energy difference between the lowest two subbands as Δ_{sas}.

Motivated by the experimental observation of a transition between two distinct $\nu=2/3$ states in wide quantum wells by Suen *et al.*[5] and also the transition between spin-polarized and spin-singlet $\nu=2/3$ states in tilted field experiments by Eisenstein *et al.*,[6] we have analyzed the double-layer model with interlayer tunneling at total filling factor $\nu=2/3$ using the edge-state formalism and a finite-size numerical diagonalization study. In the double-layer model electrons are localized on a pair of parallel planes, between which they can tunnel. Tunneling lowers the energy of the electrons in a symmetric combination of states in each plane relative to the antisymmetric state. If symmetric states are identified as pseudospin ''up'' states, and antisymmetric states as pseudospin ''down'' states, tunneling acts as a Zeeman coupling in pseudospin space. Therefore our investigation of the double-layer model will allow a very general picture to emerge of two-component systems where the pseudospin can refer either to real spin or to subband index.

We have found a rich phase diagram with three distinct phases, separated by what we identify as first-order transitions: (1) The particle-hole conjugate of the $\nu=1/3$ Laughlin state, where all the electrons are in the symmetric state of the double layer, (2) the pseudospin singlet analog of the $\nu=2/3$ spin-singlet state, where the electrons are divided equally between symmetric and antisymmetric states, and (3) a state with independent $\nu=1/3$ Laughlin states in each layer. An interesting feature is that states (1) and (2), while distinct, have been identified as having the same topological order,[7] different from that of (3). We carried out our investigation in the periodic (or toroidal) geometry, which is well adapted to exhibit these differences. For example, the long distance effective Chern-Simons (CS) theory of the Hall effect predicts a threefold ground-state degeneracy for states (1) and (2) in the periodic geometry, but a ninefold degeneracy for state (3), consistent with our observation. An interesting consequence of the fact that state (3) has different topological order from states (1) and (2) is that the boundary between state (3) and either state (1) or (2) necessarily supports a residual neutral gapless Luttinger liquid.[8] The coexistence of two, and possibly three, phases of quantum Hall states at $\nu=2/3$ raises many interesting questions, both experimental and theoretical.

II. EFFECTIVE THEORIES AND EDGE STATES

Rather than work with the wave functions themselves, we can alternatively discuss the fractional quantum Hall states in terms of the long-wavelength effective field theories describing the incompressible fluids. These low-energy effective theories capture the long-distance correlations between the particles, determining such universal quantities as the conductance and the charge and statistics of the quasiparticles. The Lagrangian density for the effective theory can be written in the form

$$2\pi\mathscr{L}=\epsilon^{\lambda\mu\nu}[\tfrac{1}{2}\hbar(\mathbf{a}_\lambda,\mathbf{K}\partial_\mu\mathbf{a}_\nu)+eA_\lambda(\mathbf{q},\partial_\mu\mathbf{a}_\nu)], \qquad (1)$$

where $\mathbf{a}_\mu$ is an n-component vector of Abelian CS gauge fields, A_μ is the electromagnetic gauge field, $\mathbf{K}$ is a nonsingular integer coupling matrix, $\mathbf{q}$ is an integer vector; $(\mathbf{a},\mathbf{b})$ is

the inner product. We note here that we use the notation $\mathbf{q}$ for the charge vector rather than $\mathbf{t}$ as used by Wen and Zee.[7] By integrating out the Chern-Simons fields $\mathbf{a}_\nu$ we can determine the Hall conductance of the above theory to be

$$\sigma^H=\frac{e^2}{h}(\mathbf{q},\mathbf{K}^{-1}\mathbf{q}). \tag{2}$$

The effective theory allows vortex defects of the Chern-Simons fields, with core energies determined by short-distance terms in the Hamiltonian which have not been included. One can determine the charge and statistics of a particular composite of vortices by specifying an integer valued vector $\mathbf{l}$ such that the composite in question is made up of l_i vortices of type i. The charge of this composite is then given by

$$Q=e(\mathbf{q},\mathbf{K}^{-1}\mathbf{l}) \tag{3}$$

and the statistical phase acquired when two such composite vortices are interchanged is given by

$$\frac{\theta}{\pi}=(\mathbf{l},\mathbf{K}^{-1}\mathbf{l}). \tag{4}$$

Furthermore, it has been shown[7] that the degeneracy of a state described by the matrix K when it is defined on a two-dimensional closed surface of genus g is given by

$$\mathbf{D}=|\mathrm{Det}\mathbf{K}|^g. \tag{5}$$

We may therefore classify an Abelian quantum Hall state by specifying an integer valued pair $\{\mathbf{K},\mathbf{q}\}$,[7] thereby determining the long-distance properties of the fluid. It is important to note that distinct quantum Hall states are represented by equivalence classes of $\{\mathbf{K},\mathbf{q}\}$ pairs as the above properties are invariant under $\mathrm{SL}(\kappa,Z)$ basis changes

$$\mathbf{K}\rightarrow\mathbf{W}\mathbf{K}\mathbf{W}^T,$$

$$\mathbf{q}\rightarrow\mathbf{W}\mathbf{q}, \tag{6}$$

where $\mathbf{W}$ is an integer matrix with $|\mathrm{Det}\mathbf{W}|=1$. Two fractional quantum Hall states described by $\{\mathbf{K},\mathbf{q}\}$ pairs that are related by an $\mathrm{SL}(\kappa,Z)$ transform belong to the same universality class and are considered topologically equivalent.

The above discussion has been limited to Abelian fractional quantum Hall fluids on the plane. If we instead define our effective theory on a sphere, there is an extra term in the effective theory that describes the coupling of the liquid to the curvature of the sphere. This manifests itself in a new topological quantum number, the flux shift, which is not determined by the long-distance effective theory described above. The shift $\mathscr{S}$, defined by the relation $N_\phi=\nu^{-1}N_e-\mathscr{S}$, is a manifestation of the coupling of the orbital angular momentum properties of the state to the curvature of the space. The effective theories employed in the composite fermion approach on the plane do not distinguish between Landau levels and spin states. As the orbital angular momentum carried by the cyclotron motion of electrons in the second Landau level is different from electrons in the lowest level, the shift $\mathscr{S}$ provides a way to distinguish between states that possess the same long-distance properties but which have different spin symmetries. By utilizing the topological character of the coupling of orbital angular momentum to curvature we can classify the spin structure of a state whose analysis would otherwise lie outside the effective theory approach.

We can write the Halperin-Laughlin wave function appropriate for multicomponent systems in the planar geometry in the form

$$\Psi_{\mathbf{K}}(\{z_i\})=\prod_{i<j}(z_i-z_j)^{K^{\mathrm{el}}(\sigma_i,\sigma_j)}\prod_i\exp\left(-\frac{1}{4l^2}\left|z_i\right|^2\right), \tag{7}$$

where σ_i is the pseudospin variable and $K_{ij}^{\mathrm{el}}=K^{\mathrm{el}}(\sigma_i,\sigma_j)$ is the symmetric matrix encoding the electron correlations. If we specialize to the double-layer system, we can identify the (m_1,m_2,n) state with the effective theory

$$\mathbf{K}=\mathbf{K}^{\mathrm{el}}=\begin{pmatrix}m_1 & n\\ n & m_2\end{pmatrix},\quad \mathbf{q}=\begin{pmatrix}1\\1\end{pmatrix}. \tag{8}$$

We shall be considering two effective theories for the system at $\nu=2/3$. The first has been identified as representing both the pseudospin singlet state and the pseudospin polarized, particle-hole conjugate of a Laughlin $\nu=1/3$ state. The effective theory is given by

$$\mathbf{K}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\quad \mathbf{q}=\begin{pmatrix}1\\1\end{pmatrix}. \tag{9}$$

The second effective theory that we will consider represents two independent $\nu=1/3$ Laughlin states

$$\mathbf{K}=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix},\quad \mathbf{q}=\begin{pmatrix}1\\1\end{pmatrix}. \tag{10}$$

Both effective theories potentially represent states at $\nu=2/3$. There is, however, a crucial difference between the two theories: they possess different ground-state degeneracies on a nontrivial closed space, and therefore possess different topological order. In the remainder of this paper we will investigate the consequences of the fact that the these two theories have different topological order.

