Quantum critical behavior of disordered itinerant ferromagnets

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The quantum ferromagnetic transition at zero temperature in disordered itinerant electron systems is considered. Nonmagnetic quenched disorder leads to diffusive electron dynamics that induces an effective long-range interaction between the spin or order parameter fluctuations of the form r^{2-2d}, with d the spatial dimension. This leads to unusual scaling behavior at the quantum critical point, which is determined exactly. In three-dimensional systems the quantum critical exponents are substantially different from their finite-temperature counterparts, a difference that should be easily observable. Experiments to check these predictions are proposed. [S0163-1829(96)00721-7]

I. INTRODUCTION

The general problem of describing quantum phase transitions is a subject of great current interest. These transitions occur at zero temperature as a function of some nonthermal control parameter, and the relevant fluctuations are of a quantum nature rather than of thermal origin. Early work in this field established that if the quantum phase transition has a classical analog at finite temperature, then in the physical dimension $d=3$ the former tends to have a simpler critical behavior than the latter. In particular, one of the most obvious examples of a quantum phase transition, namely, the ferromagnet-to-paramagnet transition of itinerant electrons at zero temperature $T=0$ as a function of the exchange interaction between the electron spins, was found to have mean-field critical behavior in $d=3$.[1] The reason for this tendency is that the coupling between statics and dynamics that is inherent to quantum statistics problems effectively increases the dimensionality of the system from d to $d+z$, with z the dynamical critical exponent. In the case of *clean* itinerant quantum ferromagnets, $z=3$ in mean-field theory. This appears to reduce the upper critical dimension d_c^+, which is the dimension above which mean-field theory yields the exact critical behavior, from $d_c^+=4$ in the classical case to $d_c^+=1$ in the quantum case.[1] However, this conclusion has recently been disputed,[2,3] and the critical behavior of clean itinerant ferromagnets in low-dimensional systems is currently under renewed investigation.[4]

It has been known for some time that if one adds quenched, nonmagnetic disorder to the system, then the critical behavior at the quantum phase transition must be substantially modified,[5] contrary to earlier suggestions.[1] It is known that the correlation length exponent ν must satisfy the inequality $\nu \geqslant 2/d$ in systems with quenched disorder.[6] The physical origin of this constraint is the requirement that disorder induced fluctuations in the location of the critical point in parameter space must be small compared to the distance from the critical point in order for the phase transition to be sharp. Any mean-field or any standard Gaussian theory yields $\nu=1/2$, which is incompatible with this lower bound in any dimension $d<4$. Technically, this implies that the disorder is a relevant perturbation with respect to the mean field or Gaussian fixed point.

From a perspective that is entirely focused on the statistical mechanics aspects of the phase transition problem (and that therefore does not take into account from the beginning the aspects that have to do with the underlying disordered electron system), it would be tempting to model this disordered quantum phase transition by making the mass term in the effective field theory for the clean case [i.e., the coefficient of the term quadratic in the order parameter field in the Landau-Ginzburg-Wilson (LGW) functional] a random variable in order to describe the fluctuations in the location of the critical point. For the quantum paramagnet-to-ferromagnet transition under consideration one can readily convince oneself that for $d<4$ the resulting disorder term is relevant in the renormalization group sense with respect to the clean Gaussian fixed point. Presumably, the presence of this relevant operator leads to a new critical fixed point with a correlation length exponent $\nu \geqslant 2/d$.

It turns out, however, that such a model is not a technically appropriate description of the quantum ferromagnetic transition in a system of disordered itinerant electrons. In order to explain this important point, let us anticipate a number of results that will be discussed in detail in Sec. II below. If one attempts to derive the LGW functional mentioned above, then the coefficient of the random mass term is a correlation function whose expansion in the random potential is very singular. The reason is that the dynamics of a quantum particle in a disordered environment in the limit of small wave numbers k and low frequencies ω is qualitatively different from that in a clean system. In the former case the motion is diffusive, while in the latter it is ballistic. Technically, the limit $k,\omega \to 0$ does not commute with the clean limit. To overcome this problem it is advantageous to not expand in powers of the random potential, but to expand instead in the fluctuations of the coefficients of the LGW functional at fixed disorder.

This procedure, by construction, automatically resums the

most obvious singularities in the disordered itinerant electron problem. Although it will turn out that the coefficients in the resulting LGW functional are still singular, these remaining singularities can be handled mathematically as we will see below. These singularities arise due to what are often called weak localization effects.[7,8] Their physical origin lies in the fact that the dynamics of diffusive electrons are intrinsically long ranged. Via mode-coupling effects, this feature couples back to the ferromagnetic transition problem. As we show in this paper, the net effect is that the interactions between the spin fluctuations in the LGW theory are of long range. The critical behavior described by the resulting nonlocal field theory can be determined exactly for all $d>2$, and satisfies $\nu \geqslant 2/d$ as required. The critical exponents obtained from this theory are d dependent for all $d<6$. In $d=3$, they are substantially different from either the mean-field exponents, or from those for a classical Heisenberg model, which has striking observable consequences.

The outline of this paper is as follows. In Sec. II we first define an itinerant disordered electron model, and then discuss how to derive an order parameter description for a ferromagnetic phase transition starting from a fermionic field theory. We also discuss in detail the coefficients in the LGW functional and show that they have long-range properties. In Sec. III the critical behavior is determined exactly for dimensions $d \geqslant 2$. For $d>6$, mean-field exponents are obtained, while for $2<d<6$, d-dependent exponents are found. In Sec. IV the results of this paper are reviewed. First we summarize the theoretical aspects of our results, and then we point out some of the experimental consequences we expect. In Appendix A we calculate the wave-number-dependent spin susceptibility for disordered interacting fermions, and in Appendix B we discuss the logarithmic corrections to scaling that exist at the critical dimensions $d_1^+=6$ and $d_2^+=4$, as well as for all $2<d<4$.

II. MODEL AND THEORETICAL FRAMEWORK

In the first part of this section we define a simple model for interacting electrons in a disordered environment. Starting with this general fermionic field theory we then derive an LGW or order parameter description, with the spin density fluctuation as the order parameter. In the last part of this section we derive and discuss the coefficients in this LGW functional. As mentioned in the Introduction, the crucial point is that the interactions in the effective LGW theory are long ranged due to the diffusive dynamics of the electrons in a disordered metal.

