Quantum transport in disordered wires: Equivalence of the one-dimensional σ model and the Dorokhov-Mello-Pereyra-Kumar equation

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The two known nonperturbative theories of localization in disordered wires, the Fokker-Planck approach due to Dorokhov, Mello, Pereyra, and Kumar, and the field-theoretic approach due to Efetov and Larkin, are shown to be equivalent for all symmetry classes. The equivalence had been questioned as a result of field-theoretic calculations of the average conductance by Zirnbauer [Phys. Rev. Lett. 69, 1584 (1992)], which disagreed with the Fokker-Planck approach in the symplectic symmetry class. We resolve this controversy by pointing to an incorrect implementation of Kramers degeneracy in these calculations, and we derive modified expressions for the first two conductance moments that agree well with existing numerical simulations from the metallic into the localized regime.

I. INTRODUCTION

There are two known approaches to the theory of phase-coherent conduction and localization in disordered wires: The first is the Fokker-Planck approach of Dorokhov, Mello, Pereyra, and Kumar.[1–5] The second is the field-theoretic approach of Efetov and Larkin, which leads to a supersymmetric nonlinear σ model.[6,7] Both approaches provide a description of quantum transport that is independent of microscopic details of the disordered wire. The only properties which enter are its length L, the elastic mean free path ℓ, the number N of propagating transverse modes at the Fermi level (referred to as "channels"), and the symmetry index $\beta \in \{1,2,4\}$ (depending on the presence or absence of time-reversal and/or spin-rotational symmetry). In the first approach, the transfer matrix is expressed as a product of a large number of random matrices. As more matrices are added to this product, the transmission eigenvalues T_n execute a Brownian motion. (The T_n are the N eigenvalues of the transmission matrix product $t^\dagger t$.) The resulting Fokker-Planck equation for the L dependence of the distribution $P(T_1, \ldots, T_N)$ is known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation. In the second approach, one starts from the random Hamiltonian of the disordered wire and then expresses averages of Green's functions[6,7] or moments of the transmission eigenvalues[8–11] as integrals over matrices Q containing both commuting and anticommuting variables. These so-called supermatrices are restricted by the nonlinear constraint $Q^2 = 1$ and give rise to a field theory known as the one-dimensional nonlinear σ model.

In the past decade, research on the Fokker-Planck and field-theoretic approach has proceeded quite independently. Recently, exact results for the average conductance $\langle G \rangle$, its variance var G, and the density $\rho(T) = \langle \Sigma_n \delta(T - T_n) \rangle$ of transmission eigenvalues were obtained from both approaches. For the unitary symmetry class (no time-reversal symmetry; $\beta = 2$), the DMPK equation was solved exactly by Beenakker and Rejaei.[12] The construction of a set of biorthogonal polynomials for this exact solution then allowed for the exact computation of $\langle G \rangle$, var G, and $\rho(T)$ for arbitrary N and L in the case $\beta = 2$.[13] Although there exists a formal solution for the other two symmetry classes [orthogonal class (time-reversal symmetry without spin-orbit scattering; $\beta = 1$) and symplectic class (time-reversal symmetry with spin-orbit scattering; $\beta = 4$)],[14] no exact results for $\langle G \rangle$, var G, and $\rho(T)$ have been obtained. Concerning the σ model, an important and substantial progress was the development of "super-Fourier analysis" by Zirnbauer.[10] This allowed the exact calculation[10,11] of $\langle G \rangle$ and var G for all β in the thick-wire limit $N \to \infty$, $L/\ell \to \infty$ at fixed ratio $N\ell/L$. The eigenvalue density $\rho(T)$ was computed from the σ model by Rejaei,[15] in the thick-wire limit and for the case $\beta = 2$.

If one takes the thick-wire limit of the $\beta = 2$ results for $\langle G \rangle$, var G, and $\rho(T)$ from the DMPK equation, they agree precisely with those from the σ model.[13,15] For $\beta = 1$ and 4, a comparison of the two approaches has only been possible in the metallic regime $\ell \ll L \ll N\ell$, where the results for $\langle G \rangle$ and var G from the DMPK equation[3–5] and from the σ model[9,11,16] agree with conventional diagrammatic perturbation theory.[17–20] The equivalence of the two approaches outside the perturbative regime has been questioned[13] as a result of recent work by Zirnbauer,[10] and by Mirlin, Müller-Groeling, and Zirnbauer.[11] Starting from the σ model in the thick-wire limit, they obtained a finite limit $\langle G \rangle \to e^2/2h$ as $L/N\ell \to \infty$ in the case $\beta = 4$. On the other hand, one can prove rigorously[13] that the DMPK equation gives $\lim_{L \to \infty} \langle G \rangle = 0$ for all β. It was this puzzling contradiction that motivated us to search for a general proof of equivalence of the DMPK equation and the σ model, without the restriction to $\beta = 2$.

In this paper, we present a general proof of the equivalence of the two approaches, which applies to all three symmetry classes β, to all length scales L, and to the

complete distribution of transmission eigenvalues described by the p-point functions $\rho_p(T_1, \ldots, T_p) = [N!/(N-p)!] \int dT_{p+1} \ldots \int dT_N P(T_1, \ldots, T_N)$ for arbitrary p. We cannot relax the assumption that the number N of propagating channels in the disordered wire is $\gg 1$, since it is needed for the derivation of the one-dimensional σ model.[11] However, we can consider the σ model formulation of a thick disordered wire, which is coupled to the leads by means of a point contact with $N_1 \leqslant N$ transmitted modes,[9] and show that it is mathematically equivalent to a DMPK equation for a wire with N_1 propagating channels. The equivalence proof demonstrates that $\lim_{L\to\infty} \langle G \rangle = 0$ in the σ model, in apparent contradiction with Zirnbauer's work. We have reexamined the calculation of Refs. 10 and 11, and argue that for $\beta = 4$ the Kramers degeneracy of the transmission eigenvalues was not taken into account properly in the super-Fourier analysis. This leads to a spurious "zero mode," which does not decay as $L \to \infty$. Restoring Kramers degeneracy, we obtain modified expressions for $\langle G \rangle$ and var G, which decrease exponentially in the localized regime and moreover agree well with numerical simulations.[21]

Both the σ model and the DMPK equation were derived from a number of different models for a disordered wire. The original derivation of the DMPK equation by Dorokhov,[1] which started from a model of N coupled chains with defects, was followed by the random-matrix formulation of Mello, Pereyra, and Kumar.[2] These authors considered a product of random transfer matrices, drawn from an ensemble of maximum entropy. Later it was shown that the DMPK equation is insensitive to the choice of the ensemble, the only relevant assumptions being weak scattering (mean free path ℓ much greater than the Fermi wave length λ_F) and equivalence of the scattering channels.[22,23] It is this latter assumption that restricts the DMPK equation to a wire geometry. From the mathematical point of view, the DMPK equation is the diffusion equation on a certain coset space of transfer matrices.[24] The one-dimensional σ model was originally derived by Efetov and Larkin[6,7] from a white noise model for the disorder potential. Two later derivations used random-matrix models for the Hamiltonian of the disordered wire. Iida, Weidenmüller, and Zuk (IWZ) adapted Wegner's n-orbital model[25] to the study of transport properties.[9] In this description, the wire is modeled by a large number of disordered segments in series, each segment having a random Hamiltonian drawn from the Gaussian ensemble. An alternative derivation of the σ model, due to Fyodorov and Mirlin,[26] uses a random band matrix to model the Hamiltonian of the disordered wire. In the present paper, we follow Ref. 11 and use the IWZ formulation of the σ model.

Our proof of equivalence of the DMPK equation and the σ model builds on the ideas which were used by Rejaei[15] to calculate $\rho(T)$ from the σ model for $\beta = 2$. Inspired by Nazarov's diagrammatic calculation of $\rho(T)$ in the metallic regime,[27] Rejaei introduced a generating function F, which depends both on the transmission eigenvalues T_n and on the radial parameters θ_i of the supermatrices in the unitary σ model. Rejaei was able to solve the $1d$ σ model exactly for $\beta = 2$ and thus obtained the density $\rho(T)$ as a function of L, by taking derivatives of F with respect to the θ_i's. The resulting $\rho(T)$ could then be compared with the result from the DPMK equation.[13] We introduce a more general generating

function, which allows us to establish the equivalence of the σ model and the DMPK equation at the level of p-point functions $\rho_p(T_1, \ldots, T_p)$, without actually having to compute this function. This approach works also for $\beta = 1$ and 4, where no explicit solution of the σ model is available.