Let us consider the edge-state theory of the Abelian quantum Hall states. The edge between Abelian Hall fluids is a one-dimensional ''Luttinger liquid,'' which can also be characterized by a $\{\mathbf{K},\mathbf{q}\}$ pair, which is the same pair of the bulk theory if the edge is between a Hall state and a nonconducting state. Generally, we may define a set of fields $\phi_i(x)$ living on the one-dimensional compact edge of an incompressible Hall sample such that they obey the equal time commutation relations

$$[\phi_i(x),\phi_j(x')]=i\pi[K_{ij}\mathrm{sgn}(x-x')+L_{ij}], \tag{11}$$

where $L_{ij}=\mathrm{sgn}(i-j)(K_{ij}+q_iq_j)$ is a Klein factor. The action for the *translationally invariant* edge fields is given by

$$S=\int dt(S_0-H_0), \tag{12}$$

where

$$S_0=-\frac{\hbar}{4\pi}\sum_{ij}\ K_{ij}^{-1}\oint dx\partial_x\phi_i\partial_t\phi_j \tag{13}$$

and

$$H_0=\frac{\hbar}{4\pi}\sum_{ij}\ V_{ij}\oint dx\partial_x\phi_i\partial_x\phi_j\,, \tag{14}$$

where V_{ij} encodes the nonuniversal interactions that determine, among other things, the velocities of the various edge modes. For stability, we require the matrix **V** to be positive definite. Let us define

$$\phi_{\mathbf{m}}=\sum_i\ m_i\phi_i \tag{15}$$

and

$$q(\mathbf{m})=(\mathbf{m},\mathbf{q}), \tag{16}$$

where **m** is an integer valued vector. We can define a set of composite local fields

$$\Psi_{\mathbf{m}}=e^{-i\phi_{\mathbf{m}}}, \tag{17}$$

which obey, for $x\neq x'$,

$$\Psi_{\mathbf{m}}(x)\Psi_{\mathbf{m}'}(x')=\eta_{\mathbf{m},\mathbf{m}'}\Psi_{\mathbf{m}'}(x')\Psi_{\mathbf{m}}(x), \tag{18}$$

where

$$\eta_{\mathbf{m},\mathbf{m}'}=(-1)^{q(\mathbf{m})q(\mathbf{m}')}. \tag{19}$$

The winding number operator, defined as

$$N_i=\frac{1}{2\pi}\oint dx\partial_x\phi_i(x) \tag{20}$$

is constrained to be integral by imposing periodic boundary conditions on the fields

$$\Psi_{\mathbf{m}}(x+L)=\Psi_{\mathbf{m}}(x). \tag{21}$$

The fields are characterized by the integer quadratic form

$$K(\mathbf{m})=(\mathbf{m},\mathbf{K}\mathbf{m})=\sum_{ij}\ m_iK_{ij}m_j\,, \tag{22}$$

which has the constraint

$$(-1)^{K(\mathbf{m})}=(-1)^{q(\mathbf{m})}. \tag{23}$$

If $K(\mathbf{m})$ is odd, the field corresponding to **m** is is fermionic, and if $K(\mathbf{m})$ is even, it is bosonic. We can define the charge density of the edge as

$$\rho(x)=\frac{e}{2\pi}\sum_{ij}\ q_iK_{ij}^{-1}\partial_x\phi_j(x), \tag{24}$$

which has the commutation relation

$$[\rho(x),\rho(x')]=i\hbar\sigma^H\delta'(x-x'), \tag{25}$$

where

$$\sigma^H=\frac{e^2}{h}\sum_{ij}\ q_iK_{ij}^{-1}q_j\,. \tag{26}$$

The physical interpretation of this quantity is that it represents the change in the quantized Hall conductivity in going from one side of the edge to the other. We can also define the total charge operator

$$Q=e\sum_{ij}\ K_{ij}^{-1}q_iN_j\,, \tag{27}$$

which obeys the commutation relation

$$[Q,\Psi_{\mathbf{m}}(x)]=eq(\mathbf{m})\Psi_{\mathbf{m}}(x). \tag{28}$$

The edge-state Hamiltonian H_0 describes n linearly independent oscillator modes propagating with velocities v_λ, which can be determined from the generalized real symmetric eigenvalue equation

$$\mathbf{V}\mathbf{u}_\lambda=v_\lambda\mathbf{K}^{-1}\mathbf{u}_\lambda\,, \tag{29}$$

where the velocities are real and the eigenmodes independent as **V** is positive definite. We can now state the condition on the matrix **K**, which ensures a set of SU(2) generating operators within the theory: If we can identify an integral vector **m** such that

$$K(\mathbf{m})=2s_{\mathbf{m}},\quad q(\mathbf{m})=0, \tag{30}$$

where $s_{\mathbf{m}}=\pm1$ then we may identify an SU(2) algebra associated with the edge field $\Psi_{\mathbf{m}}$. We can define a triplet of non-Abelian densities

$$\sigma^x(x)=\frac{1}{2}[\Psi_{\mathbf{m}}(x)+\Psi_{-\mathbf{m}}(x)],$$

$$\sigma^y(x)=(s_{\mathbf{m}}/2i)[\Psi_{\mathbf{m}}(x)-\Psi_{-\mathbf{m}}(x)],$$

$$\sigma^z(x)=\frac{1}{2}\rho_{\mathbf{m}}(x)=\frac{1}{4\pi}\partial_x\phi_{\mathbf{m}}(x), \tag{31}$$

which obey a level-1 SU(2) Kac-Moody algebra

$$[\sigma^a(x),\sigma^b(x')]=\frac{is_{\mathbf{m}}}{4\pi}\delta'(x-x')+i\epsilon^{abc}\sigma^c(x)\delta(x-x'). \tag{32}$$

We may identify the SU(2) algebra generators

$$S^a=\oint dx\sigma^a_{\mathbf{m}}(x), \tag{33}$$

which obey the commutation relations

$$[S^a,S^b]=i\epsilon^{abc}S^c. \tag{34}$$

The auxiliary constraint $q(\mathbf{m})=0$ is to ensure charge neutrality. This hidden SU(2) symmetry has been noted previously.[9–11] We note that larger symmetries [specifically SU(N) for an N-component system] may be identified using a similar analysis.