A. Model

The partition function of any fermionic system can be written in the form[9]

$$Z=\int D\overline{\psi}D\psi \exp(S[\overline{\psi},\psi]), \tag{2.1a}$$

where the functional integration measure is defined with respect to Grassmannian (i.e., anticommuting) fields $\overline{\psi}$ and ψ, and S is the action,

$$S=\int_0^\beta d\tau \int d\mathbf{x}\overline{\psi^i}(\mathbf{x},\tau)\frac{\partial}{\partial\tau}\psi^i(\mathbf{x},\tau)-\int_0^\beta d\tau H(\tau). \tag{2.1b}$$

Here $\mathbf{x}$ denotes positions and τ imaginary time, $H(\tau)$ is the Hamiltonian in imaginary time representation, $\beta=1/T$ is the inverse temperature, $i=1,2$ denotes spin labels, and summation over repeated indices is implied. Throughout this paper we use units such that $k_B=\hbar=e^2=1$. Our starting model is a fluid of interacting electrons moving in a static random potential $v(\mathbf{x})$,

$$\begin{aligned}H(\tau)=\int d\mathbf{x}\Big[&\frac{1}{2m}\nabla\overline{\psi^i}(\mathbf{x},\tau)\cdot\nabla\psi^i(\mathbf{x},\tau)\\&+[v(\mathbf{x})-\mu]\overline{\psi^i}(\mathbf{x},\tau)\psi^i(\mathbf{x},\tau)\Big]\\&+\frac{1}{2}\int d\mathbf{x}d\mathbf{y}u(\mathbf{x}-\mathbf{y})\overline{\psi^i}(\mathbf{x},\tau)\\&\times\overline{\psi^j}(\mathbf{y},\tau)\psi^j(\mathbf{y},\tau)\psi^i(\mathbf{x},\tau).\end{aligned} \tag{2.2a}$$

Here m is the electron mass, μ is the chemical potential, and $u(\mathbf{x}-\mathbf{y})$ is the electron-electron interaction potential. We assume that the random potential $v(\mathbf{x})$ is δ correlated and obeys a Gaussian distribution $P[v(\mathbf{x})]$ with second moment

$$\{v(\mathbf{x})v(\mathbf{y})\}_{\rm dis}=\frac{1}{2\pi N_F\tau_{\rm el}}\delta(\mathbf{x}-\mathbf{y}), \tag{2.2b}$$

where

$$\{\cdots\}_{\rm dis}=\int D[v]P[v](\cdots) \tag{2.2c}$$

denotes the disorder average, N_F is the bare density of states per spin at the Fermi level, and $\tau_{\rm el}$ is the bare electron elastic mean free time. More realistic models to describe itinerant electron magnetism including, e.g., band structure, can be considered along the same lines. The salient points of our results, however, are due to long-wavelength effects and hence do not depend on microscopic details like the band structure. For our purposes it therefore is sufficient to study the model defined in Eqs. (2.2).

For describing magnetism, it is convenient and standard practice to break the interaction part of the action S, which we denote by $S_{\rm int}$, into spin-singlet and spin-triplet contributions $S_{\rm int}^{(s,t)}$. For simplicity, we assume that the interactions are short ranged in both of these channels. In a metallic system this is justified due to screening, and an effective model with a short-ranged interaction in both $S_{\rm int}^{(s)}$ and $S_{\rm int}^{(t)}$ can be derived starting from a bare Coulomb interaction.[10] In order for this assumption to remain valid, our discussion applies only to cases in which the disorder is weak enough for the system to remain far from any metal-insulator transition that might be present in the phase diagram and would lead to a breakdown of screening.[11] The spin-triplet interaction $S_{\rm int}^{(t)}$ describes interactions between spin density fluctuations. This is what causes ferromagnetism, and it therefore makes sense to consider this part of the action separately. We thus write

$$S = S_0 + S_{\rm int}^{(t)}, \tag{2.3}$$

with

$$S_{\rm int}^{(t)} = \frac{\Gamma_t}{2}\int d\mathbf{x} d\tau \mathbf{n}_s(\mathbf{x},\tau)\cdot\mathbf{n}_s(\mathbf{x},\tau), \tag{2.4a}$$

where $\mathbf{n}_s$ is the electron spin density vector with components

$$n_s^a(\mathbf{x},\tau) = \frac{1}{2}\overline{\psi^i}(\mathbf{x},\tau)\sigma_{ij}^a\psi^j(\mathbf{x},\tau). \tag{2.4b}$$

Here the σ^a are the Pauli matrices, and Γ_t is the spin-triplet interaction amplitude that is related to the interaction potential u in Eq. (2.2a) via

$$\Gamma_t = \int d\mathbf{x} u(\mathbf{x}). \tag{2.4c}$$

For simplicity we have assumed a pointlike interaction so that Γ_t is simply a number. A generalization to a more realistic short-range interaction would be straightforward. S_0 in Eq. (2.3) contains all other contributions to the action. It reads, explicitly,

$$\begin{aligned} S_0 = \int_0^\beta d\tau \int d\mathbf{x}\Big[& \overline{\psi^i}(\mathbf{x},\tau)\frac{\partial}{\partial\tau}\psi^i(\mathbf{x},\tau) \\ & - \frac{1}{2m}\nabla\overline{\psi^i}(\mathbf{x},\tau)\cdot\nabla\psi^i(\mathbf{x},\tau) \\ & - [v(\mathbf{x})-\mu]\overline{\psi^i}(\mathbf{x},\tau)\psi^i(\mathbf{x},\tau)\Big] \\ & - \frac{\Gamma_s}{2}\int_0^\beta d\tau\int d\mathbf{x} n_c(\mathbf{x},\tau)n_c(\mathbf{x},\tau), \end{aligned} \tag{2.5a}$$

with n_c the electron charge or number density,

$$n_c(\mathbf{x},\tau) = \overline{\psi^i}(\mathbf{x},\tau)\psi^i(\mathbf{x},\tau), \tag{2.5b}$$

and Γ_s the spin-singlet interaction amplitude.

B. Order parameter field theory

Continuous *thermal* phase transitions are usually described by deriving a LGW theory, i.e., an effective field theory for the long-wavelength order parameter fluctuations or critical modes.[12] The physical idea behind this approach is that these fluctuations, which are slowly varying in space, determine the behavior near the critical point. The same philosophy has been applied to quantum phase transitions, with the only difference being that the critical modes are now slowly varying in both space and time. We will use this approach here, motivated in part by previous work on clean itinerant electronic systems.[1] We mention, however, that in general one should worry about both the critical modes, and all other slow or soft modes, even if these other soft modes are not 'critical' in the sense that they change their character at the phase transition. While this concern is not confined to quantum phase transitions, we will argue below that for these it poses a more serious problem than for thermal phase transitions since at $T=0$ there are more soft modes than at finite temperature. We will see that in the present problem such additional modes are indeed present and lead to complications within the framework of an LGW theory. For the problem under consideration, however, these complications can be overcome.