The outline of the paper is as follows: In Sec. II, an outline of the equivalence proof is given. The full proof for the σ model with 8×8 supermatrices follows in Secs. III and IV, with technical material in Appendixes A–C. For the p-point functions $\rho_p(T_1, \ldots, T_p)$, we have to consider the σ model with $8p \times 8p$ supermatrices. This extension is described in Appendix D. In Sec. V, we discuss the symplectic symmetry class ($\beta = 4$) in relation to Refs. 10 and 11. By accounting for Kramers degeneracy, we obtain modified expressions for $\langle G \rangle$ and var G, which we compare with numerical simulations of the IWZ model by Mirlin and Müller-Groeling.[21] We conclude in Sec. VI.

II. OUTLINE OF THE EQUIVALENCE PROOF

Although our equivalence proof is technically rather involved, the basic idea can be described in a few paragraphs. In this section, we present an outline of the equivalence proof for the small σ model (8×8 supermatrices). The details are given in the following two sections and in the Appendixes A–C. Appendix D contains the necessary modifications to extend the proof to σ models with supermatrices of arbitrary size.

Part of the complexity of the problem is that the σ model and the DMPK equation focus on totally different objects. In the σ model, transport properties are expressed as functional integrals over supermatrices Q.[9,11] (A supermatrix is a matrix containing an equal number of commuting and anticommuting elements. We follow the notation and conventions of Refs. 8, 9, and 11.) For the small σ model, the 8×8 supermatrices are parametrized as[7,8]

$$Q = T^{-1} \Lambda T, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.1a)$$

$$T = \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & \frac{1}{2}\hat{\theta} \\ \frac{1}{2}\hat{\theta} & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad (2.1b)$$

where u and v are pseudounitary 4×4 supermatrices. Notice that Q satisfies the nonlinear constraint $Q^2 = 1$, hence the name "nonlinear" σ model. (The letter σ is used for historical reasons.) The 4×4 supermatrix $\hat{\theta}$ is called the radial part of Q. It has the form

$$\hat{\theta} = \begin{pmatrix} \theta_1 & \theta_2 & 0 & 0 \\ \theta_2 & \theta_1 & 0 & 0 \\ 0 & 0 & i\theta_3 & i\theta_4 \\ 0 & 0 & i\theta_4 & i\theta_3 \end{pmatrix}, \quad (2.1c)$$

with the symmetry restrictions

$$\theta_4 = 0 \text{ if } \beta = 1,$$

$$\theta_2 = \theta_4 = 0 \text{ if } \beta = 2, \quad (2.1d)$$

$$\theta_2 = 0 \text{ if } \beta = 4.$$

While the σ model works with the radial part of a supermatrix, the DMPK equation works with the radial part of an ordinary matrix (containing only commuting elements). This is the transfer matrix X. The radial part of X is an $N \times N$ diagonal matrix $\hat{\lambda}$, related to the eigenvalues of $XX^\dagger$. The eigenvalues of $XX^\dagger$ come in N inverse pairs $e^{\pm x_n}$, related to the diagonal elements λ_n of $\hat{\lambda}$ by $\lambda_n = \sinh^2 x_n$. For $\beta = 4$, the

eigenvalues are twofold degenerate (Kramers degeneracy). The matrix $\hat{\lambda}$ then contains only the N-independent eigenvalues. The conductance G is directly related to the λ_n's by the Landauer formula[2,28]

$$G = \frac{2e^2}{h} \sum_{n=1}^{N} T_n = \frac{2e^2}{h} \sum_{n=1}^{N} \frac{1}{1+\lambda_n}, \quad (2.2)$$

since the N-independent transmission eigenvalues T_n are related to the λ_n's by $T_n = (1+\lambda_n)^{-1}$.

We connect both approaches by considering a generating function $F(\hat{\theta}, \hat{\lambda})$, which depends on both radial matrices:

$$F(\hat{\theta}, \hat{\lambda}) = \prod_{n=1}^{N} f(\hat{\theta}, \lambda_n), \quad (2.3a)$$

$$f(\hat{\theta}, \lambda) = \mathrm{Sdet}^{-d/2}[\lambda + \cosh^2(\hat{\theta}/2)] = \left[\frac{[1+2\lambda+\cos(\theta_3+\theta_4)][1+2\lambda+\cos(\theta_3-\theta_4)]}{[1+2\lambda+\cosh(\theta_1+\theta_2)][1+2\lambda+\cosh(\theta_1-\theta_2)]} \right]^{d/2}, \quad (2.3b)$$

$$d=1 \text{ if } \beta=1,2; \quad d=2 \text{ if } \beta=4. \quad (2.3c)$$

The symbol Sdet stands for the superdeterminant of a supermatrix. For $\beta = 2$, this is the generating function introduced by Rejaei.

An ensemble of disordered wires of length L provides a distribution of $\hat{\lambda}$. The ensemble average $\langle F(\hat{\theta}, \hat{\lambda}) \rangle$ contains all statistical properties that are accessible from the small σ model. These include the average conductance $\langle G \rangle$, its variance var G, and the density of transmission eigenvalues $\rho(T)$. We explain in Appendix A how to extract these quantities by taking derivatives of $\langle F(\hat{\theta}, \hat{\lambda}) \rangle$. The average $\langle F(\hat{\theta}, \hat{\lambda}) \rangle$ can be determined by each of the two approaches independently, in terms of a partial differential equation for the L dependence and an initial condition at $L=0$. For the σ model on the one hand, the evolution equation reads

$$\frac{\partial}{\partial L} \langle F(\hat{\theta}, \hat{\lambda}) \rangle = \frac{2}{\xi} \Delta_{\hat{\theta}} \langle F(\hat{\theta}, \hat{\lambda}) \rangle, \quad (2.4a)$$

where $\Delta_{\hat{\theta}}$ is the (radial part of the) Laplacian on the σ model space, and where $\xi = \beta N \ell$ is the localization length. The explicit form of $\Delta_{\hat{\theta}}$ is given by[7]

$$\Delta_{\hat{\theta}} = \frac{\beta}{2d} \sum_i J^{-1}(\hat{\theta}) \frac{\partial}{\partial \theta_i} J(\hat{\theta}) \frac{\partial}{\partial \theta_i}, \quad (2.4b)$$

where the sum runs over the independent coordinates θ_i[see Eq. (2.1d)] and $J(\hat{\theta})$ is the integration measure for the radial decomposition (2.1),

$$J(\theta_1, \theta_2, \theta_3) = \sinh\theta_1 \sinh\theta_2 \sin^3\theta_3$$

$$\times \prod_{s_1,s_2=\pm 1} \sinh^{-2}[\tfrac{1}{2}(\theta_1 + s_1\theta_2 + is_2\theta_3)]$$

$$\text{if } \beta = 1,$$

$$J(\theta_1, \theta_3) = \sinh\theta_1 \sin\theta_3 \prod_{s_1=\pm 1} \sinh^{-2}[\tfrac{1}{2}(\theta_1 + is_1\theta_3)]$$

$$\text{if } \beta = 2, \quad (2.4c)$$

$$J(\theta_1, \theta_3, \theta_4) = \sin\theta_3 \sin\theta_4 \sinh^3\theta_1$$

$$\times \prod_{s_1,s_2=\pm 1} \sinh^{-2}[\tfrac{1}{2}(\theta_1 + is_1\theta_3 + is_2\theta_4)]$$

$$\text{if } \beta = 4.$$

The DMPK equation on the other hand, yields the evolution equation

$$\frac{\partial}{\partial L} \langle F(\hat{\theta}, \hat{\lambda}) \rangle = \frac{2}{\xi} \langle D_{\hat{\lambda}} F(\hat{\theta}, \hat{\lambda}) \rangle, \quad (2.5a)$$

where $D_{\hat{\lambda}}$ is a second order differential operator in the parameters λ_n,

$$D_{\hat{\lambda}} = J^{-1}(\hat{\lambda}) \sum_{n=1}^{N} \frac{\partial}{\partial \lambda_n} J(\hat{\lambda}) \lambda_n (1+\lambda_n) \frac{\partial}{\partial \lambda_n}, \quad (2.5b)$$

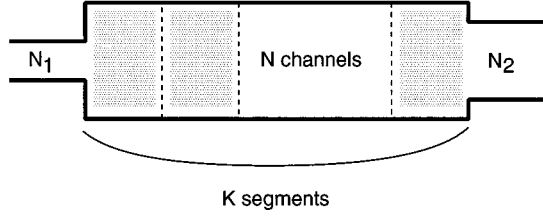$$J(\hat{\lambda}) = \prod_{n>m} |\lambda_n - \lambda_m|^\beta. \quad (2.5c)$$

FIG. 1. Schematic drawing of the disordered wire and the leads according to the IWZ model description. The left lead (lead 1) contains N_1, the right lead (lead 2) N_2 propagating channels. The number of propagating channels in the disordered wire is N. In the IWZ model, the disordered wire is divided into K segments, each having a random Hamiltonian drawn from the Gaussian ensemble. The derivation of the $1d$ σ model from the IWZ model assumes $N \gg 1$, but allows for finite N_1 and N_2.