Let us focus specifically on the double-component system, where the vector components denote different pseudospin components. We shall work in the symmetric basis where $q_i=1$ for $i=1,2$. Further, the diagonal elements K_{ii}

must be odd for Fermi statistics. We therefore find that the only effective theory that may generate an SU(2) edge symmetry is

$$\mathbf{K}=\begin{pmatrix} m\pm 1 & m \\ m & m\pm 1 \end{pmatrix}, \tag{35}$$

with m even. Diagonalizing the above edge-state system (35) we obtain the Hamiltonian

$$H=\frac{\hbar}{4\pi}\oint dx\{v_n[\partial_x\phi_n(x)]^2+v_c[\partial_x\phi_c(x)]^2 + 2v_{\rm int}\partial_x\phi_c(x)\partial_x\phi_n(x)\}, \tag{36}$$

where $\phi_n=\phi_1-\phi_2$ is a neutral mode and and $\phi_c=\phi_1+\phi_2$ is a charged mode. The two modes move with velocities that depend on the external potential, with intermode interactions $v_{\rm int}$ mixing these two modes. It is the neutral edge mode that generates the SU(2) algebra where the field

$$\Psi_{\mathbf{m}}=e^{-i(\phi_1-\phi_2)} \tag{37}$$

corresponds to the neutral physical operation of tunneling one electron from one pseudospin to the other. The effective theory defined by (35) possesses a hidden SU(2) symmetry of the bulk quantum state, in this case invariance under pseudospin rotations. We may therefore state that a bulk quantum Hall state may possess an SU(2) symmetry if and only if the effective theory corresponding to this quantum Hall state has a nontrivial solution $\mathbf{m}$ to the equation $K(\mathbf{m})=\pm 2$. As the bulk ground state is nondegenerate, apart from topological degeneracies that are not associated with the SU(2) symmetry, our states constructed in this way are SU(2) singlets. Whether or not a particular state is realized as the ground state depends, of course, on the underlying bulk Hamiltonian. It is important to realize that this symmetry is a consequence of the electron correlations rather than the underlying Hamiltonian. This algebra is realized in the bulk state independently of whether or not the Hamiltonian is strictly invariant under the symmetry.

We must note that in our construction we have assumed that the matrix $\mathbf{K}$ was non-singular. The singular case of $K_{ij}=m$ with m odd corresponds to a $\nu=1/m$ state, which is also invariant under pseudospin rotations, but which is fully polarized with $S=N_e/2$. This system has been studied previously and is found to possess many interesting features.[12]

The edge-state formalism is also capable of addressing the question of what happens at the edge between two Abelian Hall states, which is physically realized when there is phase coexistence in a first-order transition between the two states. We can form the edge state theory corresponding to the $\{\mathbf{K},\mathbf{q}\}$ effective by forming the direct sum $\mathbf{K}=\mathbf{K}_1\oplus-\mathbf{K}_2$ and $\mathbf{q}=\mathbf{q}_1\oplus\mathbf{q}_2$. We will restrict our discussion to a *clean* edge between two double-component Hall states at the same filling fraction, at least one of which possesses an SU(2) symmetry of the type discussed above. To address the stability of the edge of this system, we must consider the general nonwinding number conserving tunneling perturbation

$$T=\int dx[t(x)\Psi_{\mathbf{m}}(x)+\text{c.c.}]. \tag{38}$$

The operator must be bosonic and charge conserving with $q(\mathbf{m})=0$ and $K(\mathbf{m})$ even. The scaling dimension of this operator can be determined from the two-point function to be $2-\Delta(\mathbf{m})$, where $\Delta(\mathbf{m})$ satisfies the inequality

$$\Delta(\mathbf{m})\geqslant\frac{1}{2}|K(\mathbf{m})|. \tag{39}$$

We can construct a representation of the fields such that both K_{ij} and the nonuniversal V_{ij} are diagonal

$$(UKU^T)_{ij}=\sigma_i\delta_{ij}, \tag{40}$$

$$(UVU^T)_{ij}=\hat{v}_i\delta_{ij}, \tag{41}$$

where $\sigma_i=\pm 1$. Since stability requires $\hat{v}_i>0$, the direction of propagation of each mode is determined by σ_i. Using the transformation matrix U we can determine $\Delta(\mathbf{m})$

$$\sum_i \eta_i(\mathbf{m})=2\Delta(\mathbf{m}), \tag{42}$$

where

$$\eta_i(\mathbf{m})=\sum_{jk} U^{-1}_{ji}U^{-1}_{ki}m_jm_k. \tag{43}$$

The factors $\eta(\mathbf{m})$ obey the sum rule

$$\sum_i \eta_i(\mathbf{m})\sigma_i=K(\mathbf{m}). \tag{44}$$

One can see that if the system is maximally chiral, then the sum rule explicitly gives us the scaling dimension of the operator. If it is not maximally chiral, it only gives us a lower bound. Therefore, from a scaling perspective a tunneling operator with $K(\mathbf{m})=0$ is potentially relevant if the scaling dimension $\Delta(\mathbf{m})<2$. In general, the tunneling perturbation will be prevented from being relevant by the complex tunneling parameter $t(x)$ where

$$t(x)=|t(x)|\exp[i\alpha(x)] \tag{45}$$

in the clean case. In order for the perturbation to be relevant, the phase factor must satisfy

$$\partial_x\alpha(x)=\langle 0|\partial_x\phi_{\mathbf{m}}|0\rangle. \tag{46}$$

We will assume that such a ''phase locking'' is generically possible for the edges in question. If the $K(\mathbf{m})=0$ perturbation is relevant, the modes involved in $\Psi_{\mathbf{m}}$ become massive and are removed from the low-energy theory. The only other perturbations that can be potentially relevant have $|K(\mathbf{m})|=2$ but are not mass generating [they do form ''hidden'' SU(2) symmetries as discussed previously]. Therefore, the condition for a potentially mass generating perturbation is that we must identify a nontrivial integer valued vector $\mathbf{m}$ such that $K(\mathbf{m})=q(\mathbf{m})=0$. We shall take the point of view in this paper that if a mass-generating perturbation is potentially relevant, the system will relax so that the instability generally occurs. This is a conjecture based upon the experimental observation that the only stable Abelian Hall

states are those that do not permit such mass-generating instabilities.[8] If a mass-generating instability can occur, it appears to do so.

We can apply this analysis to the edge theory at hand. First, let us consider the edge between two states with the same topological structure. The edge-state theory will be based on the $\{\mathbf{K},\mathbf{q}\}$ pair

$$\mathbf{K}=\begin{pmatrix} m\pm 1 & m & 0 & 0 \\ m & m\pm 1 & 0 & 0 \\ 0 & 0 & -(m\pm 1) & -m \\ 0 & 0 & -m & -(m\pm 1) \end{pmatrix}, \quad (47)$$

with

$$\mathbf{q}=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (48)$$

We note that the above edge has $\sigma_H=0$ as is appropriate for an edge between two Hall states with the same filling fraction. In this case, we can identify the two operators of interest: the operator $\Psi_{\mathbf{m}}$ with $\mathbf{m}=(1,0,-1,0)$ and the operator with $\mathbf{m}=(0,1,0,-1)$. Both these operators are $K(\mathbf{m})=0$ operators, which are potentially relevant and mass generating. Interactions will tend to reduce the scaling dimension of the two operators, but if they remain relevant we expect that they will cause a mass gap to form [we note that generally diagonal elements of the interaction matrix reduce the scaling dimension of the operators, while off-diagonal elements tend to increase it, up to a maximum of $2-1/2|K(\mathbf{m})|$]. It is straightforward to see that any edge between states with the same K matrix will have two such tunneling operators with $K(\mathbf{m})=q(\mathbf{m})=0$, which potentially cause the modes to pair up and form a gap, leaving no residual gapless modes in the low-energy theory. Let us consider a single quantum Hall state where we arbitrarily choose there to be an edge in the bulk of the state. The edge state will have the same K matrix structure as the edge between any two states that have the same topological structure. Such a fictitious edge in the bulk cannot support gapless charged excitations, as is consistent with our above analysis. Generically, we then expect that at the edge between two states with the same topological structure there should be no residual gapless modes.