The techniques for deriving an order parameter field theory, starting with Eqs. (2.3)–(2.5), are standard.[1] We decouple the four-fermion term in $S_{\rm int}^{(t)}$ by introducing a classical vector field $\mathbf{M}(\mathbf{x},\tau)$ whose average is proportional to the magnetization m, and performing a Hubbard-Stratonovich transformation. All degrees of freedom other than $\mathbf{M}$ are then integrated out. This procedure in particular integrates out the soft diffusive modes or ''diffusons'' that are inherent to a disordered fermion system.[7,8] These are the additional soft modes mentioned above. We obtain the partition function Z in the form

$$Z = e^{-F_0/T}\int D[\mathbf{M}]\exp(-\Phi[\mathbf{M}]), \tag{2.6a}$$

with F_0 the noncritical part of the free energy. The LGW functional Φ reads

$$\begin{aligned} \Phi[\mathbf{M}] = & \frac{\Gamma_t}{2}\int dx \mathbf{M}(x)\cdot\mathbf{M}(x) \\ & - \ln\left\langle \exp\left[-\Gamma_t\int dx \mathbf{M}(x)\cdot\mathbf{n}_s(x)\right]\right\rangle_{S_0}, \end{aligned} \tag{2.6b}$$

where we have adopted a four-vector notation with $x=(\mathbf{x},\tau)$, and $\int dx = \int d\mathbf{x}\int_0^\beta d\tau$. Here $\langle\cdots\rangle_{S_0}$ denotes an average taken with the action S_0. A formal expansion of Φ in powers of $\mathbf{M}$ takes the form

$$\begin{aligned} \Phi[\mathbf{M}] = & \frac{1}{2}\int dx_1 dx_2 M_a(x_1)\left[\frac{\delta_{ab}}{\Gamma_t}\delta(x_1-x_2) - \chi_{ab}^{(2)}(x_1,x_2)\right]M_b(x_2) \\ & + \frac{1}{3!}\int dx_1 dx_2 dx_3 \chi_{abc}^{(3)}(x_1,x_2,x_3)M_a(x_1)M_b(x_2)M_c(x_3) \\ & - \frac{1}{4!}\int dx_1 dx_2 dx_3 dx_4 \chi_{abcd}^{(4)}(x_1,x_2,x_3,x_4)M_a(x_1)M_b(x_2)M_c(x_3)M_d(x_4) + O(M^5), \end{aligned} \tag{2.7a}$$

where we have scaled M with Γ_t^{-1}. The coefficients $\chi^{(l)}$ in Eq. (2.7a) are correlation functions for a system with a particular realization of the disorder (i.e., they are not translationally invariant). They are defined as

$$\chi^{(l)}_{a_1\cdots a_l}(x_1,\ldots,x_l)=\langle n_s^{a_1}(x_1)\cdots n_s^{a_l}(x_l)\rangle^c_{S_0}, \quad (2.7b)$$

where the superscript c denotes a cumulant or connected correlation function. For our simple model, the reference ensemble with action S_0, whose correlation functions are the $\chi^{(l)}$, consists of free electrons with disorder and a short-ranged spin-singlet model interaction. As mentioned in Sec. II A above, the model can be made more realistic, if desired, by, e.g., including a realistic band structure. In order to do so, one would simply replace the $\chi^{(l)}$ above with correlation functions for band electrons.

Equations (2.6) and (2.7) define an order parameter field theory for the paramagnet-to-ferromagnet phase transition in a disordered itinerant electronic system. The disorder has *not* been averaged over yet, and so the n-point correlation functions in Eqs. (2.7) depend explicitly on the particular realization of the randomness in the system. To proceed, we now formally carry out the disorder average of the free energy, using replicas and a cumulant expansion, and keeping terms up to $O(M^4)$ in the replicated LGW functional.[13] With $\alpha,\beta,\ldots=1,\ldots,n$ $(n\to 0)$ the replica labels, we obtain the following LGW functional for the αth copy of the replicated system:

$$\Phi^\alpha[\mathbf{M}]=\sum_{l=2}^{\infty}\Phi_l^\alpha[\mathbf{M}]=\frac{1}{2}\int dx_1dx_2X^{(2)}_{ab}(x_1,x_2)M^\alpha_a(x_1)M^\alpha_b(x_2)+\frac{1}{3!}\int dx_1dx_2dx_3X^{(3)}_{abc}(x_1,x_2,x_3)M^\alpha_a(x_1)M^\alpha_b(x_2)M^\alpha_c(x_3)$$
$$-\frac{1}{4!}\int dx_1dx_2dx_3dx_4X^{(4)\alpha\beta}_{abcd}(x_1,x_2,x_3,x_4)M^\alpha_a(x_1)M^\alpha_b(x_2)M^\beta_c(x_3)M^\beta_d(x_4)+O(M^5), \quad (2.8a)$$

where Φ_l^α denotes the contribution of order M^l, and the coefficients are given in terms of disorder-averaged correlation functions

$$X^{(2)}_{ab}(x_1,x_2)=\frac{\delta_{ab}}{\Gamma_t}\delta(x_1-x_2)-\{\chi^{(2)}_{ab}(x_1,x_2)\}_{\rm dis}, \quad (2.8b)$$

$$X^{(3)}_{abc}(x_1,x_2,x_3)=\{\chi^{(3)}_{abc}(x_1,x_2,x_3)\}_{\rm dis}, \quad (2.8c)$$

$$X^{(4)\alpha\beta}_{abcd}(x_1,x_2,x_3,x_4)=\delta_{\alpha\beta}\{\chi^{(4)}_{abcd}(x_1,x_2,x_3,x_4)\}_{\rm dis}+3\{\chi^{(2)}_{ab}(x_1,x_2)\chi^{(2)}_{cd}(x_3,x_4)\}^c_{\rm dis}. \quad (2.8d)$$

Again, the superscript c means that only the connected part, this time with respect to the disorder average, of the correlation functions $\{\cdots\}^c_{\rm dis}$ should be considered. Notice that we have separated the quantum mechanical and disorder averages, and therefore must deal with two different kinds of cumulants: In Eqs. (2.7), we have $\langle ab\rangle^c=\langle ab\rangle-\langle a\rangle\langle b\rangle$, etc., while in Eqs. (2.8) we have $\{ab\}^c_{\rm dis}=\{ab\}_{\rm dis}-\{a\}_{\rm dis}\{b\}_{\rm dis}$, etc.

C. Coefficients

As it stands, Eq. (2.8a) is just a formal nonlocal field theory that is not very useful. Normally, one would proceed by localizing the individual terms in Eq. (2.8a) about a single point in space and time, and expanding the correlation functions in powers of gradients. Due to what are often called weak localization effects,[14] this is not an option for the current case of a disordered metal, even if we stay far away from any metal-insulator transition. Structurally it is easy to show, using either many-body perturbation theory or field theoretic methods, that the disorder averaged n-point correlation function $\{\chi^{(l)}\}_{\rm dis}$ in wave number space has a weak localization correction of a form that can be written schematically as

$$\delta\chi^{(l)}(\mathbf{q},\Omega_n)\sim\frac{1}{V}\sum_{k>q}T\sum_{\omega_n>\Omega_n}\frac{1}{(\mathbf{k}^2+\omega_n)^l}, \quad (2.9)$$

with V the system volume, $k=|\mathbf{k}|$, ω_n and Ω_n Matsubara frequencies, and $(\mathbf{q},\Omega_n)$ an infrared cutoff. Although, strictly speaking, $\delta\chi^{(l)}$ depends on $l-1$ external momenta and frequencies, we have schematically represented these by a single ''typical'' momentum-frequency $(\mathbf{q},\Omega_n)$. Since we will only be interested in the scale dimension of $\{\chi^{(l)}\}_{\rm dis}$, this is sufficient. In terms of diagrams for the field theory reviewed in Ref. 8, the dominant contribution to $\{\chi^{(l)}\}_{\rm dis}(q,\Omega_n=0)$ for $q\to 0$ is shown in Fig. 1. For what