The key ingredient of the equivalence proof is the identity,

$$\Delta_{\hat{\theta}} F(\hat{\theta}, \hat{\lambda}) = D_{\hat{\lambda}} F(\hat{\theta}, \hat{\lambda}), \qquad (2.6)$$

which shows that the evolution with L of $\langle F(\hat{\theta}, \hat{\lambda}) \rangle$ is the same in both approaches. Showing that the initial conditions at $L = 0$ coincide as well, completes the equivalence proof.

III. ONE-DIMENSIONAL σ MODEL

We begin the detailed exposition of the equivalence proof with a formulation of the σ model. As in Ref. 11, we use the formulation of Iida-Weidenmüller-Zuk (IWZ).[9]

A. The IWZ model

The IWZ model[9,16] applies Wegner's n-orbital model[25] to a wire geometry and supplements it by a coupling to ideal (not disordered) leads, as in Landauer's approach to conduction.[29] The left and right leads (labeled by indices 1 and 2) contain N_1 and N_2 propagating modes each (per spin direction for $\beta = 1,2$, or per Kramers doublet for $\beta = 4$). The disordered wire of length L is assumed to consist of K segments in series (Fig. 1). The Hamiltonian H of the disordered wire without leads is represented by a matrix $H^{ij}_{\mu\nu}$, where the upper indices i, j label the segments $1 \leq i,j \leq K$ and the lower indices μ, ν label the M states (per spin direction or Kramers doublet) within each segment. The elements of H are real ($\beta = 1$), complex ($\beta = 2$), or quaternion ($\beta = 4$) numbers. The coupling between the states inside one segment is described by the matrices $H^{ii}_{\mu\nu}$, which are distributed according to the Gaussian ensemble,

$$P(H^{ii}) = \text{const} \times \exp(-\tfrac{1}{4} \beta M v_1^{-2} \, \text{Tr}(H^{ii})^2). \qquad (3.1)$$

Here, v_1 is a parameter that governs the level density at the Fermi level ($E = 0$). The coupling between the states of adjacent segments is given by another set of Gaussian distributed random matrices $H^{ij} = (H^{ji})^{\dagger}$ (with coupling parameter v_2),

$$P(H^{ij}) = \text{const} \times \exp(-\tfrac{1}{2} \beta M^2 v_2^{-2} \, \text{Tr} H^{ij} H^{ji}),$$

$$j = i \pm 1. \quad (3.2)$$

Segments that are not adjacent are uncoupled, $H^{ij} = 0$ if $|i - j| \geq 2$. The coupling to the ideal leads is described by a fixed $KM \times (N_1 + N_2)$ rectangular matrix $W = W_1 + W_2$ with real ($\beta = 1$), complex ($\beta = 2$), or quaternion ($\beta = 4$) elements. The matrix W has elements $W^i_{\mu n}$, where i labels the segment, μ the states in the segment, and n the modes in the leads. The elements of W_1 (which describes the coupling to lead 1) are nonzero only for $i = 1$ and $1 \leq n \leq N_1$; the elements of W_2 (coupling to lead 2) are nonzero only for $i = K$ and $N_1 < n \leq N_1 + N_2$.

The scattering matrix S (matrix elements S_{nm}) of the system at energy E is given by[9]

$$S = 1 - 2\pi i W^{\dagger}(E - H + i\pi WW^{\dagger})^{-1} W. \qquad (3.3)$$

The indices n, m correspond to lead 1 if $1 \leq n, m \leq N_1$ and to lead 2 if $N_1 < n, m \leq N_1 + N_2$. The reflection and transmission matrices r, r', t, t' are submatrices of S,

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \qquad (3.4)$$

Since S is unitary, the products $t^{\dagger}t$ and $t'^{\dagger}t'$ have the same set of nonzero eigenvalues, denoted by $T_n = (1 + \lambda_n)^{-1}$. (If $N_2 > N_1$, there are also $N_2 - N_1$ transmission eigenvalues which are zero, and can, therefore, be disregarded.)

B. The generating function

We now define the generating function $F(\hat{\theta}, \hat{\lambda})$ introduced in the previous section. We start from the the relationship (3.3) between the scattering matrix and the Hamiltonian in the IWZ model. We consider the generating function,

$$F = \text{Sdet}^{-1/2}(E - \mathcal{H} + i\pi W_1 W_1^{\dagger} Q + i\pi W_2 W_2^{\dagger} \Lambda), \qquad (3.5a)$$

$$\mathcal{H} = H1_8 \quad \text{if} \quad \beta = 1,4;$$

$$\mathcal{H} = (\text{Re}H)1_8 + i(\text{Im}H)\tau_3 \quad \text{if} \quad \beta = 2. \qquad (3.5b)$$

Here, 1_8 is the 8×8 supersymmetric unit matrix and τ_3 is a diagonal matrix, with elements $(1, -1, 1, -1, 1, -1, 1, -1)$. The matrix Λ was defined in Eq. (2.1a). Note that Q is an arbitrary supermatrix as in Eq. (2.1) and that it replaces the matrix Λ in the coupling term of lead 1. In Appendix B we show that F depends only on the radial part $\hat{\theta}$ of the matrix Q and that the only dependence on H is through the transmission eigenvalues $T_n = (1 + \lambda_n)^{-1}$. We also show that Eq. (3.5) reduces to the function $F(\hat{\theta}, \hat{\lambda})$ defined in Eq. (2.3) of the previous section.

In the following, we evaluate the ensemble average $\langle F \rangle$, using the supersymmetric formalism. We first express $\langle F \rangle$ as a Gaussian integral over an $8MK$-dimensional supervector ψ:

$$\langle F \rangle = \left\langle \int \mathcal{D}\psi \, \exp[\tfrac{1}{2} i \psi^{\dagger} \Lambda (E - \mathcal{H} + i\pi W_1 W_1^{\dagger} Q \right.$$

$$\left. + i\pi W_2 W_2^{\dagger} \Lambda + i\epsilon \Lambda)\psi] \right\rangle. \qquad (3.6)$$

The convergence of the Gaussian integral is assured by the parametrization (2.1) of the matrix Q. Performing the standard steps, described in Refs. 9 and 11, we obtain in the relevant limit $M \to \infty$,

$$\langle F \rangle = \int dQ_1 \int dQ_K f_1(Q,Q_1) f_2(\Lambda,Q_K) W(Q_1,Q_K), \tag{3.7a}$$

$$W(Q_1,Q_K) = \int dQ_2 \ldots \int dQ_{K-1}$$
$$\times \exp\left(-\frac{dv_2^2}{2v_1^2} \sum_{i=1}^{K-1} \text{Str}(Q_i Q_{i+1}) \right), \tag{3.7b}$$

$$f_1(Q,Q_1) = \exp\left(-\frac{1}{2}d \sum_{n=1}^{N_1} \text{Str} \ln(1 + x_n Q Q_1) \right), \tag{3.7c}$$

$$f_2(Q,Q_K) = \exp\left(-\frac{1}{2}d \sum_{n=N_1+1}^{N_1+N_2} \text{Str} \ln(1 + x_n Q Q_K) \right). \tag{3.7d}$$

The numbers x_n denote the eigenvalues of the matrices $(\pi/v_1)W_1^\dagger W_1$ (if $1 \leq n \leq N_1$) or $(\pi/v_1)W_2^\dagger W_2$ (if $N_1 < n \leq N_1 + N_2$). The integer d was defined in Eq. (2.3c).