We can also consider the edge between two states with different topological structure at the same filling fraction, as is appropriate at the edge between the pseudospin singlet state or the particle-hole conjugate state and the state with independent $\nu=1/3$ Laughlin states in each layer. Generically there is only one $K(\mathbf{m})=0$ tunneling operator at the edge between double-component states of different topological order. It is not possible for two sets of edges to pair up and form a gap, due to the mismatch in topological structure. There will always be a pair of residual *neutral* gapless edge modes left over in the low-energy theory, with a gap for making charged excitations. As an example we shall consider the edge between two states at $\nu=2/3$ possessing different topological order. This will be based on the $\{\mathbf{K},\mathbf{q}\}$ pair

$$\mathbf{K}=\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad \mathbf{q}=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (49)$$

The operator $\Psi_{\mathbf{m}}$ with $\mathbf{m}=\{1,1,-1,-1\}$ has $K(\mathbf{m})=0$ and $q(\mathbf{m})=0$, thereby allowing two charged modes to pair off and form a gap, leaving two neutral gapless modes in the low-energy effective theory. The mismatch between two distinct quantum Hall states at the same filling fraction with different topological order implies the existence of residual neutral gapless excitations at the boundary.

In considering the edge between two distinct quantum Hall states at the same filling fraction, we expect two possible scenarios. If the two states possess the same topological structure, such as the pseudospin singlet state and the particle-hole conjugate state, their respective edges will pair up and form gaps, leaving behind no residual gapless states. If the two states have different topological structure, only one set of modes will split off and form a gap. Two neutral, gapless modes will remain in the low-energy theory, a residual side effect of the mismatch in topological order.

III. PHASE DIAGRAM FROM FINITE-SIZE STUDIES

The finite-size studies that we report were carried out using periodic boundary conditions (see the Appendix) and δ function wave functions to represent the two layers in the double-layer model, with the electrons confined to the lowest Landau level as is standard practice. Our studies were performed at filling fraction $\nu=N_e/N_\phi=2/3$ with six electrons and nine flux quanta. In the following, length will be measured in units of l, the magnetic length $\sqrt{\hbar/eB}$, and energy in units of $e^2/4\pi\epsilon l$. The system of electrons interacting via the Coulomb interaction is exactly diagonalized numerically, with the spectrum as a function of $\mathbf{k}$ (see Appendix) providing the fundamental information on the system. We have diagonalized the system with the Hamiltonian given by

$$H=-\frac{\Delta_{sas}}{2}\sum_i (c^\dagger_{i,1}c_{i,2}+\text{H.c.}) + \frac{1}{2}\sum_{i,j,k,l}\langle i\alpha_i, j\alpha_j|V|k\alpha_k, l\alpha_l\rangle c^\dagger_{i\alpha_i}c^\dagger_{j\alpha_j}c_{k\alpha_k}c_{l\alpha_l}, \quad (50)$$

where

$$V(r)=\frac{1}{4\pi\epsilon}\frac{e^2}{[r^2+d^2(1-\delta_{\alpha_i,\alpha_j})]^{1/2}} \quad (51)$$

and α_i is the layer index, which denotes in which of the two layers the electron resides.

All calculations were performed using square boundary conditions where $\theta=\pi/2$ and $|\mathbf{L}_1|=|\mathbf{L}_2|$. It is known[13] that incompressible states are remarkably insensitive to the particular boundary conditions chosen, as long as the shortest length scale of the geometry is larger than the average interparticle spacing. While the exact details of the excitation spectrum in the system under investigation will depend on

our choice of conditions, the qualitative conclusions concerning the incompressible ground states should not.

In the following we shall use the term pseudospin to refer to the subband layer index, with the the up spin corresponding to the symmetric combination of layer states and the down spin the antisymmetric. Tunneling acts as a Zeeman term in the Hamiltonian, which tends to align the electron pseudospin in the up state, or the symmetric combination of layer states. We will vary both the tunneling, denoted in the Hamiltonian by Δ_{sas}, and the distance between the double-layer planes, denoted by d/l, investigating both the ground state and the dependence of the excited states on these parameters.

A. Spin-singlet state

It was realized some time ago[1] that when the electron correlations are of the same order as the Zeeman energy associated with the spin states, it is important to consider the spin degrees of freedom in constructing the ground state. Previous numerical and experimental studies[6,14] on the $\nu=2/3$ system reveal that the ground state of the non-spin-polarized system, in the limit of vanishing Zeeman energy, is plausibly a spin singlet. If we consider a phase diagram where we vary both the interlayer separation d/l and the interlayer tunneling Δ_{sas}, along the line $d=0$ the Hamiltonian is invariant under pseudospin rotations, allowing a direct mapping between pseudospin and electron spin in the presence of the Zeeman term. We know from our previous edge analysis (33) that the K matrix of the effective theory of a spin-singlet ground state must be of the form

$$\mathbf{K}=\begin{pmatrix} m\pm 1 & m \\ m & m\pm 1 \end{pmatrix}, \quad \mathbf{q}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{52}$$

where m is even. Jain and co-workers[15] have has proposed a wave function to describe the spin-singlet state at $\nu=2/3$ based upon the effective theory where $\{\mathbf{K},\mathbf{q}\}$ is given by

$$\mathbf{K}=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{q}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{53}$$

We cannot strictly interpret this K matrix in terms of the Halperin-Laughlin wave function, as thermodynamic stability requires the K matrix for such a wave function to be positive definite, so we must interpret the wave function in the planar geometry to be

$$\Psi(z_i,\sigma_i)=\prod_{i<j}(\partial_{z_i}-\partial_{z_j})^{\delta_{\sigma_i,\sigma_j}}\prod_{i<j}(z_i-z_j)^2 \times\prod_{i<j}\exp\left(i\frac{\pi}{2}\mathrm{sgn}(\sigma_i-\sigma_j)\right) \times\prod_{i}\exp\left(-\frac{1}{4l^2}\left|z_i\right|^2\right), \tag{54}$$

where we have denoted $\partial_{z_i}=\partial/\partial z_i$. This wave function is a spin singlet at the correct filling fraction with the correct topological properties.

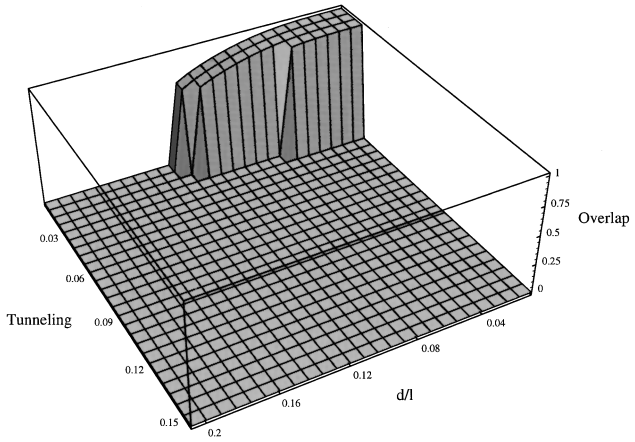

FIG. 1. Overlap of the ground state with the spin-singlet state as a function of d/l and tunneling. Tunneling is denoted in the Hamiltonian as Δ_{sas} and is measured in units of $e^2/4\pi\epsilon l$.