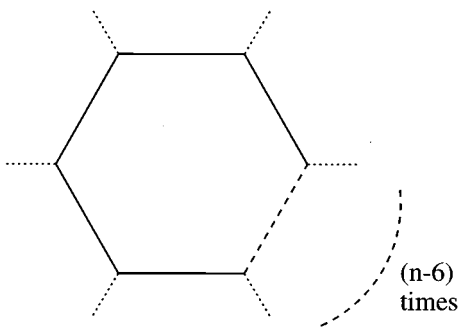

FIG. 1. Diagrammatic structure of the leading IR singular contribution to the correlation function $\chi^{(l)}(q,\Omega_n=0)$. The straight lines denote the diffusive propagators of the field theory reviewed in Ref. 8, and the dotted lines denote external legs.

follows it is very important to notice that $\delta\chi^{(l)}(q,\Omega_n=0)$ is a nonanalytic function of q.

To illustrate this point in detail, let us consider the two-point correlation function $\chi^{(2)}_{ab}$ in Eq. (2.8b). Because of translational invariance on average in space and time, as well as rotational invariance in space, we can write

$$\{\chi^{(2)}_{ab}(x_1,x_2)\}_{dis}=\delta_{ab}\chi_s(|\mathbf{x}_1-\mathbf{x}_2|,\tau_1-\tau_2), \quad (2.10a)$$

and we define the Fourier transform

$$\chi_s(\mathbf{q},\Omega_n)=\int d(\mathbf{x}_1-\mathbf{x}_2)d(\tau_1-\tau_2) \times e^{-i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)}e^{i\Omega_n(\tau_1-\tau_2)}\chi_s(x_1-x_2). \quad (2.10b)$$

χ_s is the disorder-averaged spin susceptibility of the reference system whose action is given by S_0. Particle number conservation implies that at small frequency and wave number it has a diffusive structure

$$\chi_s(\mathbf{q},\Omega_n)=\chi_0(q)\frac{Dq^2}{|\Omega_n|+Dq^2}, \quad (2.11a)$$

where D is the spin diffusion coefficient of the reference ensemble, and $\chi_0(q)$ is the static spin susceptibility. In a system with a conserved order parameter, the frequency must be taken to zero before the wave number in order to reach criticality, and so in the critical region we have $|\Omega_n|<<Dq^2$. Note that it is our restriction to nonmagnetic disorder that ensures a conserved order parameter. We will come back to this point below. In the critical limit, we can thus expand,

$$\chi_s(\mathbf{q},\Omega_n)=\chi_0(q)[1-|\Omega_n|/Dq^2+\cdots]. \quad (2.11b)$$

The static spin susceptibility $\chi_0(q)$ has been discussed in some detail before.[15] For any system with a nonvanishing interaction amplitude in the spin-triplet channel, $\Gamma_t\neq 0$, there is a diffusive or weak localization correction to the bare susceptibility. Although our bare reference ensemble does not contain such an interaction amplitude, perturbation theory will generate one as long as there is a nonvanishing interaction amplitude Γ_s in the spin-singlet channel.[15] Effectively, we therefore need to use the spin susceptibility for a system with a nonvanishing Γ_t in Eq. (2.11b). The explicit calculation is given in Appendix A, and the result is

$$\chi_0(q\to 0)=c_0-c_{d-2}q^{d-2}-c_2q^2+\cdots, \quad (2.12a)$$

where the c_i are positive constants. The nontrivial, and for our purposes most interesting, contribution in Eq. (2.12a) is the nonanalytic term $\sim q^{d-2}$. Its existence implies that the standard gradient expansion mentioned above Eq. (2.9) does not exist. The physical interpretation of this term is that, effectively, there is a long-range interaction between the order parameter fluctuations, which in real space takes the form r^{-2d+2}. This is a phenomenon that is special to zero temperature. At finite temperature, when one has to perform a frequency sum rather than a frequency integral to calculate the correlation funcion, the nonanalytic term is replaced by a term of the schematic structure

$$q^{d-2}\to(q^2+T)^{(d-2)/2}, \quad (2.12b)$$

and so for $T>0$ an analytic expansion about $q=0$ exists, and the standard local LGW functional is obtained.

Next we consider the cubic term in Eq. (2.8a). Rotational symmetry in spin space allows us to write Φ^{α}_3 as

$$\Phi^{\alpha}_3[\mathbf{M}]=\frac{1}{3!}\int dx_1dx_2dx_3u_3(x_1,x_2,x_3) \times\mathbf{M}^{\alpha}(x_1)\cdot[\mathbf{M}^{\alpha}(x_2)\times\mathbf{M}^{\alpha}(x_3)], \quad (2.13a)$$

with

$$u_3(x_1,x_2,x_3)=\frac{1}{6}\epsilon_{abc}\{\chi^{(3)}_{abc}(x_1,x_2,x_3)\}_{\rm dis}, \quad (2.13b)$$

where ϵ_{abc} is the Levi-Cività tensor. According to Eq. (2.9),

$$u_3(q\to 0,\Omega_n=0)=u_3^{(d-4)}q^{d-4}+u_3^{(0)}, \quad (2.13c)$$

with $u_3^{(d-4)}$ and $u_3^{(0)}$ finite numbers. Note that this term is divergent for $q\to 0$ for $d<4$.

The quartic term in Eq. (2.8a) is written as a sum of two terms,

$$\Phi^{\alpha}_4[\mathbf{M}]=\Phi^{\alpha(1)}_4[\mathbf{M}]+\Phi^{\alpha(2)}_4[\mathbf{M}], \quad (2.14a)$$

with

$$\Phi^{\alpha(1)}_4[\mathbf{M}]=-\frac{1}{24}\int dx_1\ldots dx_4\{\chi^{(4)}_{abcd}(x_1,\cdots,x_4)\}_{\rm dis} \times M^{\alpha}_a(x_1)M^{\alpha}_b(x_2)M^{\alpha}_c(x_3)M^{\alpha}_d(x_4), \quad (2.14b)$$

and

$$\Phi^{\alpha(2)}_4[\mathbf{M}]=\frac{1}{8}\int dx_1dx_2dx_3dx_4\{\chi^{(2)}_{ab}(x_1,x_2)\chi^{(2)}_{cd}(x_3,x_4)\}^{c}_{\rm dis} \times M^{\alpha}_a(x_1)M^{\alpha}_b(x_2)M^{\alpha}_c(x_3)M^{\alpha}_d(x_4). \quad (2.14c)$$