Following Ref. 11, we consider the limit $v_1^2 \ll v_2^2$. Then the sum in (3.7b) can be replaced by an integral and the Q_i integrals yield a path integral. The discrete number of segments K becomes the continuous (dimensionless) variable s. The propagator (3.7b) can be identified with the heat kernel of the supersymmetric space, determined by the heat equation,[11]

$$2\beta(v_2/v_1)^2 \frac{\partial}{\partial s} W(Q',Q'') = \Delta_{Q'} W(Q',Q''),$$

$$\lim_{s \to 0} W(Q',Q'') = \delta(Q',Q''). \tag{3.8}$$

The precise definition of the Laplacian Δ_Q and the detailed justification of Eq. (3.8) are contained in Ref. 11 [Δ_Q in Eq. (3.8) differs by an additional factor $\beta/(8d)$, with respect to the notations of Ref. 11]. We, thus, arrive at the expression

$$\langle F \rangle = \int dQ' \int dQ'' f_1(Q,Q') W(Q',Q'') f_2(\Lambda,Q''). \tag{3.9}$$

The next step is to notice that $f_1(Q,Q')$ has the same symmetry as the heat kernel, i.e. $f_1(T^{-1}QT, T^{-1}Q'T) = f_1(Q,Q')$, where T is an arbitrary element as described in (2.1a). This implies $\Delta_{Q'} f_1(Q,Q') = \Delta_Q f_1(Q,Q')$ and hence $\langle F \rangle$ also satisfies the heat equation,

$$2\beta(v_2/v_1)^2 \frac{\partial}{\partial s} \langle F \rangle = \Delta_Q \langle F \rangle. \tag{3.10}$$

Since $\langle F \rangle$ only depends on the radial part $\hat{\theta}$ of Q, it is sufficient to consider the radial part $\Delta_{\hat{\theta}}$ of the Laplacian

Δ_Q. This radial part $\Delta_{\hat{\theta}}$ can be written as in Eq. (2.4b). We, thus, find that the ensemble average $\langle F(\hat{\theta},\hat{\lambda}) \rangle$ of the generating function defined in Eq. (2.3) satisfies the partial differential equation,

$$2\beta(v_2/v_1)^2 \frac{\partial}{\partial s} \langle F(\hat{\theta},\hat{\lambda}) \rangle = \Delta_{\hat{\theta}} \langle F(\hat{\theta},\hat{\lambda}) \rangle, \tag{3.11}$$

with the initial condition implied by Eq. (3.8),

$$\lim_{s \to 0} \langle F(\hat{\theta},\hat{\lambda}) \rangle = \int dQ' f_1(Q,Q') f_2(\Lambda,Q'). \tag{3.12}$$

Together, Eqs. (3.11) and (3.12) determine the ensemble average of the generating function $F(\hat{\theta},\hat{\lambda})$ evaluated in the framework of the nonlinear σ model.

The two limits of the IWZ model, which were needed for the derivation of Eq. (3.12), $M \to \infty$ and $v_1^2/v_2^2 \to 0$, restrict the validity of Eq. (3.12) to the case of weak disorder ($\ell \gg \lambda_F$) and thick wires ($N \gg 1$), respectively.[9,11] Whereas the requirement of weak disorder is also needed for the DMPK equation, the requirement that the number of channels in the disordered wire be large is not. To see how the latter requirement follows from the condition $v_1^2 \ll v_2^2$, we consider the expression for the average conductance $\langle G \rangle$ in the diffusive metallic regime ($\ell \ll L \ll N\ell$),[9,11]

$$\langle G \rangle = \frac{2e^2}{h} \frac{N\ell}{L} = \frac{2e^2}{h} \frac{4v_2^2}{v_1^2 s}. \tag{3.13}$$

Taking the linear dimension of a segment of the disordered wire in the IWZ model of order ℓ (i.e., $s \approx L/\ell$, see Ref. 9), we find that $v_1^2 \ll v_2^2$ corresponds to $N \gg 1$. However, no restriction has been put to the numbers N_1 and N_2 of propagating channels in the leads in the above derivation of the σ model, which allows us to consider finite values of N_1 and N_2. This situation corresponds to the case in which the thick disordered wire is coupled to the leads 1 and 2 by means of point contacts, with N_1, N_2 open channels. As in Ref. 11, the case of a disordered wire without point contacts is recovered in the limit $N_1, N_2 \to \infty$.

We conclude this section with some remarks about the choice of initial conditions. In usual σ model calculations,[10,11,15] one considers ideal coupling ($x_n = 1$, $n = 1, \ldots, N_1 + N_2$) and identifies $N = N_1 = N_2$ (equal number of channels in the leads and in the wire). In the thick-wire limit $N \to \infty$, the function $f_i(Q,Q')$ is just the δ function[11] $\delta(Q,Q')$, and $\langle F \rangle$ becomes identical to the heat kernel itself [cf. Eq. (3.9)]:

$$\langle F \rangle = W(Q,\Lambda), \quad N_1 = N_2 = N \gg 1. \tag{3.14}$$

For $\beta = 2$, this result was derived by Rejaei.[15] In this case, $\langle F \rangle$ has the δ-function initial condition $\lim_{s \to 0} \langle F \rangle = \delta(Q,\Lambda)$. To make contact with the DMPK equation, we need a different "ballistic" initial condition, such that all T_n's are unity in the limit of zero wire length. To achieve this, we take ideal coupling and assume that one of the leads has many more channels than the other. To be specific, we fix N_1 and take the limit $N_2 \to \infty$. One then finds the initial condition,

$$\lim_{s\to 0}\langle F\rangle = \exp[-\tfrac{1}{2}N_1 d\ \mathrm{Str}\ \ln(1+Q\Lambda)]$$

$$=\left(\frac{\cos\theta_3+\cos\theta_4}{\cosh\theta_1+\cosh\theta_2}\right)^{N_1 d}, \quad 1\leqslant N_1\ll N_2. \qquad (3.15)$$

In the next section, we will see that this is precisely the ballistic initial condition of the DMPK equation.

IV. DMPK EQUATION

Let us now evaluate the ensemble average of the generating function (2.3) from the DMPK equation. The DMPK equation is a Fokker-Planck-type equation for the L evolution of the probability distribution $P(\hat{\lambda})$ of the λ_n's:[1–5]

$$\frac{1}{2}(\beta N+2-\beta)\ell\frac{\partial}{\partial L}P(\hat{\lambda})$$

$$=\sum_{n=1}^{N}\frac{\partial}{\partial\lambda_n}\lambda_n(1+\lambda_n)J(\hat{\lambda})\frac{\partial}{\partial\lambda_n}J^{-1}(\hat{\lambda})P(\hat{\lambda}), \quad (4.1a)$$

$$J(\hat{\lambda})=\prod_{n>m}|\lambda_n-\lambda_m|^{\beta}, \qquad (4.1b)$$

where ℓ denotes the mean free path in the disordered wire and N the number of propagating modes. There is *no* restriction to $N\gg 1$ in the DMPK approach. We take the ballistic initial condition,

$$\lim_{L\to 0}P(\hat{\lambda})=\prod_{n=1}^{N}\delta(\lambda_n-0^+). \qquad (4.2)$$

The DMPK equation implies for $F(\hat{\theta},\hat{\lambda})$ the evolution equation,[2,4]

$$\frac{\partial}{\partial L}\langle F(\hat{\theta},\hat{\lambda})\rangle=\frac{\partial}{\partial L}\int d\lambda_1\ldots\int d\lambda_N F(\hat{\theta},\hat{\lambda})P(\hat{\lambda})$$

$$=\frac{2}{\ell}(\beta N+2-\beta)^{-1}\langle D_{\hat{\lambda}}F(\hat{\theta},\hat{\lambda})\rangle, \qquad (4.3)$$

with the differential operator $D_{\hat{\lambda}}$ given by Eq. (2.5b). In Appendix C, we prove the algebraic identity between the two different types of Laplacians (2.4b) and (2.5b) applied to $F(\hat{\theta},\hat{\lambda})$,

$$\Delta_{\hat{\theta}}F(\hat{\theta},\hat{\lambda})=D_{\hat{\lambda}}F(\hat{\theta},\hat{\lambda}). \qquad (4.4)$$

From Eqs. (4.3), and (4.4) we conclude that the average $\langle F(\hat{\theta},\hat{\lambda})\rangle$, calculated in the framework of the DMPK equation, also fulfills the evolution equation (3.11) of the nonlinear σ model, provided that we identify [cf. Eq. (3.13)],

$$\frac{4}{s}(v_2/v_1)^2=\frac{N\ell}{L}=\frac{\xi}{\beta L}, \quad N\gg 1. \qquad (4.5)$$

Here, we introduced the localization length $\xi=\beta N\ell$ (notice that the definition of ξ in Ref. 11 differs by a factor $2/\beta$).

The comparison of the initial conditions still remains. The ballistic initial condition for the DMPK equation implies

$$\lim_{L\to 0}\langle F(\hat{\theta},\hat{\lambda})\rangle=f(\hat{\theta},\lambda=0)^N=\left(\frac{\cos\theta_3+\cos\theta_4}{\cosh\theta_1+\cosh\theta_2}\right)^{Nd},$$

$$(4.6)$$

which equals the initial condition (3.15) for the nonlinear σ model [the thick-wire limit $\lim_{L\to 0}\langle F\rangle=\delta(Q,\Lambda)$ is obtained by letting $N\to\infty$ in the above expression]. This proves the equivalence of both approaches, as far as the generating function (2.3) is concerned. In Appendix D, we extend the equivalence proof to p-point functions $\rho_p(T_1,\ldots,T_p)$ for arbitrary p.