In Fig. 1 we calculate the overlap of the exact ground state with Coulomb interactions with Jain's proposed spin-singlet wave function as we vary the Hamiltonian parameters Δ_{sas} and d/l (note that Δ_{sas} and tunneling are used interchangably). Several points should be noted. First, the system is very sensitive to the effects of tunneling, or equivalently, a magnetic field in pseudospin space. The overlap with the spin singlet state rapidly falls off with the introduction of even a slight amount of tunneling. This will tend to make the experimental observation of the spin-singlet state in multilayer samples extremely difficult. Second, the system is reasonably robust against a separation between the two layers, falling to zero at $d\approx 1.1l$. Thus we find significant overlap between the exact ground state and the proposed spin singlet even in regions where the Hamiltonian no longer commutes with the pseudospin algebra. The spin-singlet character of the state is a manifestation of the electron correlations rather than the underlying Hamiltonian.

While we expect that the numerical data give us qualitative data on the transitions discussed, it should be noted that finite-size effects will influence the exact positioning of the transitions in the d/l-Δ_{sas} plane in relation to the thermodynamic limit, and should therefore be taken with caution.

B. Particle-hole conjugate of $\nu=1/3$ Laughlin state

The particle-hole conjugate of the standard Laughlin $\nu=1/3$ gives us an incompressible liquid at filling fraction $\nu=2/3$, which is denoted in the effective theory by the same K matrix as that of the spin-singlet state (53). The matrix nature of the effective theory does not reflect the correlations between electrons of opposite pseudospin, as the particle-hole conjugate state is pseudospin polarized, but rather reflects its composite nature. While to our knowledge no simple wave function has been proposed for the exact particle-hole conjugate state, a trial hierarchy wave function, which very effectively captures the electron correlations, has been developed.[16] We may write the hierarchy $\nu=2/3$ state as

$$\Psi(\{z_i,\sigma_i\})=\prod_{i'}\int d\omega_{i'}d\omega_{i'}^{*}\prod_{i'<j'}(\omega_{i'}^{*}-\omega_{j'}^{*})^2|\omega_{i'}-\omega_{j'}|^2$$
$$\times\prod_{i,j'}(z_i-w_j')\prod_{i<j}(z_i-z_j)$$
$$\times\prod_{i'}\exp\left(-\frac{1}{2l^2}\left|\omega_{i'}\right|^2\right)$$
$$\times\prod_{i}\exp\left(-\frac{1}{4l^2}\left|z_i\right|^2\right), \tag{55}$$

where $i'=1,\ldots,N_h$ and $i=1,\ldots,N_e$ where the number of holes $N_h=N_e/2$ and N_e is the number of electrons. This hierarchical state is represented by the effective theory given by

$$\mathbf{K}=\begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix},\quad \mathbf{q}=\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{56}$$

which is equivalent to (53) up to a similarity transformation. This effective theory also possesses an SU(2) invariance. In this case, the two pseudospin variables refer to the lowest two Landau levels in the composite fermion construction. Within the composite fermion approach, to obtain the single-layer particle-hole conjugate state one starts with a spin-polarized $\nu=2$ state, a gapped system, and adiabatically attaches two flux quanta to each electron opposite to the direction of the magnetic field that generated the $\nu=2$ state. As the addition of two flux quanta does not affect the statistics, our composites are still fermions. In a mean-field sense, we start with a pseudospin polarized state at $\nu=2$ and decrease the B field by two flux tubes per electron, arriving at a pseudospin polarized state at $\nu=2/3$, which we identify as the particle-hole conjugate state. We still have residual gauge fluctuations associated with the added flux, but they should not qualitatively change the physics as we started with a gapped system. Therefore the $\nu=2/3$ polarized state can be identified with a polarized $\nu=2$ integer quantum Hall state. For the spin-singlet state, out starting point is a spin-unpolarized $\nu=2$ state with the first Landau level being filled for both the up and down spins. We then perform the same flux addition process as we did for the polarized state to arrive at a spin-singlet $\nu=2/3$ fermion state. Within this approach, which state will be realized depends on the ratio of the effective cyclotron energy to the effective Zeeman energy of the composite fermions.

In Fig. 2 we calculate the overlap of the particle-hole conjugate state with the exact ground state as a function of Δ_{sas} and d/l (Δ_{sas} and tunneling being used interchangably as in Fig. [1]). The effect of the tunneling in the spin analogy can be seen to be a turning on of a Zeeman energy term in the $\hat{z}$ direction. As this energy is increased, eventually all the spins will align themselves along the $\hat{z}$ direction, resulting in a spin-polarized Laughlin state with all the electrons occupying a symmetric combination of the layer indices. As the Zeeman term is turned on, the system abruptly reaches a transition point where the overlap of the ground state with the spin-singlet state falls to zero (Fig. 1), while the overlap with the Laughlin 2/3 particle-hole conjugate state jumps to close to unity.

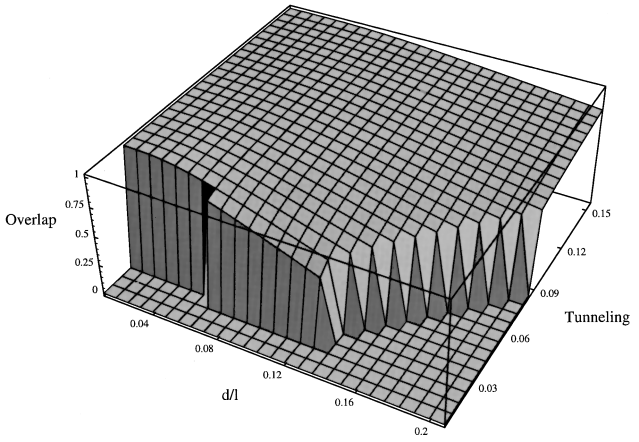

FIG. 2. Overlap of the ground state with the $\nu=2/3$ Laughlin particle hole conjugate state as a function of d/l and tunneling. Tunneling is denoted in the Hamiltonian as Δ_{sas} and is measured in units of $e^2/4\pi el$. Note the change in perspective of this figure from the previous figure.

As the spin-singlet state and the particle-hole conjugate state are described by the same pair $\{\mathbf{K},\mathbf{q}\}$, they are described by the same effective theory on the plane and possess the same edge-state structure. While the effective theory approach captures many of the long-distance properties of the fluid, it does not classify the spin of the state, which is determined by the energy associated with the spin degree of freedom. As we turn on the tunneling Δ_{sas}, we find a transition from a region where it is energetically favorable to put the composite fermions in the first two spin states in the lowest Landau level to a region where it is favorable to put them in the first two Landau levels. In both cases we have an SU(2) symmetry, in the one case between pseudospin states and in the other between Landau levels. It is important to notice that within the composite fermion approach the effective cyclotron energy is heavily renormalized, as small variation in the effective field essentially makes it energetically favorable to place the electrons in two pseudospin-polarized Landau levels, costing cyclotron energy but saving on Zeeman energy. The composite fermion process of attaching flux has the effect of enhancing the ratio of the effective energy associated with a pseudospin flip to the effective cyclotron energy.