For our purposes we only need to know the degrees of divergence of the coefficients in Eqs. (2.14). We can therefore formally consider the local, static limit of Eq. (2.14b) (even though it does not necessarily exist), and write

$$\Phi^{\alpha(1)}_4[\mathbf{M}]=\frac{T}{24V}u_4\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\sum_{n_1,n_2,n_3} \times[\mathbf{M}^{\alpha}(\mathbf{q}_1,\omega_{n_1})\cdot\mathbf{M}^{\alpha}(\mathbf{q}_2,\omega_{n_2})] \times[\mathbf{M}^{\alpha}(\mathbf{q}_3,\omega_{n_3})\cdot\mathbf{M}^{\alpha}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3, -\omega_{n_1}-\omega_{n_2}-\omega_{n_3})], \quad (2.15a)$$

where

$$M^{\alpha}_a(\mathbf{q},\omega_n)=\frac{1}{\sqrt{\beta V}}\int d\mathbf{x}d\tau e^{-i\mathbf{q}\cdot\mathbf{x}+i\omega_n\tau}M^{\alpha}_a(\mathbf{x},\tau) \quad (2.15b)$$

and

$$u_4=-\lim_{\{\mathbf{q}\}\to 0}\{\chi^{(4)}_{aaaa}\}_{\rm dis}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3;\omega_{n_1}=\omega_{n_2}=\omega_{n_3}=0). \quad (2.15c)$$

According to Eq. (2.9),

$$u_4(q\to 0,\Omega_n=0)=u_4^{(d-6)}q^{d-6}+u_4^{(0)}, \qquad (2.15d)$$

and so u_4 diverges in this limit for $d<6$, and is finite for $d>6$. With a more accurate representation for u_4 than Eq. (2.9) one also finds a term $\sim q^{d-4}$, but this will be of no relevance for what follows.

We now consider $\Phi_4^{\alpha(2)}$ given by Eq. (2.14c). Its most interesting feature is its frequency structure that follows from $\{\chi_{ab}^{(2)}(x_1,x_2)\chi_{cd}^{(2)}(x_3,x_4)\}_{\rm dis}^c$ being a function of $\tau_1-\tau_2$ and $\tau_3-\tau_4$.[16] This implies that there are effectively two free τ integrals in $\Phi_4^{\alpha(2)}$, unlike the case of $\Phi_4^{\alpha(1)}$, where there is only one. In frequency space, $\Phi_4^{\alpha(2)}$ can be written

$$\begin{aligned}\Phi_4^{\alpha(2)}[\mathbf{M}]=-\frac{1}{8}\int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 \\ \times\sum_{n_1 n_2}\{\chi_{ab}^{(2)}(\mathbf{x}_1,\mathbf{x}_2;\omega_{n_1})\chi_{cd}^{(2)}(\mathbf{x}_3,\mathbf{x}_4;\omega_{n_2})\}_{\rm dis}^c \\ \times M_a^{\alpha}(\mathbf{x}_1,\omega_{n_1})M_b^{\alpha}(\mathbf{x}_2,-\omega_{n_1})M_c^{\alpha}(\mathbf{x}_3,\omega_{n_2}) \\ \times M_d^{\alpha}(\mathbf{x}_4,-\omega_{n_2}). \qquad (2.16)\end{aligned}$$

We see that $\Phi_4^{\alpha(1)}$ carries an extra factor of T compared to $\Phi_4^{\alpha(2)}$. The correlation function in Eq. (2.16) has been calculated for noninteracting disordered electrons.[17] The important result of these authors was that the Fourier transform of the correlation function at $\omega_{n_1}=\omega_{n_2}=0$ is finite. We have convinced ourselves that including interactions does not change this result. $\Phi_4^{\alpha(2)}[\mathbf{M}]$ can then be replaced by

$$\Phi_4^{\alpha(2)}[\mathbf{M}]=-\frac{v_4}{8}\int d\mathbf{x}\sum_{n_1 n_2}|\mathbf{M}^{\alpha}(\mathbf{x},\omega_{n_1})|^2|\mathbf{M}^{\beta}(\mathbf{x},\omega_{n_2})|^2, \qquad (2.17)$$

with v_4 a finite coefficient. The physical meaning of $\Phi_4^{\alpha(2)}$ is easily determined. Consider the local in space and time contribution to the disorder average of the coefficient $X^{(2)}$ in Eq. (2.8a), i.e., the term $t_0=1/\Gamma_t-\chi_0(q=0)$, which determines the distance from the critical point in the Gaussian theory. If we make t_0 a random function of space with a Gaussian distribution and integrate out that randomness, then we obtain a term with the structure of $\Phi_4^{\alpha(2)}$. This term in our action thus represents a ''random mass'' term, reflecting the fluctuations in the location of the critical point mentioned in the Introduction.

We conclude this section with a discussion of why the diagram shown in Fig. 1 gives the most important contribution to the correlation function $\{\chi^{(l)}\}_{\rm dis}$. The field theory of Ref. 8 allows for a systematic loop expansion for these correlation functions, with the diagram shown in Fig. 1 the one-loop contribution. If we consider higher-loop corrections to this, we need to distinguish between skeleton diagrams and insertions. Insertions will produce finite (in $d\geq 2$) renormalizations of the one-loop result. In the skeletons, each additional loop adds one frequency-momentum integral, and one independent propagator. Since the propagators are at most a diffusion pole squared,[8] this means that each loop adds effectively a factor of q^{d-2} to $\{\chi^{(l)}\}_{\rm dis}$, and hence we need to consider the lowest-loop contribution for a given type of term. For the static terms, i.e., those of zeroth order in the frequency Ω_n, the lowest nonvanishing contribution is at one-loop order since the zero-loop terms are of $O(\Omega_n)$. For instance, the zero-loop contribution to $\{\chi^{(2)}\}_{\rm dis}$ is $\sim|\Omega_n|/(|\Omega_n|+Dq^2)$, while the one-loop contribution adds the static piece to yield Eq. (2.11a). We thus conclude that the most infrared-divergent contribution to the static l-point correlation function is given by the diagram shown in Fig. 1 with dressed internal propagators, and for just getting the power of the leading divergence it suffices to use bare internal propagators.