V. THE CONTROVERSIAL SYMPLECTIC ENSEMBLE

The main motivation of this work was to resolve a controversy between the DMPK equation and the one-dimensional σ model in the symplectic symmetry class ($\beta=4$). On the one hand, the DMPK equation implies[13] $\langle G\rangle\to 0$ as $L\to\infty$. On the other hand, Zirnbauer[10] finds from the σ model that $\langle G\rangle\to\frac{1}{2}e^2/h$ as $L\to\infty$.

The equivalence proof presented in this paper has as a logical consequence that $\langle G\rangle\to 0$, as $L\to\infty$, if $\langle G\rangle$ is evaluated in the framework of the σ model. To demonstrate this, we apply the argument of Ref. 13. The DMPK equation implies for the average dimensionless conductance $g=\Sigma_n(1+\lambda_n)^{-1}$ the evolution equation,[4]

$$\xi\frac{\partial\langle g\rangle}{\partial L}=-\beta\langle g^2\rangle-(2-\beta)\langle g_2\rangle, \qquad (5.1)$$

with $g_2=\Sigma_n(1+\lambda_n)^{-2}$. This relation also follows from the evolution equation (3.11) of the σ model (expanding the generating function for small θ_i and applying the results of Appendix A). Since $0\leqslant g_2\leqslant g^2$, we have

$$\xi\frac{\partial\langle g\rangle}{\partial L}\leqslant-\frac{1}{2}\beta\langle g^2\rangle\leqslant 0. \qquad (5.2)$$

We suppose that $\lim_{L\to\infty}\langle g\rangle$ exists. Since $\partial\langle g\rangle/\partial L\leqslant 0$ [Eq. (5.2)], this implies $\lim_{L\to\infty}\partial\langle g\rangle/\partial L=0$. Hence $\lim_{L\to\infty}\langle g^2\rangle=0$ by Eq. (5.2). Since $\langle g\rangle^2\leqslant\langle g^2\rangle$, this implies that also $\lim_{L\to\infty}\langle g\rangle=0$.

Where does the nonzero limit in Refs. 10 and 11 come from? The ground-breaking contribution of Zirnbauer was to use a "super-Fourier expansion" of the heat kernel $W(Q,Q')$ in terms of eigenfunctions of the Laplacian in the space of the σ model. This type of Fourier analysis is well understood for classical symmetric spaces.[30] The development and application of the supersymmetric analog for the σ model enabled Zirnbauer, Mirlin, and Müller-Groeling to compute nonperturbatively the first two moments of the conductance for any β. The nonzero limiting value $\lim_{L\to\infty}\langle g\rangle=1/4$ for $\beta=4$ resulted from a "zero mode," a nontrivial eigenfunction of the Laplacian with zero eigenvalue. Since this zero mode does not decay as $L\to\infty$, it led to the surprising conclusion of absence of localization in a wire with spin-orbit scattering in zero magnetic field.[10]

An explicit expression for the zero mode was not obtained in Refs. 10 and 11, but only its contribution to the moments of the conductance was computed. By inspecting the initial condition (3.15) of the generating function for the σ model, we have been able to construct a zero mode for $\beta=4$, but

only if we ignore the Kramers degeneracy of the transmission eigenvalues. This unphysical zero mode, given by

$$\phi_0(\theta_1, \theta_3, \theta_4) = \frac{\cos\theta_3 + \cos\theta_4}{2 + 2\,\cosh\theta_1}, \qquad (5.3)$$

arises by taking the initial condition (3.15), with $N_1 = 1$ and $\beta = 4$, but *without* the extra factor two in the exponent, required by Kramers degeneracy. This unphysical initial condition solves the evolution equation (3.11) for the ensemble average of the generating function and implies an L-independent average conductance $\langle g \rangle = 1/4$. Although we can not prove that Eq. (5.3) is Zirnbauer's zero mode, the coincidence with the limiting value $\lim_{L\to\infty}\langle g \rangle = 1/4$, $\lim_{L\to\infty}$ var $g = 1/16$ is quite suggestive.

The reason why we have to exclude the zero mode (5.3) from the Fourier expansion of the heat kernel is that it is not single valued on the σ model space of supermatrices Q, although it is a well-defined function of $\hat{\theta}$. The parametrization (2.1) of Q is 2π periodic in the angles $\theta_\pm = \theta_3 \pm \theta_4$. We can then consider on the space of angles θ_3, θ_4 a parity operation P, which consists of adding π to both of these angles. This parity operation does not change Q, but it changes the zero mode (5.3). The Laplacian (2.4b) commutes with P and the eigenfunctions have, therefore, either even or odd parity (eigenvalues $+1$ or -1 of P, respectively). The physical modes of the σ model must have even parity, since only these functions are single valued. For $\beta = 4$, it is the Kramers degeneracy that ensures that the initial condition (3.15) has even parity.

This observation led us to check the parity of the eigenfunctions $\phi_\nu(Q)$ of the Laplacian in the super-Fourier analysis of Refs. 10 and 11. We consider the eigenvalue equation,

$$\Delta_{\hat{\theta}}\phi_\nu(\theta_1, \theta_3, \theta_4) = -\varepsilon(\nu)\phi_\nu(\theta_1, \theta_3, \theta_4), \qquad (5.4)$$

for $\beta = 4$ in the limit $\theta_1 \to \infty$ at fixed θ_3, θ_4. In this limit, the Laplace operator simplifies considerably,

$$\Delta_{\hat{\theta}} \to e^{\theta_1}\frac{\partial}{\partial\theta_1}e^{-\theta_1}\frac{\partial}{\partial\theta_1} + \frac{1}{\sin\theta_3}\frac{\partial}{\partial\theta_3}\sin\theta_3\frac{\partial}{\partial\theta_3}$$
$$+ \frac{1}{\sin\theta_4}\frac{\partial}{\partial\theta_4}\sin\theta_4\frac{\partial}{\partial\theta_4}. \qquad (5.5)$$

From this expression, one may identify the set of quantum numbers $\nu = (\lambda, 1 + 2n_1, 1 + 2n_2)$, where λ is a real number and n_1, n_2 are non-negative integers. The asymptotic behavior of the eigenfunctions $\phi_\nu(\theta_1, \theta_3, \theta_4)$ is given by

$$\phi_\nu(\theta_1, \theta_3, \theta_4) \sim \exp[\tfrac{1}{2}(1 + i\lambda)\theta_1][P_{n_1}(\cos\theta_3)P_{n_2}(\cos\theta_4)$$
$$+ P_{n_2}(\cos\theta_3)P_{n_1}(\cos\theta_4)], \qquad (5.6)$$

with the Legendre polynomials $P_n(x)$ and the eigenvalues

$$\varepsilon(\lambda, 1 + 2n_1, 1 + 2n_2) = \tfrac{1}{4}[\lambda^2 + (1 + 2n_1)^2 + (1 + 2n_2)^2 - 1]. \qquad (5.7)$$

The parity of this eigenfunction is just $(-1)^{n_1 + n_2}$ and we have to restrict ourselves to those n_1 and n_2 with $n_1 + n_2$ even. Applying this selection rule to the expressions for $\langle g \rangle$ and $\langle g^2 \rangle$ of Refs. 10 and 11, omitting the zero mode [and
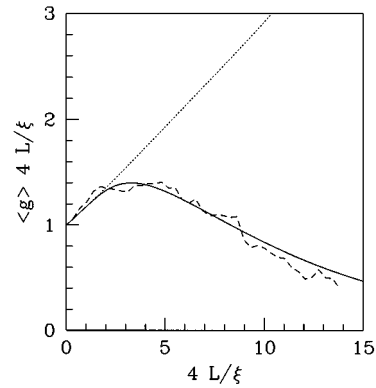

FIG. 2. The average conductance $\langle g \rangle$ multiplied by $4L/\xi = L/N\ell$ for the symplectic symmetry class as a function of $4L/\xi$ for $N \gg 1$. Shown are our result (5.8) (solid), the numerical simulation of Ref. 21 ($M = 100$, $N = 25$) (dashed), and the result of Ref. 11 (dotted).

