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Josephson current and proximity effect in Luttinger liquids

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A theory describing a one-dimensional Luttinger liquid in contact with a superconductor is developed. Boundary conditions for the fermion fields describing Andreev reflection at the contacts are derived and used to construct a bosonic representation of the fermions. The Josephson current through a superconductor/Luttinger liquid/superconductor junction is considered for both perfectly and poorly transmitting interfaces. In the former case, the Josephson current at low temperatures is found to be essentially unaffected by electron-electron interactions. In the latter case, significant renormalization of the Josephson current occurs. The profile of the (induced) condensate wave function in a semi-infinite Luttinger liquid in contact with a superconductor is shown to decay as a power law, the exponent depending on the sign and strength of the interactions. In the case of repulsive (attractive) interactions the decay is faster (slower) than in their absence. An equivalent method of calculating the Josephson current through a Luttinger liquid, which employs the bosonization of the system as a whole (i.e., superconductor, as well as Luttinger liquid) is developed and shown to give the results equivalent to those obtained via boundary conditions describing Andreev reflection.

I. INTRODUCTION

When a normal metal (N) is put in good electrical contact with a superconductor (S), superconducting order is induced in the normal metal over distances far greater than any microscopic lengthscale, either of the normal metal or the superconductor. This induced order leads to a number of remarkable phenomena, such as the Josephson effect in SNS junctions[1] and the induced Meissner effect in SN bilayers,[2] collectively known as "proximity effects."[2] Until very recently, all work on such effects, both experimental and theoretical, has concentrated on systems in which N is in the Fermi-liquid (FL) state. It has long been appreciated theoretically that, in contrast with their higher-dimensional analog, (effectively) one-dimensional systems of interacting electrons are not Fermi liquids. Instead, they exhibit a number of possible regimes,[3] among which the Luttinger liquid (LL) provides a one-dimensional (1D) metallic counterpart to the (higher-dimensional) FL state, albeit differing in several important respects, most notably in the absence of single-particle excitations in the low-energy part of the spectrum. The basic features of LL's have been understood mainly in the context of 1D organic charge-transfer and mixed-valence conductors.[4] In addition, the prediction of the suppression of the tunneling conductance of LL's (Refs. 5 and 6) has stimulated the experimental search for Luttinger liquids in mesoscopic systems, in particular, in the edge channels of fractional quantum Hall systems[7] and in semiconductor quantum wires.[8]

The purpose of this paper is to address the issue of proximity effects at Luttinger-liquid/superconductor interfaces, including the Josephson effect in superconductor/Luttinger-liquid/superconductor (SLS) junctions. Our motivation is twofold. First, experimentally, such a study is relevant in view of the rapid progress in the fabrication of superconductor/semiconductor interfaces,[9] especially those with high interface transparency (such as, e.g., the Nb/InAs interface), and also in view of the recently reported observations of LL-like behavior in GaAs quantum wires.[8] Thus, the fabrication and investigation of SLS systems may reasonably be anticipated in the near future. Second, theoretically, we aim to understand the interplay between electron-electron interactions and induced superconducting order in 1D electronic systems. Furthermore, one of the possible scenarios of high-temperature superconductivity in oxide materials is built on the assumption of the LL-like character of the normal electronic state in these materials.[11] The existence of LL's in dimensions higher than one, however, is not yet established, in contrast to the 1D case. Thus, a 1D LL in which the superconductivity is induced via the proximity effect may provide a model system for superconductivity in 2D.[12]

Our main results can be formulated as follows: (i) At low temperatures, the Josephson current through an SLS junction having perfectly transmitting interfaces has the same phase—and length—dependences as in the noninteracting case, the only difference being a renormalization of the effective Fermi velocity. The reason for this is that, using the bosonic language, the nondissipative (topological) currents, including the Josephson current, are carried in LL's by the topological modes of the boson fields, which are not sensitive to the interactions. At temperatures above a certain crossover value, interactions lead to the additional suppression of the Josephson current. (ii) The (induced) superconducting condensate wave function in a LL in good electrical contact with a superconductor decays as $x^{-\gamma}$ with the distance x from the LL/S interface, with γ depending on the strength of the interactions. ($\gamma=1$ corresponds to the noninteracting case, whereas $\gamma>1$ for repulsive and $\gamma<1$ for attractive interactions.) (iii) For the case of imperfectly transmitting interfaces, the renormalization of the interface-transmission coefficients (via a mechanism known from studies of LL's coupled by weak links[5,6]) is reflected in the renormalization of the Josephson current, which gets strongly suppressed in the case of the repulsive interactions. Along the way, we have also (iv) derived effective boundary conditions describing Andreev reflection at the interface with a superconductor, which we have then used as boundary conditions in the bosonization procedure; (v) determined the structure of the topological (Haldane) excitations in an SLS system; and (vi) confirmed result (i) via an alternative approach, in which bosonization is applied to both the superconducting and normal parts of the system.

The issue of the Josephson current through a LL has also been studied in a recent paper by Fazio, Hekking, and Odintsov[13] for the case of poor interface transmittance (see also Ref. 10). By using the tunneling Hamiltonian method, it was found that the Josephson current through an SLS junction is suppressed compared to the noninteracting case. The present paper takes a different approach. This approach originates from work on SNS junctions with perfect interface transmittance,[14–17] in which the Josephson current was related to the spectrum of electronic states confined to the N region by Andreev reflection. Our results concerning poorly transmitting interfaces agree with those of Ref. 13 (up to nonuniversal numerical prefactors, which we have not attempted to calculate).

The present paper is organized as follows. In Sec. II we derive the boundary conditions for the fermion field operators at the NS interface in the absence of interactions. In Sec. III we develop a bosonization procedure for interacting fermions confined to the normal 1D region of an SLS system, which makes use of the boundary conditions derived in Sec. II. We calculate the Josephson current through an SLS junction in Sec. IV. In Sec. V, we analyze the profile of the condensate amplitude in a semi-infinite LL connected to a superconductor. Up to this stage, we will have applied bosonization only to the normal part of the system, the presence of a superconductor being implemented as a boundary condition. An alternative approach, in which both the normal and the superconducting parts of the system are bosonized, is presented in Sec. VI.