In the critical limit the zero-loop contribution to $\{\chi^{(2)}\}_{\rm dis}$ is proportional to Ω_n/q^2; see Eq. (2.11b). In the next section we will discuss a fixed point where the dynamical exponent is $z=d$; i.e., Ω_n scales like q^d at criticality. In the critical limit, the term of $O(\Omega_n)$ therefore shows the same scaling behavior as the term of $O(\Omega_n^0)$, Eq. (2.12a). The same is true for all higher correlation functions. It is easy to show that the zero-loop contribution to $\{\chi^{(l)}\}_{\rm dis}$ goes like Ω_n/q^{2l-2}, while the one-loop contribution goes like $q^{d-(2l-2)}$. Anticipating again that Ω_n scales like q^d at criticality, these two contributions have the same scaling behavior for all values of l. All higher-loop contributions will be less important, as we have seen above. Also, higher orders in an expansion in powers of Ω can be neglected, since $\Omega_n\sim q^2$ in the diffusive propagators. Hence, each additional power of Ω_n will lead to a factor that scales like q^{d-2} at criticality and is less relevant than the terms of $O(\Omega_n^0)$ and $O(\Omega_n)$. We thus conclude that the most relevant term of $O(M^l)$ in the LGW functional has a coefficient u_l that behaves effectively like

$$u_l\sim q^{d-2(l-1)}. \qquad (2.18)$$

This covers both the terms of $O(\Omega_n^0)$ and $O(\Omega_n)$, and all higher powers of Ω_n are less important. The leading term for $q\to 0$, shown in Eq. (2.18), comes from the contribution of the lth moment to the cumulant $X^{(l)}$, i.e., from $\{\chi^{(l)}\}_{\rm dis}$, while the other contributions, i.e., the subtraction terms in a given cumulant, are less divergent. However, due to the frequency structure of these terms that was explained above using the example of $X^{(4)}$, it is not obvious that they are unimportant for the critical behavior. We will analyze this point in the next section, and will find that only the subtraction term in the quartic cumulant, which is given explicitly in Eq. (2.17), is important in that respect and thus needs to be kept.

III. CRITICAL BEHAVIOR

In the first part of this section we discuss the critical behavior of the Gaussian part of the theory defined by Eqs. (2.8) – (2.17). The renormalization group properties of the Gaussian fixed point are also discussed. We then analyze the non-Gaussian terms in the field theory, and show that they are irrelevant, in the renormalization group sense, with respect to the Gaussian fixed point for all dimensions $d>2$, except for a marginal operator in $d=4$. This implies that the Gaussian theory yields the exact critical behavior for all of these dimensions, except for logarithmic corrections to scaling in $d=4$ and $d=6$ that are discussed in Appendix B. We then construct the equation of state near the critical point, which requires a more detailed knowledge of the non-Gaussian terms in the field theory since it is determined in

part by dangerous irrelevant variables. We conclude this section with a discussion of the specific heat near the quantum critical point, and with a discussion of the quantum-to-classical crossover behavior.

A. Gaussian fixed point

According to Eqs. (2.8)–(2.12), the Gaussian part of Φ^α is

$$\Phi_2^\alpha[\mathbf{M}]=\frac{1}{2}\sum_{\mathbf{q}}\sum_{\omega_n}\mathbf{M}^\alpha(\mathbf{q},\omega_n)[t_0+a_{d-2}q^{d-2}+a_2q^2+a_\omega|\omega_n|/q^2]\cdot\mathbf{M}^\alpha(-\mathbf{q},-\omega_n), \tag{3.1a}$$

where

$$t_0=1-\Gamma_t\chi_s(\mathbf{q}\to 0,\omega_n=0) \tag{3.1b}$$

is the bare distance from the critical point, and a_{d-2}, a_2, and a_ω are positive constants.

We first analyze the critical behavior implied by Eqs. (3.1). Later we will show that for $d>2$ fluctuations are irrelevant, and the critical behavior found this way is exact for these dimensions. By inspection of the Gaussian LGW functional in Eq. (3.1a) one obtains

$$\nu=\begin{cases}1/(d-2) & \text{for } 2<d<4,\\ 1/2 & \text{for } d>4,\end{cases} \tag{3.2a}$$

$$\eta=\begin{cases}4-d & \text{for } 2<d<4,\\ 0 & \text{for } d>4,\end{cases} \tag{3.2b}$$

$$z=\begin{cases}d & \text{for } 2<d<4,\\ 4 & \text{for } d>4.\end{cases} \tag{3.2c}$$

Here ν is the correlation length exponent, defined by $\xi\sim t^{-\nu}$, with t the dimensionless distance from the critical point. η is the exponent that determines the wave number dependence of the order parameter susceptibility at criticality, $\langle M_a(\mathbf{q},0)M_a(-\mathbf{q},0)\rangle\sim\mathbf{q}^{-2+\eta}$. z is the dynamical scaling exponent that characterizes critical slowing down by relating the divergence of the relaxation time τ_r to that of the correlation length, $\tau_r\sim\xi^z$.

Let us discuss, for later reference, the critical behavior given by Eqs. (3.1) and (3.2) from a renormalization group point of view. Let b be the renormalization group length rescaling factor. Under renormalization, all quantities change according to $A\to A(b)=b^{[A]}A$, with $[A]$ the scale dimension of A. The scale dimension of the order parameter is

$$[\mathbf{M}(\mathbf{q},\omega_n)]=-1+\eta/2 \tag{3.3a}$$

or, equivalently,

$$[\mathbf{M}(\mathbf{x},\tau)]=(d+2)/2. \tag{3.3b}$$

At the critical fixed point, a_ω and either a_{d-2} (for $2<d<4$) or a_2 (for $d>4$) are not renormalized; i.e., there scale dimensions are zero. Using this and $[q]=1$, $[\omega_n]=z$ immediately yields Eqs. (3.2b), (3.2c). Equation (3.2a) follows from the relevance of t_0, or its renormalized counterpart t, at the critical fixed point. That is, the scale dimension of t is positive and given by $1/\nu\equiv[t]=2-\eta=d-2$.

B. Non-Gaussian terms

We now show that all of the non-Gaussian terms in the field theory are renormalization group irrelevant with respect to the Gaussian fixed point discussed in the last subsection. Let us first consider the term of order M^3. From Eq. (2.13c) we see that the most relevant coefficient at that order is $u_3^{(d-4)}$ for $d<4$ and $u_3^{(0)}$ for $d>4$. For simplicity, we will use the symbol u_3 for the most relevant coefficient in a given dimension; i.e., u_3 denotes $u_3^{(d-4)}$ for $d<4$ and $u_3^{(0)}$ for $d>4$. For $d<4$ we then have to assign a scale dimension to the cutoff wave number q. The most obvious choice is to identify q with the inverse correlation length ξ^{-1}, which makes $[q]=1$. While we will see later that this identification is not correct for all values of d, it will turn out that it provides an upper limit for $[q]$. Assuming, then, $[q]\leqslant 1$, which will be justified in Sec. III.C. below, and using Eq. (3.3), we see that the scale dimension of the effective coefficient of the term of order M^3 is bounded by

$$[u_3]\leqslant-(d-2)/2. \tag{3.4}$$

Similarly, using Eqs. (2.15d) and (3.3), one sees that the coefficient u_4 in Eq. (2.15a) has a scale dimension bounded by

$$[u_4]\leqslant\begin{cases}-(d-2) & \text{for } 2<d<4,\\ -2(d-3) & \text{for } 4<d<6,\\ -d & \text{for } d>6,\end{cases} \tag{3.5a}$$

and that the scale dimension of v_4 in Eq. (2.17) is

$$[v_4]=-|4-d|. \tag{3.5b}$$

Here again we denote the most relevant coefficient in Eq. (2.15d) by u_4 for simplicity. Equations (3.4) and (3.5) imply that the cubic and quartic terms in the field theory are renormalization group irrelevant with respect to the Gaussian fixed point for all $d>2$ except for $d=4$, where the coefficient v_4 is a marginal operator. As we explained after Eq. (2.17), v_4 reflects the ''random mass'' contribution to the action of the disordered magnet. We stress that this operator, while being strongly relevant with respect to the *clean* Gaussian fixed point, is not relevant with respect to the Gaussian fixed point of the present LGW functional.