the subsidiary series with quantum numbers $\nu = (i, l, l \pm 2)$ of Refs. 10 and 11, for which the asymptotic behavior (5.6) is also valid], and multiplying the surviving terms with a factor of 2 to account for Kramers degeneracy, yields for $\beta = 4$ and in the limit $N_1 = N_2 = N \to \infty$, the expression

$$\langle g^n \rangle = 2^{5-n}\sum_{\substack{l_1, l_2 = 1,3,5,\ldots, \\ l_1 + l_2 \equiv 2\ (\mathrm{mod}4)}}\int_0^\infty d\lambda\ \lambda(\lambda^2 + 1)$$

$$\times \tanh(\pi\lambda/2)l_1 l_2 p_n(\lambda, l_1, l_2)$$

$$\times \prod_{\sigma, \sigma_1, \sigma_2 = \pm 1}(-1 + i\sigma\lambda + \sigma_1 l_1 + \sigma_2 l_2)^{-1}$$

$$\times \exp[-(\lambda^2 + l_1^2 + l_2^2 - 1)L/(2\xi)], \qquad (5.8a)$$

where $n = 1, 2$ and

$$p_1(\lambda, l_1, l_2) = \lambda^2 + l_1^2 + l_2^2 - 1, \qquad (5.8b)$$

$$p_2(\lambda, l_1, l_2) = \tfrac{1}{4}[2\lambda^4 + l_1^4 + l_2^4 + 3\lambda^2(l_1^2 + l_2^2)$$
$$- 2\lambda^2 + l_1^2 + l_2^2 - 2]. \qquad (5.8c)$$

Note that in our notations, the dimensionless conductance g is by a factor 2 smaller than g in the notations of Ref. 11. Comparison of Eq. (5.8) with the $\beta = 4$ result of Ref. 11, where the parity selection rule was not implemented, shows that the perturbation expansion around $L/\xi = 0$ is the same. [We checked this numerically up to order $(L/\xi)^3$.] Outside the perturbative regime, the two expressions are completely different. Instead of a nonzero limit $\langle g \rangle = 1/4$ for $L/\xi \gg 1$, we find from Eq. (5.8) the exponential decay

$$\langle g \rangle \approx \tfrac{16}{9}(2L/\pi\xi)^{-3/2}e^{-L/2\xi}. \qquad (5.9)$$

To test our result, we have compared it with a direct numerical simulation of the IWZ model by Mirlin and Müller-Groeling[21] (with $M = 100$, $N = 25$, and an average over 100 different samples). The comparison is shown in Figs. 2 and 3. It is clear that our Eq. (5.8) (solid curve)
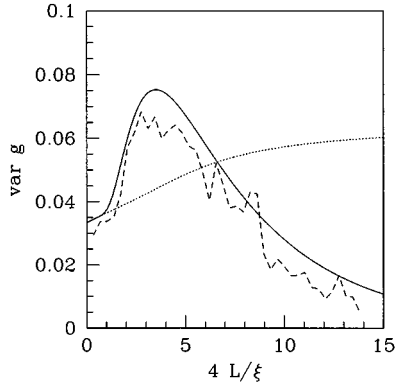
FIG. 3. As in Fig. 2 for the variance var g of the conductance.

FIG. 5. Same as Fig. 4, for the variance var g of the conductance.

agrees quite well with the simulation, while the result of Ref. 11 does not (dotted curve).

Notice that this issue of the parity of the eigenfunctions does not occur for $\beta=1,2$, since there is only one compact angle (θ_3) in those cases. The parity operation on the $\hat{\theta}$ matrices exists only for $\beta=4$. For completeness, we collect in Figs. 4 and 5 the results for $\langle g \rangle$ and var g for all three symmetry classes. The $\beta=1,2$ results are from Ref. 11, the $\beta=4$ result is our Eq. (5.8).

VI. CONCLUSION

We have established the exact mathematical equivalence of the two nonperturbative theoretical approaches to phase-coherent transport and localization in disordered wires: The Fokker-Planck equation of Dorokhov, Mello, Pereyra, and Kumar[1–5] and the one-dimensional supersymmetric nonlinear σ model.[6,7,9,11,26] The equivalence has the logical consequence that the absence of localization in the symplectic symmetry class, obtained by Zirnbauer by super-Fourier analysis of the σ model, is not correct. By applying a selection rule enforced by Kramers degeneracy to the eigenfunctions of Refs. 10 and 11, we have obtained modified expres-

sions for $\langle G \rangle$ and var G, which decay exponentially as $L \to \infty$, and which agree well with existing numerical simulations.[21]

Our equivalence proof has both conceptual and practical implications. The DMPK equation and the $1d$ σ model originated almost simultaneously in the early eighties, and at the same institute.[1,6] Nevertheless, work on both approaches proceeded independently in the next decade. Knowing that, instead of two theories, there is only one, seems to us a considerable conceptual simplification of the field. It implies that the microscopic derivations and random-matrix models developed for the σ model apply as well to the DMPK equation, and vice versa. (We see only the restriction, that the σ model requires the thick-wire limit $N \to \infty$, while the DMPK equation applies to any number of channels N.) Practically, each of the two approaches has its own advantages, and now that we know that they are equivalent, we can choose the approach which is best suited to our needs and skills.
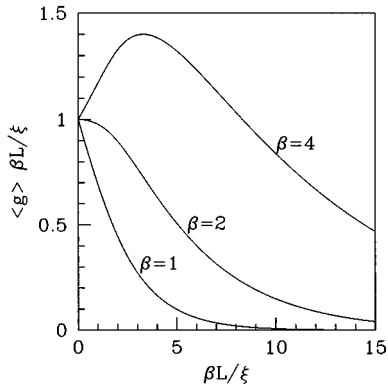
FIG. 4. The average conductance $\langle g \rangle$ multiplied by $\beta L/\xi = L/N\ell$ for the three symmetry classes, as a function of $\beta L/\xi$ for $N \gg 1$. The curves for $\beta=1,2$ are taken from Refs. 10 and 11 and the curve for $\beta=4$ is calculated from Eq. (5.8). Notice that $\xi = \beta N\ell$ is proportional to β, so that the scaling of the axes is β independent.
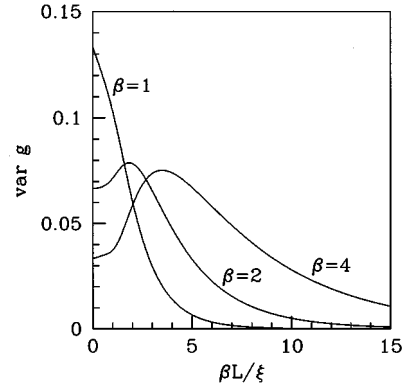
APPENDIX A: TRANSPORT PROPERTIES DETERMINED BY THE GENERATING FUNCTION

We list the transport properties of interest that can be generated from $F(\hat{\theta}, \hat{\lambda})$, following Rejaei.[15] Let us consider the function

$$f(z_1, z_2) = \left\langle \frac{\det(1 + z_2 t^\dagger t)}{\det(1 + z_1 t^\dagger t)} \right\rangle, \qquad \text{(A1)}$$

which equals $\langle F(\hat{\theta}, \hat{\lambda}) \rangle$ at $z_2 = -\sin^2(\frac{1}{2}\theta_3)$, $z_1 = \sinh^2(\frac{1}{2}\theta_1)$, and $\theta_2 = \theta_4 = 0$. We write Eq. (A1) in the form

$$f(z_1,z_2)=\langle\exp[\operatorname{Tr}\ln(1+z_2 t^\dagger t)-\operatorname{Tr}\ln(1+z_1 t^\dagger t)]\rangle.$$
(A2)

The standard expansion,[8,9] with respect to small z_1 and z_2, yields the first two moments of the dimensionless conductance $g=(1/d)\operatorname{Tr}t^\dagger t$ (with $d=1+\delta_{4,\beta}$),

$$\langle g\rangle=\frac{1}{d}\frac{\partial}{\partial z_2}f(z_1,z_2)|_{z_1=0=z_2}$$

$$=-\frac{1}{d}\frac{\partial}{\partial z_1}f(z_1,z_2)|_{z_1=0=z_2},$$
(A3)

$$\langle g^2\rangle=-\frac{1}{d^2}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}f(z_1,z_2)|_{z_1=0=z_2}.$$
(A4)

We may also consider[15,27] derivatives of $f(z_1,z_2)$ at $z_1=z_2$. This may require the analytic continuation of θ_1, θ_3 to complex values, if $z_1<0$, $z_2>0$, or $z_2<-1$. Therefore, we introduce the function $f(z_1)$ as

$$f(z_1)=\frac{\partial}{\partial z_2}f(z_1,z_2)|_{z_2=z_1}$$

$$=\langle\operatorname{Tr}[(1+z_1 t^\dagger t)^{-1}t^\dagger t]\rangle$$

$$=\sum_{n=0}^\infty(-z_1)^n\langle\operatorname{Tr}(t^\dagger t)^{n+1}\rangle$$

$$=z_1^{-1}(\operatorname{Tr}(1)-\langle\operatorname{Tr}[(1+z_1 t^\dagger t)^{-1}]\rangle).$$
(A5)

The average density of transmission eigenvalues now follows from

$$\rho(T)=\langle\operatorname{Tr}\delta(T-t^\dagger t)\rangle$$

$$=-\frac{1}{\pi T^2}\operatorname{Im}f[-(T+i0^+)^{-1}].$$
(A6)

The application of Eq. (A6) requires the analytical continuation of both variables z_1 and z_2 to values <-1.