II. ANDREEV BOUNDARY CONDITIONS

A. Andreev reflection at the NS interface: Qualitative picture

Electronic excitations in a normal metal having energies smaller than the superconducting energy gap Δ_0 suffer Andreev reflection at the interface, i.e., electronlike excitations are reflected as holelike excitations, with a Cooper pair being injected into the superconductor, and vice versa.[18,19] The single-particle excitations in S are mixtures of electronlike and holelike states with weights determined by the self-consistency condition. In the bulk of N, the electronlike and holelike states are uncorrelated. Near the boundary, however, Andreev reflection mixes the electronlike and holelike states precisely in the same proportion as they are mixed in S, which leads to the formation of a condensate, the amplitude of which decays into the bulk of N. The decay length of this condensate is the length L_T over which superconducting correlations in the motion of bulk normal-state electrons exist. In the case of perfect metals, $L_T=\hbar v_F/T$, where v_F is the Fermi velocity and T is the temperature (we choose units in which $k_B=1$). [The same length determines the thermal disruption (in the absence of inelastic processes) of mesoscopic phase coherence, as is manifested in the phenomenon of universal conductance fluctuations.[20]) For $T\ll\Delta\simeq T_c$ (where T_c is the critical temperature of S), $L_T\gg\xi_S$ (where $\xi_S\simeq\hbar v_F/\Delta_0$ is the coherence length of S). In order to describe the influence of the superconductors on the N region, we now derive effective boundary conditions that account for the Andreev reflection suffered by the low-energy components of the fermion fields at the NS interfaces. Our strategy is as follows: in Sec. II B, we derive these boundary conditions for the case of the noninteracting electron gas in N; then, in Sec. III, we implement these boundary conditions into the bosonization scheme for interacting electrons.

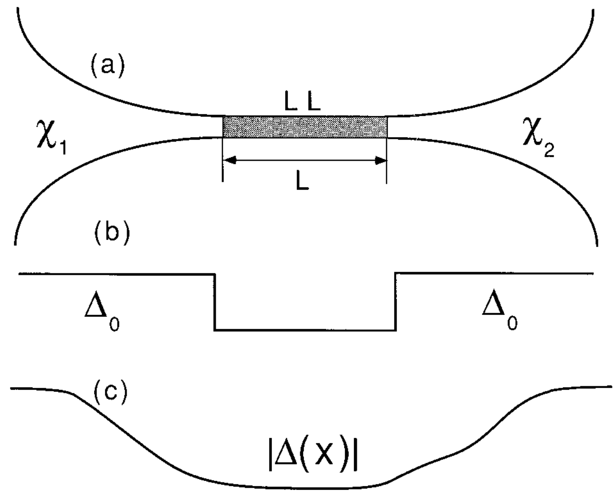

FIG. 1. (a) A Luttinger-liquid (LL) conductor connecting two superconducting electrodes with phases of the order parameters χ_1 and χ_2. (b) The model profile of the pair potential used for the derivation of Andreev boundary conditions (Sec. II). (c) Generic profile of the pair potential appropriate for the bosonization of the system as a whole (Sec. VI).

B. Derivation of Andreev boundary conditions

We consider a one-dimensional electronic conductor (i.e., a quantum wire) of length L, adiabatically connected to superconducting leads [see Fig. 1(a)]. We begin by analyzing the ideal case, in which the single-electron parameters (Fermi velocities, effective masses, etc.) are the same in the

N and S parts of the structure, the only difference between N and S being the presence of a pairing potential in S. We adopt the conventional model[14–17] in which the pairing potentials in the leads are assumed to be unaffected by the presence of N. Although this is a non-self-consistent approximation, it is known to reflect correctly the aspects of the problem relevant for the present treatment.[14–17]

To derive the boundary conditions, we replace the adiabatically widening 3D superconducting leads by effective 1D leads. The profile of the pair potential is then given by [cf. Fig. 1(b)]

$$\Delta(x)=\begin{cases}\Delta_0 e^{i\chi_1} & \text{for } x\leqslant 0\\ 0 & \text{for } 0<x<L\\ \Delta_0 e^{i\chi_2} & \text{for } x\geqslant L.\end{cases} \tag{1}$$

In the Andreev (semiclassical) approximation,[18] which is valid for $\Delta_0 \ll \epsilon_F \equiv \hbar k_F v_F/2$, the spinor of Bogoliubov amplitudes

$$\mathbf{w}=\begin{pmatrix} u\\ v\end{pmatrix} \tag{2}$$

satisfying the Bogoliubov–de Gennes equations[22] is decomposed into left- and right-moving components,

$$\mathbf{w}=e^{ik_F x}\mathbf{w}_+ + e^{-ik_F x}\mathbf{w}_-, \tag{3}$$

where k_F is the Fermi wave vector. The components $\mathbf{w}_\pm$ now satisfy the (formally) relativistic (first-order) Bogoliubov–de Gennes equations: $H_D^\pm \mathbf{w}_\pm = \epsilon \mathbf{w}_\pm$, with the Hamiltonians

$$H_D^\pm=\begin{pmatrix}\mp i\hbar v_F\partial_x & \Delta(x)\\ \Delta^*(x) & \pm i\hbar v_F \partial_x\end{pmatrix}.$$

The full solution of these equations is obtained [14] by finding the solutions in the N and S regions and then matching them at the interfaces. (In the semiclassical approximation, only the wave functions need be continuous.) The solution in N for $\epsilon<\Delta_0$ can be written as

$$\mathbf{w}_\pm = A_\pm \begin{pmatrix} e^{\pm ikx}\\ \mathscr{R}^{\mp 1}(\epsilon)e^{-i\chi_1}e^{\mp ikx}\end{pmatrix}, \tag{4}$$

where

$$\mathscr{R}(\epsilon)=e^{-i\eta(\epsilon)} \quad \text{and} \quad \eta(\epsilon)=\cos^{-1}(\epsilon/\Delta_0), \tag{5}$$

in which $\mathscr{R}$ is the Andreev reflection coefficient, whose phase is η. The quasiparticle momentum $\hbar k=\epsilon/v_F$ satisfies the quantization condition

$$\mathscr{R}(\epsilon)^2 e^{\pm i(\chi_1-\chi_2)}e^{2ikd}=1, \tag{6}$$

where $\pm$ corresponds to two sets of energy levels.[14] In Eq. (4), $A_\pm$ are overall normalizations which, without the loss of generality, can be chosen to be real. Evaluating Eq. (4) at $x=0$ and $x=L$ and using Eq. (6), one can see that at the boundaries the left and right components of the Bogoliubov amplitudes satisfy

$$\begin{cases} v_\pm = \mathscr{R}^{\mp 1}e^{-i\chi_1}u_\pm, & \text{for } x=0\\ v_\pm=\mathscr{R}^{\pm 1}e^{-i\chi_2}u_\pm, & \text{for } x=L.\end{cases} \tag{7}$$

Equations (7) describe the essence of Andreev reflection: the electronlike excitations ($u_\pm$) are converted into holelike excitations ($v_\pm$), at the same time acquiring the phase of the order parameter ($\chi_{1,2}$) together with the phase shift of the Andreev reflection coefficient (η).