It is an easy matter to analyze the behavior of the higher-order terms in the Landau expansion. The coefficient of the general term of order M^l in the LGW functional has a scale dimension bounded by

$$[u_l]\leqslant\begin{cases}-(d-2)(l-2)/2 & \text{for } 2<d<4,\\ -(l-2)-l(d-4)/2 & \text{for } 4<d<6,\\ -(l-4)-d(l-2)/2 & \text{for } d>6.\end{cases} \tag{3.5c}$$

This holds for the pure moment contribution to the lth cumulant, and we see that all of the u_l are irrelevant for $d>2$. For the coefficients of the subtraction terms in the cumulants one easily convinces oneself that, while their scale dimension initially increases with d increasing from $d=2$ [as does the scale dimension of v_4; see Eq. (3.5b)], it stays negative for all $d>2$. All of these terms are therefore irrelevant operators as well. We conclude that for $d>2$ there is a

critical Gaussian fixed point corresponding to a phase transition with Gaussian exponents (except for logarithmic corrections to the Gaussian critical behavior in $d=4$ due to the marginal operator v and, as we will see later, also in $d=6$), while for $d\leqslant 2$ the non-Gaussian terms are potentially relevant. We will see in Sec. III C below that the inequality in Eq. (3.5c) is actually an equality for $2<d<4$, so that $d=2$ is the upper critical dimension, and in $d=2$ an *infinite* number of operators become marginal.

C. Magnetization and the magnetic susceptibility

Although u_4 is an irrelevant operator for $d>2$, it is dangerously irrelevant[18] for the magnetization, since it determines the critical behavior of the average magnetization m as a function of t and an external magnetic field H. The technical reason for this is that m is a singular function of u for $u\to 0$. Schematically, the equation of state in mean-field theory is of the form

$$tm+u_4m^3=H, \tag{3.6}$$

where we have suppressed all numerical prefactors. According to Eq. (2.15d), u_4 diverges for $d<6$ as q^{d-6}. Furthermore, if we were to keep higher terms of order m^l in the equation of state, their coefficients would diverge as q^{d-2l}. This implies that the cutoff q scales like $m^{1/2}$, and effectively Eq. (3.6) reads

$$tm+\overline{u}m^{d/2}+u_4^{(0)}m^3=H, \tag{3.7}$$

with $u_4^{(0)}$ from Eq. (2.15d), and $\overline{u}$ another finite coefficient. From Eq. (3.7) we immediately obtain the exponents β and δ, defined as $m(t,H=0)\sim t^{\beta}$, $m(t=0,H)\sim H^{1/\delta}$,

$$\beta=\begin{cases} 2/(d-2) & \text{for } 2<d<6,\\ 1/2 & \text{for } d>6, \end{cases} \tag{3.8a}$$

$$\delta=\begin{cases} d/2 & \text{for } 2<d<6,\\ 3 & \text{for } d>6. \end{cases} \tag{3.8b}$$

In $d=6$ logarithmic corrections to scaling occur; see Appendix B.

Above we have used a general scaling argument to obtain Eq. (3.7). For small disorder, the same result can also be derived explicitly by means of an infinite resummation. As mentioned after Eq. (3.6), the term of $O(m^3)$ with its divergent coefficient $u_4\sim q^{d-6}$ in Eq. (3.6) is only the first in an infinite series of terms that behave like $q^{d-2l}m^l$. Calculating the prefactors of the divergent coefficients, one realizes that the divergences are the consequence of an illegal expansion of an equation of state of the form

$$tm+\frac{1}{V}\sum_{\mathbf{k}}\ T\sum_{\omega_n}\ \frac{\text{const}\times m^3}{[(\omega_n+k^2)^2+m^2]^2}=H. \tag{3.6$'$}$$

Performing the integral one recovers Eq. (3.7).

Next we determine the functional form of the equation of state at nonzero temperature in order to obtain a scaling equation for m as a function of t, H, and T. This can most easily be done by utilizing Eq. (2.12b). From Eq. (2.12a) it follows that the tm term in the equation of state has a correction of the form $m(q^2+T)^{(d-2)/2}\sim m(m+T)^{(d-2)/2}$, where we have used $q^2\sim m$ as explained after Eq. (3.6). Similarly, the term um^3 in Eq. (3.6) with $u\sim q^{d-6}$ gets replaced by $m^3(m+T)^{(d-6)/2}$. At $T=0$ we recover Eq. (3.7). For $T<m$ in suitable units, there are corrections of $O(T/m)$ to the term $\sim m^{d/2}$ in that equation, while for $T>m$, t gets replaced by $t+T^{1/2\nu}$. All of these observations can be summarized in the homogeneity law

$$m(t,T,H)=b^{-\beta/\nu}m(tb^{1/\nu},Tb^{\phi/\nu},Hb^{\delta\beta/\nu}), \tag{3.9a}$$

with

$$\phi=2\nu. \tag{3.9b}$$

Similarly, the magnetic susceptibility χ_m satisfies a homogeneity law

$$\chi_m(t,T,H)=b^{\gamma/\nu}\chi_m(tb^{1/\nu},Tb^{\phi/\nu},Hb^{\delta\beta/\nu}), \tag{3.10a}$$

with

$$\gamma=\beta(\delta-1)=1. \tag{3.10b}$$

Equations (3.10) and (3.11) warrant some discussion. The scaling of T in these equations follows directly from Eq. (2.12b). The effective scale dimension of T in m and χ_m is therefore 2 and *not* z. The salient point is that z is determined by the scaling of Ω_n or T with q in the Gaussian action, and hence in the critical propagator. However, the magnetization is calculated at $\Omega_n=q=0$, and its leading temperature dependence is determined by the diffusive modes, which feature $\Omega_n\sim T\sim q^2$, rather than by the critical ones. This leads to $[T]=\phi/\nu$, with ϕ as given in Eq. (3.10b). The proper interpretation of ϕ is that of a crossover exponent associated with the crossover from the quantum to the thermal fixed point that occurs at any $T>0$. Since $z>\phi/\nu$, the critical scaling $T\sim b^z$ would be the dominant temperature dependence if m and χ_m depended on the critical modes. That they do not can also been seen from a determination of the magnetic susceptibility directly from the Gaussian action: Recognizing that the coefficient of M^2 in Φ_2, Eq. (3.1a), is the inverse spin susceptibility, and using Eq. (2.12b) again, we obtain

$$\chi_m(t,T)=\frac{1}{t+T^{1/2\nu}}, \tag{3.10c}$$

in agreement with Eqs. (3.10a), (3.10b).