APPENDIX B: THE GENERATING FUNCTION IN TERMS OF THE TRANSMISSION MATRIX

In this appendix, we show that Eq. (3.5) for the generating function in the IWZ model equals Eq. (2.3). We first consider the two cases $\beta=1,4$ of time-reversal symmetry, when $\mathscr{H}=H1_8$ in Eq. (3.5a). The necessary modifications for $\beta=2$ are described at the end.

We make use of the folding identity,

$$\operatorname{Sdet}\begin{pmatrix}1_n & A\\ B & 1_m\end{pmatrix}=\operatorname{Sdet}(1_n-AB)=\operatorname{Sdet}(1_m-BA).$$
(B1)

We abbreviate $G_\pm=(E-H\pm i\pi WW^\dagger)^{-1}$. Taking out the factor $(E-\mathscr{H}+i\pi WW^\dagger\Lambda)$ (with unit superdeterminant), we may rewrite Eq. (3.5a) as

$$F=\operatorname{Sdet}^{-1/2}\left[1+\begin{pmatrix}G_+ & 0\\ 0 & G_-\end{pmatrix}i\pi W_1 W_1^\dagger(Q-\Lambda)\right]$$

$$=\operatorname{Sdet}^{-1/2}\left[1+i\pi\begin{pmatrix}W_1^\dagger G_+ W_1 & 0\\ 0 & W_1^\dagger G_- W_1\end{pmatrix}(Q-\Lambda)\right],$$
(B2)

where we have applied Eq. (B1) on $B=W_1^\dagger$. We now insert the reflection matrix $r=1-2\pi i W_1^\dagger G_+ W_1$ [see Eqs. (3.4) and (3.3)] into Eq. (B2) and obtain

$$F=\operatorname{Sdet}^{-1/2}\left[\frac{1}{2}(1+\Lambda Q)+\frac{1}{2}\begin{pmatrix}r & 0\\ 0 & r^\dagger\end{pmatrix}(1-\Lambda Q)\right].$$
(B3)

Now we use the parametrization (2.1) for Q. Notice that Eq. (B3) does not depend on the angular part of Q [the matrices u,v in (2.1b)]. Hence, we may choose Q as

$$Q=T^{-1}\Lambda T,\quad T=\exp(\tfrac{1}{2}\Theta),\quad \Theta=\begin{pmatrix}0 & \hat\theta\\ \hat\theta & 0\end{pmatrix},$$
(B4)

which leads to

$$\tfrac{1}{2}(1+\Lambda Q)=\cosh(\tfrac{1}{2}\Theta)T,\quad \tfrac{1}{2}(1-\Lambda Q)=-\sinh(\tfrac{1}{2}\Theta)T.$$

Inserting this in Eq. (B3) and taking out the factor T (with unit superdeterminant), we get

$$F=\operatorname{Sdet}^{-1/2}\begin{pmatrix}\cosh(\tfrac{1}{2}\hat\theta) & -r\,\sinh(\tfrac{1}{2}\hat\theta)\\ -r^\dagger\sinh(\tfrac{1}{2}\hat\theta) & \cosh(\tfrac{1}{2}\hat\theta)\end{pmatrix}$$

$$=\operatorname{Sdet}^{-1/2}[\cosh^2(\tfrac{1}{2}\hat\theta)-\sinh^2(\tfrac{1}{2}\hat\theta)r^\dagger r]$$

$$=\operatorname{Sdet}^{-1/2}[1+\sinh^2(\tfrac{1}{2}\hat\theta)t^\dagger t],$$
(B5)

where we have again used (B1) and the relation $r^\dagger r=1-t^\dagger t$ imposed by unitarity of the scattering matrix.

The matrix $t^\dagger t$ has eigenvalues $T_n=(1+\lambda_n)^{-1}$ ($n=1,\dots,N_1$), which are twofold degenerate for $\beta=4$, hence

$$F=\prod_{n=1}^{N_1}\operatorname{Sdet}^{-d/2}[1+\sinh^2(\tfrac{1}{2}\hat\theta)T_n]$$

$$=\prod_{n=1}^{N_1}\operatorname{Sdet}^{-d/2}[\lambda_n+\cosh^2(\tfrac{1}{2}\hat\theta)].$$
(B6)

Here, $d=1+\delta_{\beta,4}$. Equation (B6) establishes the connection between the two expressions (2.3) and (3.5) for the generating function.

The above calculation assumes $\mathscr{H}=H1_8$, hence $\beta=1$ or $\beta=4$. In the case $\beta=2$, one has instead $\mathscr{H}=\operatorname{Re}(H)1_8+i\operatorname{Im}(H)\tau_3$. Instead of two subblocks with r and $r^\dagger$ [see Eq. (B3)], one now needs four subblocks with r, r^T, $r^\dagger$, and r^*. Repeating the calculation, one finds that the final result (B6) remains the same.

APPENDIX C: IDENTITY OF LAPLACIANS

The goal of this appendix is to prove Eq. (2.6). Hereto, we first analyze the structure of the left-hand side of Eq. (2.6) in more detail.

The derivatives of $F(\hat{\theta},\hat{\lambda})$, with respect to θ_j, are calculated using

$$\frac{\partial F(\hat{\theta},\hat{\lambda})}{\partial \theta_j} = \left(\sum_{n=1}^{N} \frac{1}{f(\hat{\theta},\lambda_n)} \frac{\partial f(\hat{\theta},\lambda_n)}{\partial \theta_j} \right) F(\hat{\theta},\hat{\lambda}), \quad \text{(C1)}$$

$$\frac{\partial^2 F(\hat{\theta},\hat{\lambda})}{\partial \theta_j^2} = \left(\sum_{n=1}^{N} \frac{1}{f(\hat{\theta},\lambda_n)} \frac{\partial^2 f(\hat{\theta},\lambda_n)}{\partial \theta_j^2} \right) F(\hat{\theta},\hat{\lambda})$$

$$+ \left(\sum_{n \neq m}^{N} \frac{1}{f(\hat{\theta},\lambda_n) f(\hat{\theta},\lambda_m)} \frac{\partial f(\hat{\theta},\lambda_n)}{\partial \theta_j} \frac{\partial f(\hat{\theta},\lambda_m)}{\partial \theta_j} \right)$$

$$\times F(\hat{\theta},\hat{\lambda}). \quad \text{(C2)}$$

Inspection of Eqs. (2.4b), (C1), and (C2) shows that $\Delta_{\hat{\theta}} F(\hat{\theta},\hat{\lambda})$ has two contributions, one involving a single summation over the channel indices n, and another one involving a double summation over channel indices $n \neq m$,

$$\Delta_{\hat{\theta}} F(\hat{\theta},\hat{\lambda}) = \left(\sum_{n=1}^{N} g_1(\hat{\theta},\lambda_n) \right) F(\hat{\theta},\hat{\lambda})$$

$$+ \left(\sum_{n \neq m}^{N} g_2(\hat{\theta},\lambda_n,\lambda_m) \right) F(\hat{\theta},\hat{\lambda}). \quad \text{(C3)}$$

Using the definition (2.3b) of $f(\hat{\theta},\hat{\lambda})$, one may straightforwardly calculate the functions g_1 and g_2. The expressions are rather lengthy and will not be given here.

The right-hand side of Eq. (2.6) contains the differential operator $D_{\hat{\lambda}}$, which is given by Eq. (2.5b). Simple algebra yields

$$D_{\hat{\lambda}} = \sum_{n=1}^{N} \left(\lambda_n(1+\lambda_n)\frac{\partial^2}{\partial \lambda_n^2} + (1+2\lambda_n)\frac{\partial}{\partial \lambda_n} \right)$$

$$+ \frac{\beta}{2} \sum_{n \neq m}^{N} (\lambda_n - \lambda_m)^{-1} \left(\lambda_n(1+\lambda_n)\frac{\partial}{\partial \lambda_n} \right.$$

$$\left. - \lambda_m(1+\lambda_m)\frac{\partial}{\partial \lambda_m} \right). \quad \text{(C4)}$$

As a consequence, $D_{\hat{\lambda}} F(\hat{\theta},\hat{\lambda})$ has again the structure of Eq. (C3), with g_1 and g_2 now given by

$$g_1(\hat{\theta},\lambda) = \frac{1}{f(\hat{\theta},\lambda)} \left[\lambda(1+\lambda) \frac{\partial^2 f(\hat{\theta},\lambda)}{\partial \lambda^2} + (1+2\lambda) \frac{\partial f(\hat{\theta},\lambda)}{\partial \lambda} \right], \quad \text{(C5a)}$$

$$g_2(\hat{\theta},\lambda_1,\lambda_2) = \frac{\beta/2}{\lambda_1 - \lambda_2} \left[\frac{\lambda_1(1+\lambda_1)}{f(\hat{\theta},\lambda_1)} \frac{\partial f(\hat{\theta},\lambda_1)}{\partial \lambda_1} \right.$$

$$\left. - \frac{\lambda_2(1+\lambda_2)}{f(\hat{\theta},\lambda_2)} \frac{\partial f(\hat{\theta},\lambda_2)}{\partial \lambda_2} \right]. \quad \text{(C5b)}$$

Comparison of Eqs. (C3) and (C5) shows that the two definitions of the functions g_1 and g_2 are identical. This completes the proof of Eq. (2.6).