In the limit $\epsilon \ll \Delta_0$, the phase shift $\eta \to \pi/2$,[21] and the boundary conditions (7) become energy independent. This enables one to derive from Eqs. (7) the boundary conditions for the real-space fermion operators $\psi_s(x)$, where $s=\uparrow,\downarrow$ denotes the spin projection. These field operators are related to the (u,v) amplitudes via the Bogoliubov transformation[22]

$$\psi_s(x)=\sum [c_s u(x) - s c_{-s}^\dagger v_s^*(x)], \tag{8}$$

where c_s ($c_s^\dagger$) is the fermion annihilation (creation) operator, the sum runs over all single-particle quantum numbers, and the variable s takes on the values $+1(-1)$ for the $\uparrow(\downarrow)$ spin projections. We decompose $\psi_s(x)$ into the left and right movers:

$$\psi_s(x)=e^{ik_F x}\psi_{+,s}(x)+e^{-ik_F x}\psi_{-,s}(x). \tag{9}$$

Substituting decompositions (3) and (9) into Eq. (8), we obtain the Bogoliubov transformation for $\psi_{\pm,s}$:

$$\psi_{\pm,s}(x)=\sum [c_s u_\pm(x) - s c_{-s}^\dagger v_\mp^*(x)]. \tag{10}$$

The boundary conditions for the (Pauli) spinors $\psi_{\pm,s}$ then follow upon substitution of Eqs. (7) into Eq. (10), and using $\eta=\pi/2$. After some algebra, we obtain

$$\psi_{+,\uparrow}|_{x=0,L} = \mp i e^{i\chi_{1,2}}\psi_{-,\downarrow}^\dagger|_{x=0,L}, \tag{11a}$$

$$\psi_{+,\downarrow}|_{x=0,L} = \pm i e^{i\chi_{1,2}}\psi_{-,\uparrow}^\dagger|_{x=0,L}, \tag{11b}$$

or, more compactly,

$$\begin{pmatrix}\psi_{+,\uparrow}\\ \psi_{+,\downarrow}\end{pmatrix}\Bigg|_{x=0,L} = \mp i e^{i\chi_{1,2}}\hat{T}\begin{pmatrix}\psi_{-,\uparrow}\\ \psi_{-,\downarrow}\end{pmatrix}\Bigg|_{x=0,L}. \tag{12}$$

Here, $\hat{T}=\hat{g}\hat{C}$ is the time-reversal operator,[22] with

$$\hat{g}=i\hat{\sigma}_y=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{13}$$

and $\hat{C}$ being the Hermitian conjugation operator. The presence of $\hat{T}$ in Eq. (12) signals an important property of Andreev reflection:[18] a reflected excitation is the time-reversed version of an incident one.

Further insight into the meaning of the boundary conditions (12) can be obtained by employing the chiral symmetry of left-right fermion fields $\psi_{\pm,s}$. In what follows, we adopt the methods of Refs. 23 and 24, in which the chiral symmetry of boson fields satisfying Dirichlet or Neumann boundary conditions was used to derive effective periodic boundary conditions. The right (left) field describes the propagation of the (formally) relativistic fermions to the right (left) with the Fermi velocity. Consequently, in the Heisenberg representation, the space-time dependence of these fields is given by

$$\psi_{\pm,s}(x,t)=\psi_{\pm,s}(x\mp v_F t). \tag{14}$$

Using the boundary conditions (11a) and (11b), one sees that at any instant of time the time-dependent left-moving fermion fields satisfy

$$\psi^{\dagger}_{-,s}(v_F t) = -sie^{-i\chi_1}\psi_{+,-s}(-v_F t), \tag{15a}$$

$$\psi^{\dagger}_{-,s}(L+v_F t) = sie^{-i\chi_2}\psi_{+,-s}(L-v_F t). \tag{15b}$$

Choosing $t=(L+x)/v_{\rm F}$ in Eq. (15b), we obtain $\psi^{\dagger}_{-,s}(x+2L) = sie^{-i\chi_2}\psi_{+,-s}(-x)$. Equation (15a) gives $\psi_{+,s}(-x) = -sie^{i\chi_1}\psi^{\dagger}_{-,-s}(x)$ which, in combination with the previous equation, leads to

$$\psi_{-,s}(x+2L,t) = e^{i\pi\vartheta}\psi_{-,s}(x,t), \tag{16a}$$

$$\psi_{+,s}(x,t) = -sie^{i\chi_1}\psi^{\dagger}_{-,-s}(-x,t), \tag{16b}$$

where $\vartheta \equiv 1 + (\chi/\pi)$ and $\chi \equiv \chi_2 - \chi_1$. Thus, we see that the Andreev boundary conditions (11a) and (11b) are equivalent to twisted *periodic* boundary conditions for $\psi_{-,s}$, Eq. (16a), on an interval of length twice the length of the original system, supplemented by the connection between the $\psi_{+,s}$ and $\psi_{-,s}$ fields following from the chiral symmetry, Eq. (16b). (Equivalently, the periodic boundary conditions can be derived for $\psi_{+,s}$ and the chiral symmetry can be used to obtain $\psi_{-,s}$.) The problem thus becomes very similar to one of fermions on a ring of circumference $2L$, threaded by an effective Aharonov-Bohm flux $\vartheta/2$, the persistent current[25–27] being the analog of the Josephson current. A detailed treatment of persistent currents in Luttinger liquids was given in Ref. 28 (for the case of spinless electrons), and we shall adopt this treatment in what follows.[29]

We now discuss the range of validity of the Andreev boundary conditions. The condition $\epsilon \ll \Delta_0$, which we needed in order to arrive at Eq. (12), means that our boundary conditions are capable of describing only excitations with wavelengths $1/k \gg \hbar v_F/\Delta_0 \simeq \xi_S$. Such excitations exists only in "long" junctions, i.e., $L \gg \xi_S$; thus our treatment is valid only for this case. On the other hand, as follows from self-consistent calculations,[2] the order parameter in S gets reduced from its bulk value over the scale ξ_S near the boundary, which also affects the excitations in N in the boundary region of the thickness of ξ_S. Thus, the model of a steplike profile of Δ, Eq. (1), can adequately describe only processes taking place in the interior of N (i.e., for x outside boundary layers of width ξ_S), where the exact shape of the profile of Δ in S is irrelevant. The latter condition can be satisfied only if $L \gg \xi_S$. Therefore, the range of validity of our boundary conditions is the same as that of the non-self-consistent model itself. We can also view Eq. (12) as the minimal-model boundary conditions that describe the time-reversal process associated with Andreev reflection.