C. (3,3,0) double-layer state

At nonzero d/l, as the Hamiltonian no longer commutes with psuedospin rotations the ground state need not be an eigenstate of pseudospin nor need it possess an SU(2) symmetry as the particle-hole conjugate state does. A proposed effective theory for the ground state at large layer separation d/l is given by

$$\mathbf{K}=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix},\quad \mathbf{q}=\begin{pmatrix}1\\ 1\end{pmatrix}, \tag{57}$$

which has a direct interpretation as a (3,3,0) Halperin-Laughlin multicomponent wave function representing independent $\nu=1/3$ Laughlin states in each layer. In Fig. 3 we calculate the overlap of the exact ground state with the

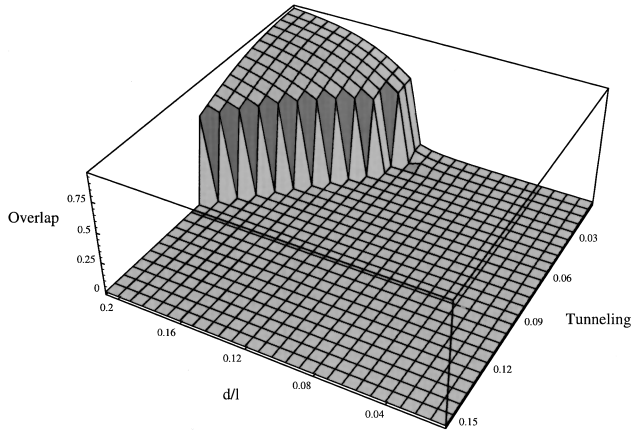

FIG. 3. Overlap of the ground state with the (3,3,0) state as a function of d/l and tunneling. Tunneling is denoted in the Hamiltonian as Δ_{sas} and is measured in units of $e^2/4\pi\epsilon l$. Note the change in perspective of this figure from the previous two figures.

(3,3,0) state as a function of Δ_{sas} and d/l (note again that tunneling and Δ_{sas} are being used interchangably). The transition between the (3,3,0) state and the particle-hole conjugate state (Fig. 2) represents a competition between minimizing the Coulomb energy between the electrons and the tunneling energy cost of localizing the electrons on independent planes. For any given tunneling Δ_{sas} it is possible to find a large enough d/l such that it is energetically favorable to form the independent $\nu=1/3$ states in each layer, costing in tunneling energy but gaining even more in minimizing the electrostatic ground-state energy. Equivalently, given any interlayer distance we can find a tunneling energy that will outweigh the favorable interaction energy associated with having the electrons as far apart as possible.

D. Phase diagram

We can identify three separate phases for the $\nu=2/3$ system as a function of tunneling Δ_{sas} and the layer separation d/l. We calculate the phase diagram for the $\nu=2/3$ system in Fig. 4 from the overlap data by matching a system with a particular phase if the overlap of the ground state with the characteristic state is greater than some cutoff, chosen to be 0.75 in Fig. 4. One important feature to note is the existence of a triple point in our numerical studies at $d\approx 1.1l$ and $\Delta_{sas}\approx 0.01$ where all three phases will be in coexistence. It is difficult to extract the experimental parameters where such a triple point might occur from our finite-size numerical data, as such quantitative information will be sensitive to finite-size effects such as geometry and particle number. We conjecture that the triple point where all three discussed phases will be in coexistence will persist in the thermodynamic limit, in the neighborhood of the physical parameters suggested by our studies. The observation of such a triple point remains an interesting experimental possibility.

IV. FINITE-SIZE STUDY OF TRANSITIONS

Whenever a system exhibits different macroscopic phases as a function of system parameters, it is natural to ask questions about the transitions between such phases. While finite-size studies are unable to address questions about the thermodynamic features of such transitions, they are able to shed light on qualitative changes in structure that a bulk liquid may undergo in going from one phase to another. In order to investigate the transitions involved in our double-layer $\nu=2/3$ system, we have exactly diagonalized our finite system using periodic boundary conditions as a function of the system parameters Δ_{sas} and d/l. We can follow the development of individual energy levels of the system as these parameters are varied by identifying the quantum numbers of the state in question such as parity under reflection and the translational quantum number $\mathbf{k}$ and noting that under adiabatic perturbation the energy levels should be smooth and connected in our finite system. In the following figures we plot the energy levels of the system as a function of either Δ_{sas} and d/l, at each step subtracting off the average energy of the system in order eliminate background energies of the system. The energy levels that are relevant to each transition are connected for emphasis during their development.

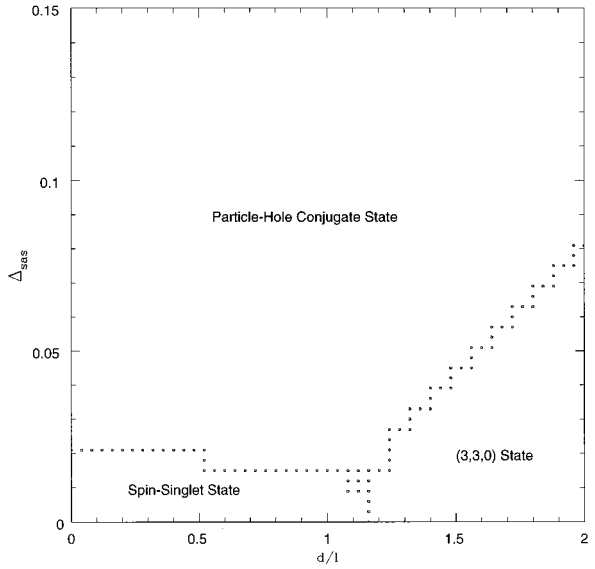

FIG. 4. Phase diagram of $\nu=2/3$ system as a function of Δ_{sas} and d/l. Δ_{sas} is measured in units of $e^2/4\pi\epsilon l$.

When using periodic boundary conditions it is important to note that each eigenstate has a generic degeneracy associated with center-of-mass translations given by q if the filling fraction is $\nu=p/q$. The formalism used to classify states (see the Appendix) extracts this degeneracy explicitly. Since we are always working at fixed filling fraction, this degeneracy will be unimportant.

A. Variation of d/l at $\Delta_{sas}=0$

In Fig. 5, the variation of the energy levels as a function of d/l for $\Delta_{sas}=0$ is shown, with the average energy at each d/l subtracted off. The transition being witnessed is from the spin-singlet phase to the (3,3,0) phase as d/l is increased. At small values of d/l the ground state is well represented by Jain's spin-singlet state, with a well-defined energy gap to all excitations indicative of an incompressible phase.

At large values of d/l the ground state is given by a threefold degenerate multiplet of states, each being a $\mathbf{k}=0$ eigen-

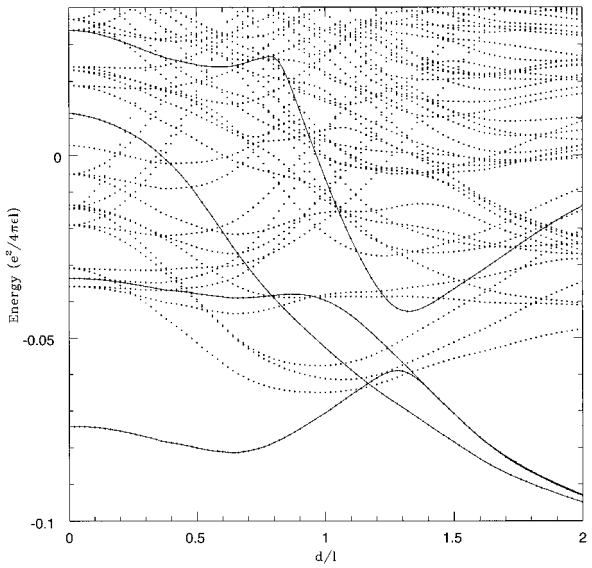

FIG. 5. Variation in energy levels as d/l is varied at $\Delta_{sas}=0$.