APPENDIX D: EXTENSION TO HIGHER-DIMENSIONAL SUPERMATRICES

The argumentation presented in Secs. III and IV can be generalized to σ models, with Q matrices of arbitrary dimension $8p$, with $p \geq 1$. This generalized equivalence proof applies to the p-point functions $\rho_p(T_1, \ldots, T_p)$, instead of to the limited number of statistical quantities that can be generated by the "small" σ model with $p=1$ (compare Appendix A). Here, we briefly present the modifications with respect to the $p=1$ case. The modifications concern the parametrization (2.1) and the generating function (2.3b).

The main technical difficulty in such a generalization is due to the radial part of the Laplace operator. The procedure to calculate it on conventional symmetric spaces is standard[30] and is carried over to the supersymmetric σ models, as described in Appendix B of Ref. 11. It is now more convenient to use a slightly modified form of the parametrization of the Q matrices, where $\hat{\theta}$ in Eq. (2.1a) is fully diagonal (rather than block diagonal):

$$\hat{\theta} = \begin{pmatrix} \hat{x} & 0 \\ 0 & i\hat{y} \end{pmatrix},$$

$$(\hat{x})_{nm} = x_n \delta_{nm}, \quad (\hat{y})_{nm} = y_n \delta_{nm}, \quad 1 \leq n,m \leq 2p. \quad \text{(D1)}$$

The symmetry restrictions are [cf. Eq. (2.1d)]

$$y_i = y_{i+p} \quad \text{if} \quad \beta = 1,$$

$$x_i = x_{i+p}, \quad y_i = y_{i+p} \quad \text{if} \quad \beta = 2, \quad \text{(D2)}$$

$$x_i = x_{i+p} \quad \text{if} \quad \beta = 4,$$

for $i = 1, \ldots, p$. In the case $p=1$, we have the relations $x_1 = \theta_1 + \theta_2$, $x_2 = \theta_1 - \theta_2$, $y_1 = \theta_3 + \theta_4$, and $y_2 = \theta_3 - \theta_4$ between these parameters and the θ_i used in (2.1c).

We can directly apply the results of Appendix B in Ref. 11, which are given in terms of the so-called *roots* $\alpha(\Theta)$ [with Θ given as in (B4)]. The roots are linear functions of Θ, which are the eigenvalues of the linear mapping $\text{ad}(\Theta)(X_\alpha) := [\Theta, X_\alpha] = \alpha(\Theta) X_\alpha$, defined on a certain super-Lie algebra.[11] The eigenvectors X_α of the mapping are the *root vectors,* which do not depend on Θ. The radial integration measure $J(\hat{\theta})$ in Eq. (2.4b) can be expressed as[30]

$$J(\hat{\theta}) = \prod_{\alpha > 0} \sinh^{m_\alpha}[\tfrac{1}{2}\alpha(\Theta)], \quad \text{(D3)}$$

where the integer m_α is the multiplicity of the root α (the dimension of the root space). Both positive and negative values of m_α can occur. The factor $\frac{1}{2}$ is due to the difference between the normalization (2.1a) of Θ and the one used in Ref. 11. In Appendix A of Ref. 11, explicit formulas for the roots as well as for the root vectors are given for the case $\beta = 1$, $p=1$.

We have calculated the roots and the root vectors for all β and arbitrary dimension $8p$ of the Q matrices. For simplicity, we only present the results for the roots and their multiplicities. Let us denote with p_x (p_y) the number of independent x_i (y_i) parameters, i.e.: $p_x = 2p, p, p$ and $p_y = p, p, 2p$ for $\beta = 1, 2, 4$, respectively. Note that $\beta p_x = 2 p_y$. We find eight different types of (positive) roots:

$$\alpha(\Theta) = x_j - x_l \quad (1 \leq j < l \leq p_x), \quad m_\alpha = \beta,$$

$$\alpha(\Theta) = x_j - i y_l \quad (1 \leq j \leq p_x, 1 \leq l \leq p_y), \quad m_\alpha = -2,$$

$$\alpha(\Theta) = i(y_j - y_l) \quad (1 \leq j < l \leq p_y), \quad m_\alpha = 4/\beta,$$

$$\alpha(\Theta) = 2 x_j \quad (1 \leq j \leq p_x), \quad m_\alpha = \beta - 1,$$

$$\alpha(\Theta) = 2 i y_j \quad (1 \leq j \leq p_y), \quad m_\alpha = 4/\beta - 1, \tag{D4}$$

$$\alpha(\Theta) = x_j + x_l \quad (1 \leq j < l \leq p_x), \quad m_\alpha = \beta,$$

$$\alpha(\Theta) = x_j + i y_l \quad (1 \leq j \leq p_x, 1 \leq l \leq p_y), \quad m_\alpha = -2,$$

$$\alpha(\Theta) = i(y_j + y_l) \quad (1 \leq j < l \leq p_y), \quad m_\alpha = 4/\beta.$$

The radial part of the Laplacian takes the form

$$\Delta_{\hat\theta} = \sum_{j=1}^{p_x} J^{-1}(\hat\theta) \frac{\partial}{\partial x_j} J(\hat\theta) \frac{\partial}{\partial x_j} + \frac{\beta}{2} \sum_{j=1}^{p_y} J^{-1}(\hat\theta) \frac{\partial}{\partial y_j} J(\hat\theta) \frac{\partial}{\partial y_j}. \tag{D5}$$

The expressions (2.3a) and (3.5a) for the generating function now remain valid with the modified $\hat\theta$ of Eq. (D1), and with Eq. (2.3b) replaced by

$$f(\hat\theta, \lambda) = \prod_{i=1}^{p_y} [1 + 2\lambda + \cos(y_i)] \prod_{i=1}^{p_x} [1 + 2\lambda + \cosh(x_i)]^{-\beta/2}. \tag{D6}$$

It is convenient to use the variables $u_i = \sinh^2(\frac{1}{2} x_i)$ and $v_i = -\sin^2(\frac{1}{2} y_i)$ in terms of which the Laplacian has the form

$$\Delta_{\hat\theta} = \sum_{j=1}^{p_x} \tilde{J}^{-1} \frac{\partial}{\partial u_j} u_j(1 + u_j) \tilde{J} \frac{\partial}{\partial u_j}$$

$$- \frac{\beta}{2} \sum_{j=1}^{p_y} \tilde{J}^{-1} \frac{\partial}{\partial v_j} v_j(1 + v_j) \tilde{J} \frac{\partial}{\partial v_j}, \tag{D7a}$$

$$\tilde{J} = \prod_{1 \leq i < j \leq p_x} (u_i - u_j)^\beta \prod_{1 \leq i < j \leq p_y} (v_i - v_j)^{4/\beta}$$

$$\times \prod_{i=1}^{p_x} \prod_{j=1}^{p_y} (u_i - v_j)^{-2}$$

$$\times \prod_{i=1}^{p_x} [u_i(1 + u_i)]^{\beta/2 - 1} \prod_{i=1}^{p_y} [v_i(1 + v_i)]^{2/\beta - 1}. \tag{D7b}$$

The generating function $F(\hat\theta, \hat\lambda)$ is given by

$$F(\hat\theta, \hat\lambda) = \prod_{n=1}^{N} \left(\prod_{i=1}^{p_y} (1 + \lambda_n + v_i) \prod_{i=1}^{p_x} (1 + \lambda_n + u_i)^{-\beta/2} \right). \tag{D8}$$

We have verified that the identity of Laplacians [Eq. (2.6)] remains true for the modified expressions (D7a) and (D8). The calculations go in a similar way as shown in Appendix C for $p = 1$. Now, we have to keep track of seven different types of contributions with double and triple sums over functions of λ_n, u_i, v_j.

In Appendix A, we have shown that the average density of transmission eigenvalues $\rho(T)$ can be obtained from the generating function (2.3a). Using the corresponding function for the higher-dimensional σ model considered here, it is straightforward to get the p-point correlation functions $\rho_p(T_1, \ldots, T_p)$.

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