III. BOSONIZATION OF LUTTINGER LIQUID IN CONTACT WITH SUPERCONDUCTORS

We now turn to the bosonization of an interacting 1D electronic system[31] in contact with superconductors. We represent the free fermion fields in the conventional bosonic form:

$$\psi_{\pm,s}(x) = \frac{1}{\sqrt{\alpha L}}\exp[\pm i\sqrt{\pi}\phi_{\pm,s}(x)], \tag{17}$$

where $\alpha \to +0$ is a convergence factor and the chiral bosons $\phi_{\pm,s}$ are expressed through the density (phase) bosons $\phi_s(\theta_s)$ via

$$\phi_{\pm,s}(x) = \phi_s(x) \mp \theta_s(x). \tag{18}$$

We construct mode expansions for $\phi_{\pm,s}(x)$ in such a way that the twisted boundary conditions (16a) and the auxiliary conditions (16b) are satisfied:

$$\phi_{-,s}(x) = \frac{\varphi_s}{\sqrt{\pi}} + \sqrt{\pi}(N_s+\vartheta)\frac{x}{2L} + \bar{\phi}_s(x), \tag{19a}$$

$$\phi_{+,s}(x) = \frac{\varphi_{-s}}{\sqrt{\pi}} - \sqrt{\pi}(N_{-s}+\vartheta)\frac{x}{2L} + \bar{\phi}_{-s}(-x). \tag{19b}$$

(The additive c-number terms have been omitted in the expansion for $\phi_{+,s}$.) Here, φ_s are zero-mode operators, N_s are operators whose eigenvalues give the winding numbers of the Haldane (topological) excitations,[32] and $\bar{\phi}_s(x)$ are the non-zero-mode components of the chiral boson fields, which are periodic on the interval $(0,2L)$:

$$\bar{\phi}_s(x) = \sum_{k>0}\gamma_k(e^{-ikx}a^{\dagger}_{k,s} + e^{ikx}a_{k,s}), \tag{20}$$

where $k=\pi n/L$ (with $n=1,2,\ldots,$), $\gamma_k = \exp(-\alpha n/2)/\sqrt{kL}$, and $a_{k,s}$ satisfy the canonical commutation relations $[a_{k,s}, a^{\dagger}_{k',s'}] = \delta_{ss'}\delta_{kk'}$. In Eqs. (19a) and (19b), the terms linear in x describe the topological excitations of the bosonic system, which do not conserve the total number of fermions. The eigenvalues of N_s give the numbers of fermions added to or removed from the Luttinger liquid. The non-zero-mode components $\bar{\phi}_s$ describe the quantum fluctuations around the topological excitations. These correspond to the fluctuations in the fermion density that conserve the total number of fermions.

We require that the chiral bosons obey the canonical commutation relations,[33]

$$[\phi_{\pm,s}(x), \partial_{x'}\phi_{\pm,s'}(x')] = \mp i\delta_{ss'}\sum_{n=-\infty}^{\infty}\delta(x-x'+2nL), \tag{21}$$

where the summation over n reflects periodicity on the interval $(0,2L)$. The non-zero-mode components of, e.g., expansion (19a) obey

$$[\bar{\phi}_{-,s}(x), \partial_{x'}\bar{\phi}_{-,s'}(x')] = \delta_{ss'}\left\{ix/L + \sum_{n=-\infty}^{\infty}\delta(x-x'+2nL)\right\}. \tag{22}$$

Thus, in order for Eq. (21) to be satisfied, the zero-mode operators φ_s and the winding-number operators N_s must obey

$$[\varphi_s, N_{s'}] = 2i\delta_{ss'}. \tag{23}$$

(The same result can certainly be found by considering the commutation relations of $\phi_{+,s}$.) *A posteriori*, we can also justify the choice of the coefficients γ_k in Eqs. (19a) and (19b): they were chosen in such a way that the commutation relations (21) are satisfied.

Next, we introduce the charge (ρ) and spin (σ) components of the boson fields: $\phi_{\rho,\sigma} \equiv (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$ and $\theta_{\rho/\sigma} \equiv (\theta_\uparrow \pm \theta_\downarrow)/\sqrt{2}$. The mode expansions for ϕ_μ and θ_μ (where $\mu = \rho,\sigma$) follow from the expansions (19a) and (19b):

$$\phi_\rho(x) = \frac{\varphi_\rho}{\sqrt{\pi}} + \sum_{k>0} \gamma_k \cos kx (a^\dagger_{k\rho} + a_{k\rho}), \tag{24a}$$

$$\phi_\sigma(x) = \sqrt{\frac{\pi}{2}} M \frac{x}{L} + i \sum_{k>0} \gamma_k \sin kx (a^\dagger_{k\sigma} - a_{k\sigma}), \tag{24b}$$

$$\theta_\rho(x) = \sqrt{\frac{\pi}{2}} (J + \vartheta) \frac{x}{L} + i \sum_{k>0} \gamma_k \sin kx (a^\dagger_{k\rho} - a_{k\rho\dagger}), \tag{24c}$$

$$\theta_\sigma(x) = \frac{\varphi_\sigma}{\sqrt{\pi}} + \sum_{k>0} \gamma_k \cos kx (a^\dagger_{k\sigma} + a_{k\sigma}), \tag{24d}$$

where $\varphi_{\rho/\sigma} \equiv (\varphi_\uparrow \pm \varphi_\downarrow)/\sqrt{2}$, $M \equiv (N_\uparrow - N_\downarrow)/2$, $J \equiv (N_\uparrow + N_\downarrow)/2$, and $a_{k\rho/\sigma} \equiv (a_{k,\uparrow} \pm a_{k,\downarrow})/\sqrt{2}$. It is natural that the phase difference of the superconducting order parameters χ, which determines the charge flow between the superconductors, appears only in the field associated with the charge current, i.e., θ_ρ.

We now have to determine the topological constraints imposed on the winding numbers N_s (and, consequently, on M and J). This can be done by substituting the expansion, e.g., for $\phi_{-,s}$, Eq. (19a), into the bosonization formula (17), and requiring that the boundary conditions for fermions (16a) be satisfied.[28] [When disentangling the operators in the exponent of Eq. (17), one must recall that ϕ_s and N_s do not commute, and use Eq. (23).] One thus finds that N_s satisfies

$$(-)^{N_s+1} = 1, \tag{25}$$

i.e., that the eigenvalues of N_s are odd. (Neglecting the operator nature of the zero modes and the winding numbers would have led to N_s being even.) Consequently, $J+M$ must be odd. It is convenient to introduce an effective winding number $J' = J+1$, so that $J + \vartheta = J' + \chi/\pi$ in Eq. (24c). Then, $J' + M$ must be even. Comparing this constraint with the similar constraint on the topological numbers in the persistent-current problem,[28] we see that our constraint effectively corresponds to the case of an *odd* number of fermions on the ring, in which case the response of the system to the twist in boundary conditions is diamagnetic, i.e., the free energy is minimal at zero twist. Tracing back through our calculations, we note that the diamagnetic nature of the Josephson current is guaranteed by the Andreev phase shift ($\pi/2$), which ultimately shifts J to $J+1$. The physical meaning of the topological constraint is quite simple: the energy of the LL is minimal when the left- and right-moving branches of the spectrum are populated symmetrically; changing the total number of fermions in the LL by an even (odd) number results in excitations with even (odd) total momentum quanta.