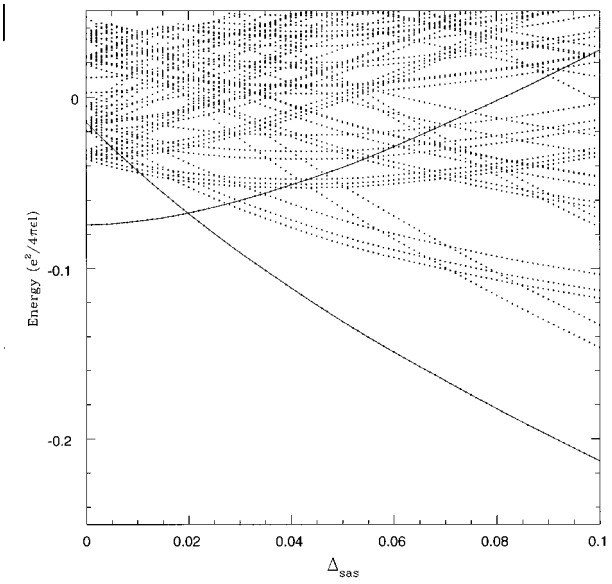

FIG. 6. Variation in energy levels as Δ_{sas} is varied at $d/l=0$. Δ_{sas} is measured in units of $e^2/4\pi\epsilon l$.

state that in the thermodynamic limit becomes rotationally invariant. There is again a well-defined energy gap between the ground-state multiplet and all excited states. There is a slight splitting of the degeneracy due to finite-size effects but this feature will disappear as the size of the system is increased.

The key feature to note is the change in ground-state degeneracy in going from the spin-singlet state to the (3,3,0) state. Such a change is to be expected from our effective theory considerations. As noted previously the degeneracy of a state given by the effective theory with the pair $\{\mathbf{K},\mathbf{q}\}$ on a torus is given by $|\mathrm{Det}K|$. The spin-singlet state must then have an overall degeneracy of three while the (3,3,0) state has a degeneracy of nine. As mentioned above our formalism extracts a threefold center-of-mass degeneracy generic to states at $\nu=2/3$, leaving us with a residual threefold degeneracy for the (3,3,0) state and a nondegenerate spin-singlet state, consistent with our numerical data.

Another point to be noted is that one of the energy levels coming down in the (3,3,0) state triplet has the same symmetry as the spin-singlet state, resulting in an energy level repulsion as they cross. As the transition point approaches, the triplet of states comes down, crossing with the spin-singlet state.

It is interesting to note that even in considering bulk transitions, as we are in our finite-size studies, the systems display a residual side effect from the topological mismatch between the two states in coexistence. In our edge-state analysis we found that at the edge between the spin-singlet and (3,3,0) state there will be residual neutral gapless modes due to the difference in topological structure, while in our bulk finite-size studies we find a change in ground-state degeneracy. Both features are generic and stable against perturbation, indicative of the true topological character of the incompressible fluids.

B. Variation of Δ_{sas} at $d/l=0$

In Fig. 6 we investigate the energy-level structure by fixing $d/l=0$ and varying Δ_{sas}. We are witnessing the transition from the spin-singlet ground state to the particle-hole conjugate state as a function of the Zeeman energy associated with the pseudospin degree of freedom. The system quickly undergoes a transition from the spin-singlet state to the particle-hole conjugate state indicated by a simple level crossing. The spin-singlet state is extremely sensitive to the effects of the effective magnetic field, rapidly finding it energetically favorable to place the composite fermions in the second Landau level rather than the first spin-reversed Landau level.

As the two phases possess the same topological structure they also possess the ground-state degeneracy. They do, however, have different pseudospin symmetries, allowing the energy levels to cross without repulsion. Such an energy-level crossing will become a first-order transition in the thermodynamic limit.

C. Variation of Δ_{sas} at $d=2.0l$

In Fig. 7 we investigate the energy structure at $d=2.0l$ as we vary Δ_{sas}. We are seeing the transition from the (3,3,0) state to the particle-hole conjugate state as we turn up the tunneling. The transition is qualitatively the same as the spin singlet to (3,3,0) transition, with a degeneracy transition due to the topological mismatch between the two states. Again we see a level crossing driven by variation in the sample parameters, where two states involved in the crossing mix and cause energy-level repulsion. This transition is particularly relevant as experimental evidence supporting such a phase transition already exists.[5] The experiments of Suen *et al.*[5] were performed using a single wide quantum well geometry, making quantitative comparison with our idealized double-layer calculations difficult.[17]

V. NATURE OF THE TRANSITIONS

We believe that the transitions seen in our finite-size studies represent first-order phase transitions in the thermody-

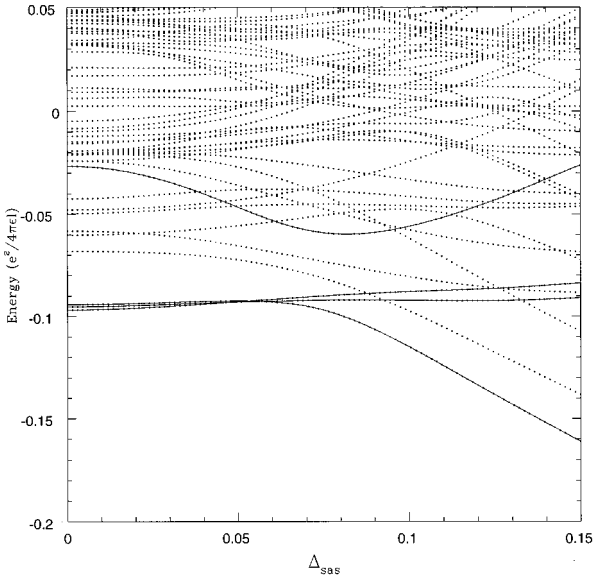

FIG. 7. Variation in energy levels as Δ_{sas} is varied at $d/l=2.0$. Δ_{sas} is measured in units of $e^2/4\pi\epsilon l$.

namic limit at low temperatures in a clean system. Traditionally, one uses broken symmetries and their associated order parameters to classify and organize many-body systems, often allowing the construction of low-energy effective theories based on these order parameters, which capture the essence of the correlated states as well as predicting effects that are not accessible in a microscopic approach. The effective theories employed in the description of the fractional quantum Hall effect are not based on an order parameter derived from a broken symmetry. Rather, these effective theories embody a type of order called topological order, which is stable against perturbation and manifests itself in such properties as the ground-state degeneracy when the system is defined on a topologically nontrivial closed space. We therefore do not consider the fractional quantum Hall states to be broken symmetry states.