We note that expansions similar to Eqs. (24a)–(24d) could have been obtained by first deriving the boundary conditions directly for the charge and spin bosons from the boundary conditions for the fermions (in the same way that the boundary conditions for bosons are derived from the Dirichlet boundary conditions for fermions in Ref. 23), and then constructing expansions satisfying these boundary conditions. In this way, however, the zero modes of the expansions, which are crucial for the topological constraints on the eigenvalues of M and J, might have been missed. (We will derive and use the boundary conditions for charge/spin bosons in Sec. V, when the topological structure of the boson fields will not be important.)

The bosonized Hamiltonian of the LL is given by

$$\mathscr{H} = \frac{\hbar}{2} \sum_{\mu=\rho,\sigma} \int_{-L/2}^{L/2} dx \left\{ \frac{v_\mu}{K_\mu} (\partial_x \phi_\mu)^2 + v_\mu K_\mu (\partial_x \theta_\mu)^2 \right\}. \tag{26}$$

If the LL model originates from the Hubbard model then $K_{\rho/\sigma} = 1/\sqrt{1 \pm g}$, where $g \equiv Ua/\pi v_F$, with U being the strength of on-site interactions and a the microscopic length cutoff (of order the Fermi wavelength), and $v_\mu \equiv v_F/K_\mu$. In addition, if the underlying SU(2) symmetry of the Hubbard model is intact, then $K_\sigma = 1$.[34]

IV. JOSEPHSON CURRENT THROUGH A LUTTINGER LIQUID

One of the most important consequences of induced coherence in the N part of an SNS system is the Josephson current through it. This current differs from its counterpart in tunnel junctions in that the critical current $\mathscr{J}_c$ decays with the junction length L according to the power law $1/L$ (for $L \ll L_T$), rather than exponentially.

The Josephson current in SNS junctions is affected strongly by the quality of the interface. The transmittance of interfaces between semiconductors and superconductors varies widely, depending on the nature of the junction. The interface that has been studied most intensively in recent years, particularly in the context of mesoscopic effects, is the Nb/InAs interface. This interface is unique in the sense that a charge-accumulation layer is formed instead of a Schottky barrier and, as a result, the interface transparency is quite high. More commonly, however, the transmittance may be quite low, both because of interface roughness and Schottky-barrier formation. Below, we calculate the Josephson current through the LL in two limiting cases: perfectly transmitting interfaces (Sec. IV A) and poorly transmitting interfaces (Sec. IV B). The latter case has been investigated in Ref. 13.

A. Perfectly transmitting interfaces

First, we consider the case of perfectly transmitting interfaces, in which the only scattering that takes place at the

S/LL boundaries is Andreev reflection, single-particle reflection being absent. The Josephson current $\mathscr{J}$ is given by

$$\mathscr{J}=\frac{2e}{\hbar}\frac{\partial}{\partial\chi}\Omega, \tag{27}$$

where $\Omega=-k_BT\ln Z$ is the grand potential, and Z is the grand partition function. Substituting Eqs. (24a)–(24d) into Eq. (26) and diagonalizing the non-zero-mode part via a canonical transformation, we get for the many-body eigenenergies of the system

$$\mathscr{E}=\frac{\pi\hbar}{4L}\left[v_\rho K_\rho\left(J'+\frac{\chi}{\pi}\right)^2+\frac{v_\sigma}{K_\sigma}M^2\right]+\hbar\sum_{k>0}\sum_{\mu=\rho,\sigma}v_\mu k(n_{k\mu}+1/2), \tag{28}$$

where $n_{k\mu}\equiv b^\dagger_{k\mu}b_{k\mu}$, and the new boson operators $b_{k\mu}$ are connected to the old ones via $a_\mu=b_\mu\cosh\lambda_\mu-b^\dagger_\mu\sinh\lambda_\mu$, in which

$$\lambda_{\rho/\sigma}=\pm\frac{1}{2}\tanh^{-1}\frac{1-K^2_{\rho/\sigma}}{1+K^2_{\rho/\sigma}}. \tag{29}$$

We see that the phase difference χ appears only in the topological part of $\mathscr{E}$, as it should, because the non-zero-mode excitations are neutral, and therefore do not contribute to the (equilibrium) charge current. We also note that only two of the four charge/spin bosons, viz., ϕ_σ and θ_ρ, contribute to the topological part of $\mathscr{E}$. (The Josephson-current problem differs in this respect from the persistent-current problem, in which all four bosons contain topological excitations.) The combination $\{\phi_\sigma,\theta_\rho\}$ commonly arises in the study of superconductivity in LL's.[34]

The partition function factorizes as $Z=Z_t(\chi)Z_n$, where $Z_{t/n}$ is the contribution from the topological (non-zero-mode) part of $\mathscr{E}$. To calculate $\mathscr{J}$, we need only know Z_t, which is given by

$$Z_t(\chi)=\sum_{J',M}{}' \; e^{-\varepsilon_\rho(J'+\chi/\pi)^2}e^{-\varepsilon_\sigma M^2}, \tag{30}$$

where $\varepsilon_\rho\equiv\pi L_T v_\rho K_\rho/4v_F L$ and $\varepsilon_\sigma\equiv\pi v_\sigma L_T/4K_\sigma v_F L$, and the primed sum indicates that J' and M are connected via the constraint found in Sec. III (i.e., $J'+M$ even). (Although the spin part of Z_t does not depend on χ, it does not simply reduce to an overall multiplicative factor because of this constraint.) It is convenient to represent the winding numbers J' and M in the following form: $J'=2j+\kappa_J$ and $M=2m+\kappa_M$ (with $\kappa_J=0,1$ and $\kappa_M=0,1$).[28] The topological constraint is then satisfied for $j,m=0,\pm1,\ldots,$ and $\kappa_J=\kappa_M$. We can then rewrite Eq. (30) in the unconstrained form:

$$Z_t(\chi)=f_{0,\rho}(\chi)f_{0,\sigma}(0)+f_{1,\rho}(\chi)f_{1,\sigma}(0), \tag{31}$$

where

$$f_{\kappa,\mu}(\chi)\equiv\sum_{n=-\infty}^{\infty}e^{-\varepsilon_\mu(2n+\kappa+\chi)^2}. \tag{32}$$

The Josephson current can now readily be calculated. Without writing down the exact expression (which contains, as usual for this kind of problem, Jacobi ϑ_3 functions[28]) we consider only the asymptotic cases of low ($L\ll L_T$) and high ($L\gg L_T$) temperatures. In the former case, one finds

$$\mathscr{J}=\frac{ev_\rho K_\rho}{L}\frac{\chi}{\pi},\quad\text{for } |\chi|\leqslant\pi, \tag{33}$$

with $\mathscr{J}(\chi+2\pi)=\mathscr{J}(\chi)$. We note that the interaction renormalization of the Josephson current is the same as that of the persistent current.[28] When the Luttinger-liquid Hamiltonian (26) is obtained as the long-wavelength limit of the Hubbard Hamiltonian then $v_\rho K_\rho=v_F$. Comparing Eq. (33) for $v_\rho K_\rho=v_F$ with the corresponding expressions for the noninteracting electrons,[15–17] we see that *the Josephson current through the Luttinger liquid is precisely the same as through the noninteracting electron gas*. A word of caution is necessary, however: this conclusion is only valid if backscattering and umklapp scattering are not taken into account. Even if these types of scattering are irrelevant (in the renormalization-group sense), they will modify the parameters of the LL entering Eq. (26), so that the equality $v_\rho K_\rho=v_F$ will no longer hold.[35] Nevertheless, the deviations from this equality are expected to be small. (For instance, in the spinless case, the maximal reduction in the product $v_\rho K_\rho$ due to umklapp scattering is 20%, even at half-filling, when such processes are most effective.[28,36]) Also, there is a much more significant source of the renormalization of $\mathscr{J}$ which we have not yet taken into account, viz., the nonperfectness of the interfaces (see Sec. IV B).

For high temperatures ($L\gg L_T$) we find

$$\mathscr{J}=8e\hbar^{-1}T\exp(-2\pi\alpha L/L_T)\sin\chi, \tag{34}$$

where

$$\alpha\equiv\frac{1}{2}\left(\frac{v_\rho K_\rho}{v_F}+\frac{v_\sigma}{v_F K_\sigma}\right). \tag{35}$$

In the Hubbard model with the SU(2) symmetry, $\alpha=(1+1/\sqrt{1-g})/2>1$. Thus, at high temperatures, interactions lead to the further suppression of $\mathscr{J}$ (in addition to the thermal disruption of the phase coherence).

B. Poorly transmitting interfaces

Having discussed the case of perfectly transmitting interfaces, we now give a brief discussion of the case of poorly transmitting interfaces. In this case, qualitative information can be obtained by making use of the known results on the interaction-induced renormalizations of the transmission coefficient. (For the analogous treatment of persistent current in imperfect LL rings, cf. Ref. 37.)

First, consider a noninteracting SNS ''clean'' system (i.e., the elastic mean free path being far greater than L), with interface-transmission coefficients $T_{1,2}\ll1$. For simplicity, we now restrict attention to the low-temperature case ($L_T\gg L$). The result for the Josephson current can be obtained from the general formula of Ref. 38, Eq. (16), which expresses $\mathscr{J}$ through the probability for an excitation to propagate from one interface to another within a certain time. Substituting the probability of ballistic propagation into

Eq. (16) of Ref. 38, we find, after some simple algebra, that the critical current $\mathcal{J}_c^i$ for the structure with imperfect interfaces is

$$\mathcal{J}_c^i \simeq T_1 T_2 \mathcal{J}_c, \tag{36}$$

where $\mathcal{J}_c$ is the critical current for $T_1 = T_2 = 1$, which is given by the χ-independent factor in Eq. (33). In the interacting case, the low-transparency interfaces can be described within the weak-link model of Kane and Fisher.[5,6] In this model, the weak links (in our case, the interfaces) are treated as perturbations that transfer electrons between the disconnected (in zeroth order) parts of the LL, and the renormalization group (RG) of the boundary sine-Gordon model is used to find the effective values of the hopping (transmission) probabilities. The case of the double barrier has been considered in Ref. 6. We note that (i) because in our Hamiltonian (26), and hence in our action, the topological excitations are decoupled from the nonzero modes, the RG flow equations for the transmission coefficients are the same as in Ref. 6; (ii) the effective cutoff for the RG flow is provided in our case by the junction length L. Therefore, we can borrow the result for the renormalized product $T_1 T_2$ from the Kane-Fisher result for the double barrier away from the resonance:

$$T_1 T_2 \to T_1 T_2 (k_F L)^{-(1/K_\rho + 1/K_\sigma - 2)}. \tag{37}$$

If the SU(2) symmetry of the underlying Hubbard model is intact, i.e., $K_\sigma = 1$, we find

$$\mathcal{J}_c^i \simeq T_1 T_2 \left(\frac{1}{k_F L} \right)^{K_\rho^{-1} - 1} \mathcal{J}_c, \tag{38}$$

which is in agreement with Ref. 13, up to a (nonuniversal) numerical coefficient.

Whether Eq. (33) or Eq. (38) is relevant to a given experimental situation depends on the bare (i.e., unrenormalized) values of $T_{1,2}$ and on L. Suppose that $T_1 \approx T_2 \equiv T_0 \approx 1$. Then the interface barriers can be treated according to the weak-barrier model.[5,6] Assume, for simplicity, that the potential barriers are δ functions with the (bare) amplitude V_0. As V_0 is small, its RG flow at distinct interfaces is independent, and is given by $V = V_0 (L/a)^{(1-K_\rho)/2}$. Then T_0 is renormalized to

$$T = \frac{1}{1 + (mV/\hbar k_{\mathrm{F}})^2} = \frac{1}{1 + \dfrac{1 - T_0}{T_0} \left(\dfrac{L}{a} \right)^{(1-K_\rho)}}. \tag{39}$$

For relatively short junctions, i.e.,

$$L \ll L^* \simeq a \left(\frac{T_0}{1 - T_0} \right)^{1/(1-K_\rho)}, \tag{40}$$

the renormalization of T_0 due to interactions is small, and Eq. (33) applies. The better the interface the larger L^*. In particular, as $T_0 \to 1$, $L^* \to \infty$, in accordance with the previously found virtual absence of the renormalization of $\mathcal{J}$ for perfect interfaces (cf. Sec. IV A). For longer junctions, i.e., $L \gg L^*$, Eq. (38) applies.

Choosing $g = Ua/\pi v_F = 1/2$ (i.e., $K_\rho \approx 0.8$), $T_0 = 0.7$, and recalling that $a \simeq 2\pi/k_F \simeq 400$ Å in the relevant semiconductor structures, we find $L^* \simeq 1$ μm. Junctions of lengths in the range 0.1–10 μm are quite common in experiments,[9] so both Eqs. (33) and (38) may be relevant in experimental situation.

Indirectly, one also can appreciate the extent to which interactions renormalize the Josephson current by using recent experimental results on the (dissipative) conductance of the ultrahigh mobility GaAs quantum wires.[8] As was shown in Ref. 8, the conductance of the wire is reduced from the conductance quantum (i.e., e^2/h per spin projection) as the temperature is lowered, the temperature dependence being consistent with the theory of charge transport in dirty Luttinger liquids.[39–41] The absolute value of this reduction is quite small, however: it amounts to 1–5 % for wires of length 2–10 μm.