As the fractional quantum Hall states are not broken symmetry states, they cannot undergo the usual second-order phase transition where the order parameter goes smoothly to zero in the vicinity of a transition point, and is zero on the other side of the transition. It would appear that due to the fact that the order embodied by the fractional quantum Hall states is discrete and topological in origin that it would be impossible to go smoothly from one state to another with different topological order as a function of system parameters in a clean system. Let us consider a point in the d/l-Δ_{sas} plane where either the $\nu=2/3$ spin-singlet state or the particle-hole conjugate state is in coexistence with the (3,3,0) state. From our edge-state analysis we know that when two phases having different topological order are in coexistence there will be at least one pair of residual neutral gapless modes at the boundary between phases. The finite energy of these modes will localize the boundary between the two phases, forcing the transition to occur by nucleation, resulting in a first-order phase transition. If two states have different topological order, there is no way to go smoothly from one state to another due to the discrete nature of topological order.

In the transition between two states that possess the same topological order there is no such mismatch. If we consider the interface between two such states, such as the spin-singlet state and the particle-hole conjugate state, we expect that the two pairs of edge modes will pair up and form a gap, leaving no residual gapless modes. In a clean system the two states have different pseudospin symmetries, as the particle-hole conjugate state is spin polarized while the spin singlet state is a true pseudospin singlet. This difference is reflected in the flux-number shift of the two states on the sphere: $\mathcal{S}=0$ for the particle-hole conjugate state and $\mathcal{S}=1$ for the spin singlet. The transition can then go by a simple level crossing, resulting in a first-order transition in the thermodynamic limit.

There exists the possibility that there might be some intermediate state that exists between the two principal states undergoing the transition. This case is really a two-step process rather than a direct transition: Hall state to intermediate state, and then intermediate state to Hall state. In principle, the intermediate state could be incompressible, but we can then apply the same arguments used above to show that the two principal states undergoing the transition cannot be connected smoothly. While this scenario is an experimental possibility, in the $\nu=2/3$ system no evidence exists for this type of intermediate transition. As such, we conjecture that all the transitions involved in the $\nu=2/3$ system will be first order.

VI. CONCLUSIONS

In conclusion, we have examined the structure of the phase diagram of the $\nu=2/3$ double-layer electron system as a function of d/l, the distance between the layers, and Δ_{sas}, the tunneling parameter for the system. A phase diagram consisting of three different phases, each belonging to a different, distinct universality class, was calculated. A triple point is conjectured where all three phases are stable. A gapless, neutral Luttinger liquid structure is predicted at the interface between either the spin-singlet state or the particle-hole conjugate state and the (3,3,0) state. At the interface between the spin-singlet state and the particle-hole conjugate state no residual gapless modes are expected. It is conjectured that there should be first-order transitions between all three phases, indicated by distinct level crossings in the finite system energy levels.

ACKNOWLEDGMENTS

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APPENDIX: PERIODIC BOUNDARY CONDITIONS

We wish to impose generalized boundary conditions by requiring that all physical quantities be invariant under translation of any particle by the set of translations $\mathbf{L}_{mn}=m\mathbf{L}_1+n\mathbf{L}_2$ such that

$$|\mathbf{L}_1\times\mathbf{L}_2|=2\pi N_\phi l^2, \tag{A1}$$

where N_ϕ is the number of flux quanta. We impose the general boundary conditions on the wave function for any particle i:

$$t_i(\mathbf{L}_{mn})|\Psi\rangle=(\eta_{mn})^{N_\phi}e^{i\mathbf{\Phi}_0\cdot\mathbf{L}_{mn}}|\Psi\rangle, \tag{A2}$$

where $\eta_{mn}=(-1)^{(m+n+mn)}$ and we will choose $\mathbf{\Phi}_0=0$ as our boundary condition. The operators $t(\mathbf{a})$ are the translation operators in the presence of a magnetic field,[19] which obey the noncommutative algebra

$$t(\mathbf{a})t(\mathbf{a}')=t(\mathbf{a}+\mathbf{a}')e^{i\mathbf{a}\times\mathbf{a}'/2l^2}. \tag{A3}$$

We shall denote the two-dimensional coordinate as $a=a_x+ia_y$ and $\overline{a}=a_x-ia_y$.

In the periodic geometry, Halperin's extension of Laughlin's wave function, suitable for double-layer systems, can be written as

$$\Psi^{(m_1,m_2,n)}[z_i,z_i']=\Psi_{\text{c.m.}}[Z,Z']\Psi_{\text{rel}}^{(m_1,m_2,n)}[z_i,z_i'], \tag{A4}$$

where

$$\Psi_{\text{rel}}^{(m_1,m_2,n)}[z_i,z_i']=\prod_{i<j}[\phi(z_i-z_j)]^{m_1}\prod_{i<j}[\phi(z_i'-z_j')]^{m_2}\times\prod_{i,j}[\phi(z_i-z_j')]^{n}. \tag{A5}$$

$Z=\Sigma_i z_i$, $Z'=\Sigma_i z'$, and the unprimed coordinates refer to electrons in the first layer while the primed refer to electrons in the second. This wave function is denoted (m_1,m_2,n). The basic building block that we have built our wave function from is the quasiperiodic function $\phi(z)$, which can be written as

$$\phi(z)=w(z)\exp\left[-\left(\frac{z^*z}{4N_\phi l^2}\right)\right], \tag{A6}$$

where

$$w(z)=\exp\left(\frac{z^2}{4N_\phi l^2}\right)\Theta_1(\kappa z|\tau) \tag{A7}$$

and $\Theta_1(u|\tau)$ is the odd elliptic Θ function, $L_{mn}=\kappa^{-1}(m+n\tau)$, and $\tau=(L_2/L_1)e^{i\theta}$, where $\mathbf{L}_1\cdot\mathbf{L}_2=|\mathbf{L}_1||\mathbf{L}_2|\cos\theta$. We have used the symmetric gauge $\mathbf{A}=(B/2)\mathbf{r}\times\hat{z}$ in expressing the function ϕ. We constrain m_1 and m_2 to be odd for Fermi statistics. Note that in writing the wave function in this form we have expressed the correlations between the electrons but for notational simplicity suppressed the pseudospin part of the wave function, which would properly antisymmetrize the overall wave function. This form of the double-layer wave functions has been discussed previously.[18]

We consider a system of electrons confined to two parallel planes subject to periodic boundary conditions confined to the lowest Landau level. The symmetry analysis of this system, as introduced by Haldane,[19] allows us to construct a Hilbert space that extracts the center-of-mass degeneracy as well as providing a correct classification of states allowing comparison with studies performed in other geometries. We can therefore classify the eigenstates of a translationally invariant Hamiltonian obeying $[H,T(\mathbf{a})]=0$ by the quantum number $\mathbf{k}$ defined to be

$$T\left(\frac{\mathbf{L}_{mn}}{\overline{N}}\right)|\Psi\rangle=(\eta_{mn})^{pq}\exp\left(i\frac{\mathbf{k}\cdot\mathbf{L}_{mn}}{\overline{N}}\right)|\Psi\rangle, \tag{A8}$$

where $N_e=\overline{Np}$ and $N_\phi=\overline{Nq}$, allowing us to write $\nu=N_e/N_\phi=p/q$. In the thermodynamic limit the states characterized by $\mathbf{k}=0$ become rotationally invariant, implying that the signature of an incompressible quantum Hall state is a $\mathbf{k}=0$ ground state with an energy gap to all excited states. As the operator (65) commutes with the center-of-mass operator $T(\mathbf{L}_{mn}/N_\phi)=\Pi_i t_i(\mathbf{L}_{mn}/N_\phi)$ each eigenstate has a q-fold degeneracy associated with the action of the center-of-mass operator.

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