V. PROXIMITY EFFECT IN LUTTINGER LIQUIDS

As has been mentioned in Sec. II A, Andreev reflection at the NS interface gives rise to correlations between electron-like and holelike excitations in N. These correlations are similar to those between the single-particle excitations in S, which can be viewed as the induction of superconducting off-diagonal long-range order in N due to the proximity of S. The presence of such order is usually described by the (inhomogeneous) condensate wave function[2] $F(x)$, defined by

$$F(x) \equiv \langle \psi_\uparrow(x) \psi_\downarrow(x) \rangle. \tag{41}$$

In the bulk of N, $F = 0$. The scale over which F (exponentially) changes from its value at the NS boundary to zero in the bulk is given by L_T. As $T \to 0$, the length $L_T \to \infty$, and the exponential decay of F crosses over to a slower (power-law) decay. In particular, if N is a Fermi-liquid metal, F decays with the distance from the interface as $1/x$ (at $T = 0$).[43] We now explore how this decay law is changed if N is in the LL state.

Consider a semi-infinite LL occupying the half line $x > 0$ and connected to S at $x = 0$. The bosonized form of $F(x)$ is given by

$$F(x) = \frac{1}{\pi \delta a} \langle e^{-i\sqrt{2\pi}\theta_\rho(x,0)} \cos(\sqrt{2\pi}\phi_\sigma(x,0)) \rangle, \tag{42}$$

where $\delta \to +0$ is a (dimensionless) cutoff parameter, a is the microscopic scale of the system, and $\theta_\rho(x,\tau)$ and $\phi_\sigma(x,\tau)$ are boson fields in the (imaginary time) Heisenberg representation. In Eq. (42), the average is taken with respect to Boltzmann factor $e^{-S/\hbar}$, where $S = S_\rho + S_\sigma$ is the (Euclidean) action corresponding to Hamiltonian Eq. (26), and

$$S_\rho \equiv \frac{\hbar K_\rho}{2} \int dx d\tau \frac{1}{v_\rho} (\partial_\tau \theta_\rho)^2 + v_\rho (\partial_x \theta_\rho)^2, \tag{43a}$$

$$S_\sigma \equiv \frac{\hbar}{2K_\sigma} \int dx d\tau \frac{1}{v_\sigma} (\partial_\tau \phi_\sigma)^2 + v_\sigma (\partial_x \phi_\sigma)^2. \tag{43b}$$

[Note that we have deliberately expressed S_μ via those boson fields that enter the bosonized form of $F(x)$.] The presence of S at $x = 0$ imposes certain boundary conditions on these fields. We derive these boundary conditions directly from the boundary conditions for fermions Eqs. (11a) and (11b) by using the bosonized form of the fermion fields (17). (The phase of the order parameter in S is now taken to be zero, as we do not consider charge flow through the interface.) Simple algebra then leads to

$$\phi_\sigma(0,\tau) = -\sqrt{2\pi}/4, \quad \theta_\rho(0,\tau) = 0. \tag{44}$$

In the (semi-) infinite geometry, the energy of topological excitations is infinitesimally small,[42] and therefore we do not have to incorporate the winding numbers of such excitations in boundary conditions (44). As one might have anticipated, Andreev boundary conditions for fermions (11a) and (11b) impose boundary conditions only on those components of the boson fields that occur in the bosonized form of the condensate wave function (42).

In order to remove the divergence in Eq. (42) as $\delta \to +0$, we use the following trick. Consider the modified boundary condition for the ϕ_σ field: $\phi_\sigma(0,\tau) = -\sqrt{2\pi}/4 + \delta$. Introduce a new field $\tilde{\phi}_\sigma \equiv \sqrt{2\pi}/4 - \delta + \phi_\sigma$ satisfying the homogeneous boundary condition $\tilde{\phi}_\sigma(0,\tau) = 0$. After this, F takes the form

$$F(x) = \frac{\sin\delta}{\delta}\frac{1}{\pi a}\langle e^{-\sqrt{2\pi}\theta_\rho}\rangle\langle e^{-\sqrt{2\pi}\tilde{\phi}_\sigma}\rangle|_{\delta\to 0}$$

$$= \frac{1}{\pi a}\exp\{-\pi(G_\rho(x,x,0) + G_\sigma(x,x,0))\}, \tag{45}$$

where $G_{\rho/\sigma}(x,x',\tau)$ is the propagator of the charge (spin) boson field, which satisfies

$$K_\mu^{\alpha_\mu}(\partial_x^2 + v_\mu^{-2}\partial_\tau^2)G_\mu = -\delta(x-x')\,\delta(\tau), \tag{46}$$

in which $\alpha_{\rho/\sigma} = \pm 1$. G_μ obeys the following boundary conditions: $G_\mu(0,x',\tau) = 0$, $G_\mu(x,x',\tau)|_{x\to\infty} \to 0$, and $G_\mu(x,x',\tau+\beta) = G_\mu(x,x',\tau)$, where $\beta = 1/T$. The Fourier transform in τ of the solution of Eq. (46) is given by

$$G_\mu(x,x',\omega) = K_\mu^{-\alpha_\mu}|\bar{\omega}|^{-1}\sinh(|\bar{\omega}|x_<)\exp(-|\bar{\omega}|x_>), \tag{47}$$

where $x_< \equiv \min\{x,x'\}$ and $x_> \equiv \max\{x,x'\}$. Inverting the transform, we get

$$G_\mu(x,x,0) = \frac{1}{2\pi K^{\alpha_\mu}}\ln(x/a), \tag{48}$$

where, in order to regularize G_μ, we have chosen the same short-distance cutoff a as in Eq. (42). Substituting Eq. (48) into Eq. (45), we find

$$F(x) = \frac{C}{a}\left(\frac{a}{x}\right)^\gamma, \text{ with } \gamma \equiv \frac{1}{2}(K_\sigma + K_\rho^{-1}), \tag{49}$$

where C is a (nonuniversal) numerical coefficient. In the absence of interactions, $K_\rho = K_\sigma = 1$ and we return to the $1/x$ scaling. In the presence of repulsive (attractive) interactions, $\gamma > 1$ ($\gamma < 1$), and the condensate amplitude in the LL decays faster (slower) than in the FL. This result is in accord with one's intuition: the repulsive (attractive) Coulomb interaction weakens (strengthens) the superconducting state induced in N by Andreev reflection. The exponent γ is one-half of the exponent determining the spatial decay of the (singlet) superconducting fluctuations in the infinite LL.[34]

At first sight, the result that the profile of the condensate wave function in the LL decays faster than in the FL seems to contradict to the results of Sec. IV, in which it was found that the junction-length dependence of the critical current is the same in the LL and the FL. Indeed, it seems natural to connect the $1/x$ decay law of the condensate in the FL with the $1/L$ dependence of $\mathcal{J}_c$; then, it would be reasonable to expect that the $1/x^\gamma$ decay law of the condensate in the LL would be transformed into a $1/L^\gamma$ dependence of $\mathcal{J}_c$, even if the interfaces are perfect.[44] In fact, this conclusion would not be valid and, as we show below, the $1/L$ dependence of $\mathcal{J}_c$ (at $T=0$) is universal, and not connected with the profile of the condensate in the N region of an SNS junction. Consider, again, an SNS junction of length $L \gg \xi_S$. Our main argument is that at $T=0$ the only relevant length scale in the problem is L; therefore, at distances from the interface larger than ξ_S, the profile of condensate wave function N is described by a single dimensionless parameter x/L. Therefore, $F(x)$ can be represented in the following form:

$$F(x) = F_0^-\Phi^-(x/L) + F_0^+\Phi^+(x/L), \tag{50}$$

where $F_0^\pm = F_0 e^{i\chi_{1,2}}$ are the values of F at $x = 0(L)$, and the scaling functions $\Phi^\pm(z)$ satisfy the following boundary conditions: $\Phi^\pm(1) = 1(0)$, $\Phi^\pm(0) = 0(1)$. The supercurrent flowing through the junction is given by

$$\mathcal{J} = iA\left(F(x)\frac{dF^*(x)}{dx} - \text{c.c.}\right), \tag{51}$$

where A is an L independent constant. Substituting Eq. (50) into Eq. (51), we see that $\mathcal{J}$ can be represented in the form

$$\mathcal{J} = AL^{-1}\Phi(x/L), \tag{52}$$

where $\Phi(z)$ is a scaling function that is a combination of the functions $\Phi^\pm(z)$ and their derivatives. Due to charge conservation, $\mathcal{J}(x)$ does not depend on x, which can be satisfied only if $\Phi(z)$ is z independent. Thus, we see that $J \propto 1/L$ regardless of the particular form of the condensate wave function, the latter determining only the numerical coefficient in front of the $1/L$ dependence.

VI. SOLUTION VIA THE BOSONIZATION OF THE WHOLE SYSTEM

So far, we have applied bosonization only to the Luttinger-liquid part of the system, i.e., to the interval $0 < x < L$. We can gain some further insight into our results by comparison with a system in which the LL occupies the entire real line, but, by some mechanism, has acquired a superconducting gap when $x<0$ and $x>L$ [cf. Fig. 1(c)]. The existence of a gap means that the usual Luttinger Hamiltonian is modified by the addition of the term

$$H_{\text{gap}} = \int dx|\Delta(x)|e^{i\chi(x)}(\psi_{+\uparrow}^\dagger\psi_{-\downarrow}^\dagger + \psi_{-\uparrow}^\dagger\psi_{+\downarrow}^\dagger) + \text{H.c.} \tag{53}$$

The corresponding (Minkowski) bosonic action is

$$S=\int dxdt\Bigg\{\frac{K_\rho}{2}\left(\frac{1}{v_\rho}(\partial_t\theta_\rho)^2-v_\rho(\partial_x\theta_\rho)^2\right)$$
$$+\frac{1}{2K_\sigma}\left(\frac{1}{v_\sigma}(\partial_t\phi_\sigma)^2-v_\sigma(\partial_x\phi_\sigma)^2\right)$$
$$+|\Delta|(:\cos[\chi(x)+\sqrt{2\pi}(\theta_\rho-\phi_\sigma)]:$$
$$+:\cos[\chi(x)+\sqrt{2\pi}(\theta_\rho+\phi_\sigma)]:)\Bigg\}, \qquad (54)$$

where $\chi(x)$ is the local phase of the order parameter. In regions where $\Delta(x)$ is large, the principal effect of the nonlinear terms is to constrain the values of $\theta_\rho(x)$ and $\varphi_\sigma(x)$ to the minima of the cosine potential, so that

$$\chi+\sqrt{2\pi}(\theta_\rho-\phi_\sigma)=2n\pi, \qquad (55a)$$

$$\chi+\sqrt{2\pi}(\theta_\rho+\phi_\sigma)=2m\pi. \qquad (55b)$$

Equivalently

$$\theta_\rho=\frac{1}{\sqrt{2\pi}}[-\chi+\pi(n+m)], \qquad (56a)$$

$$\phi_\sigma=\frac{1}{\sqrt{2\pi}}[+\pi(n-m)]. \qquad (56b)$$

There are no constraints on θ_σ or ϕ_ρ. The fields θ_ρ and ϕ_σ are thus locked (modulo winding numbers) to the condensate phase in the superconducting regions. In the purely LL part of the system all four fields are free to fluctuate. The condensate therefore imposes boundary conditions that are essentially the same as those in Eq. (44).

The bosonized form of the number density current is

$$j(x)=-2v_\rho K_\rho\frac{1}{\sqrt{2\pi}}\partial_x\theta_\rho. \qquad (57)$$

Substituting θ_ρ from Eq. (56a) into Eq. (57) we find the $T=0$ supercurrent to be

$$j(x)=2v_\rho K_\rho\frac{1}{2\pi}\partial_x\chi. \qquad (58)$$

We can confirm this result by considering the case of a Galilean-invariant system. For such a system we know that

$$j(x)=\rho_s v_s=\rho_s\frac{\hbar}{2m}\partial_x\chi, \qquad (59)$$

where ρ_s is the density of superconducting electrons. At $T=0$ we will have $\rho_s=\rho$. Comparing Eqs. (58) and (59), we see that consistency requires the equilibrium number density in the Galilean-invariant liquid to be given by

$$\rho=2K_\rho v_\rho m/\pi\hbar. \qquad (60)$$

(The factor of 2 in this equation arises from the two spin projections.) That this is correct is shown by comparing the commutator

$$\left[\psi^\dagger\psi(x),\frac{\hbar}{2mi}\psi^\dagger(x')\overleftrightarrow{\partial}_{x'}\psi(x')\right]=\frac{\hbar}{mi}\psi^\dagger\psi(x)\partial_x\delta(x-x') \qquad (61)$$

of the charge and current in a Galilean-invariant system with the corresponding commutator in our Luttinger system, viz.,

$$[\rho(x),j(x')]=-2iK_\rho v_\rho\frac{1}{\pi}\partial_x\delta(x-x'). \qquad (62)$$

The Luttinger model approximates the Galilean-invariant system by the replacement of the charge-density operator on the right-hand side of Eq. (61) by its expectation value. This confirms that Eq. (60) is correct.

We now apply Eq. (59). In the purely LL segment of the line (i.e., $0<x<L$) the θ_ρ and ϕ_σ fields are no longer constrained by the condensate. However, as we mentioned earlier, their values at the ends of the interval are fixed, just as in Eq. (44):

$$\int_{x_1}^{x_2}j(x)dx=2v_\rho K_\rho\frac{1}{2\pi}(\chi_2-\chi_1). \qquad (63)$$

This is the same result as Eq. (33), because the quantity found by the thermodynamic trick of differentiating the free energy with respect to χ is the spatial average of the current. The advantage of Eq. (60) is that we can see that this average current is independent of the precise way in which the gap goes to zero as we enter the Luttinger link. Indeed, because the duality map between one-dimensional charge-density waves (CDW) and superconductors interchanges the charge and current densities, the results we have just described are just the dual of the well-known result in the theory of CDW systems that the total charge induced in a region is a topological quantity depending only on the asymptotic values of the CDW condensate phase.[45]

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