We are now in a position to determine the exact scale dimension of u_4, and of the other coefficients in the field theory, and to thus verify the assumption made in the last subsection. As we have seen after Eq. (3.6), the cutoff q scales like $m^{1/2}$, so that $[q]=[m]_{\text{eff}}/2$, where $[m]_{\text{eff}}=\beta/\nu$ is the *effective* scale dimension of m, i.e., the scale dimension after the effects of the dangerous irrelevant variable u_4 have been taken into account. From Eqs. (3.8a), (3.2a) we see that $[m]_{\text{eff}}\leqslant 2$, which justifies the assumption made in Sec. III B that led to the upper bounds on the scale dimensions of the u_l. Repeating the power counting arguments that led to the inequalities, Eqs. (3.4) and (3.5a), we obtain

$$[u_3]=-(d-2)/2 \tag{3.4$'$}$$

and

$$[u_4]=\begin{cases} -(d-2) & \text{for } 2<d<4, \\ -(d^2-12)/(d-2) & \text{for } 4<d<6,' \\ -d & \text{for } d>6. \end{cases} \quad (3.5')$$

The exact scale dimension of the general coefficient u_l, if desired, can be obtained by the same argument. Again, the upper bound given in Eq. (3.5c) is equal to the exact value for $2<d<4$. These results complete the proof that $d=2$ is the upper critical dimension for our problem, and that an infinite number of operators become marginal in $d=2$, and relevant in $d<2$.

Finally, we mention how to reconcile the value for the exponent β, Eq. (3.8a), with scaling. Putting $T=H=0$ for simplicity, but keeping the dependence of m on u_4 explicitly, and using Eq. (3.3b), we write

$$m(t,u_4)=b^{-(d+2)/2}m(tb^{1/\nu},u_4 b^{[u_4]}). \quad (3.9')$$

From Eq. (3.6) we know that $m\sim u_4^{-1/2}$. Using this in Eq. (3.9a) changes the scale dimension of m from $[m]=(d+2)/2$ to an effective value, $[m]_{\text{eff}}=[m]-[u_4]/2$. With Eq. (3.5′) we then obtain $[m]_{\text{eff}}=\beta/\nu$ with the correct values of β and ν.

D. Specific heat

The scaling equation for the specific heat is determined by the sum of the mean-field and the Gaussian fluctuation contributions to the free energy density, f. The mean-field contribution follows immediately from Eq. (3.7). The Gaussian fluctuation contribution f_G, which gives the leading nonanalyticity at the critical point, can be calculated by standard methods.[19] Neglecting an uninteresting constant contribution to f_G, we obtain

$$f_G=\frac{T}{2V}\sum_{\mathbf{q},\omega_n}\{2\ln(H/m+a_{d-2}q^{d-2}+a_2q^2+a_\omega|\omega_n|/q^2) + \ln[x_dH/m-(x_d-1)t+a_{d-2}q^{d-2}+a_2q^2 + a_\omega|\omega_n|/q^2]\}. \quad (3.11)$$

Here $x_d=d/2$ for $2<d<6$ and $x_d=3$ for $d>6$. The specific heat coefficient γ_V is conventionally defined by

$$\gamma_V=c_V/T=-\partial^2 f/\partial T^2. \quad (3.12)$$

Again we are interested only in scaling properties and not in exact coefficients. Schematically, Eqs. (3.11) and (3.12) give

$$\gamma_V=\int_0^\Lambda dq\,\frac{q^{d-1}}{T+q^d+q^4+Hq^2/m}, \quad (3.13)$$

with Λ an ultraviolet cutoff.

Several points should be noted. First, for all d that obey $2<d<4$, γ_V given by Eq. (3.13), or exactly by Eqs. (3.11) and (3.12), is logarithmically singular for $T,H\to 0$. This d-independent logarithmic singularity is somewhat unusual. Wegner has discussed how logarithmic corrections to scaling arise if a set of scale dimensions fulfills some resonance condition.[20] In the present case the appearance of a logarithm can be traced to the fact that the scale dimension of the free energy, $d+z=2z$ for $2<d<4$, is a multiple of the scale dimension of T, which is $z=d$ in this region. The unusual feature of the logarithm appearing in a *range* of dimensions, rather than only for a special value of d, is due to the dynamical exponent being exactly d in that range. Second, in Eq. (3.13) two different temperature scales appear. The first two terms in the denominator indicate that $T\sim\xi^{-d}$, as one would expect from dynamical scaling. However, the last term in Eq. (3.13) contains the magnetization, which in turn depends on the crossover temperature scale $T\sim\xi^{-2}$; see Sec. III C above. These two features imply that the scaling equation for γ_V should be written

$$\gamma_V(t,T,H)=\Theta(4-d)\ln b+F_\gamma(tb^{1/\nu},Tb^z,Hb^{\delta\beta/\nu},Tb^2). \quad (3.14)$$

Since $z>2$, one can formally ignore the fourth entry in the scaling function since it is subleading compared to the second entry. The corrections to the resulting theory can be considered as ''corrections to scaling.'' Notice that in contrast to the magnetization and the magnetic susceptibility, the specific heat does depend on the critical modes, and hence contains the critical temperature scale. As mentioned in the last subsection, the latter is dominant when it is present, and γ_V provides an example for that.

IV. DISCUSSION

A. Theoretical aspects

In this paper we have shown that in disordered itinerant quantum ferromagnets the diffusive nature of the electrons leads to long-range interactions between spin fluctuations in an order parameter field theory. As for classical models with long-range interactions,[21] the critical behavior of this field theory can be determined exactly. For $d>6$, standard mean-field results are obtained, but for $2<d<6$ one finds nontrivial critical behavior with dimensionality-dependent exponents. For $d\leqslant 2$ our approach breaks down because the electrons are localized, and because in our field theory an infinite number of operators becomes marginal in $d=2$. The exact critical exponents for $d>2$ are given by Eqs. (3.2), (3.8), and (3.10b), and the scaling properties of some of the more interesting physical properties are given by Eqs. (3.9), (3.10a), and (3.14). In this subsection we discuss various aspects of these results that have not been covered yet.

First of all, there is an important conceptual question that should be considered. In our approach we have assumed that it is sensible to construct an order parameter field theory to describe the critical behavior of the order parameter. In general this procedure will break down if there are other soft or slow modes that couple to the order parameter fluctuations. That is, a more complete low-energy theory might be needed. In the present case, the diffusons that lead to the nonanalyticities in the bare field theory and to, e.g., Eqs. (2.12) and (2.15) are such soft modes. One should ask why we were able to proceed with an order parameter description anyway, without running into unsurmountable difficulties due to the additional soft modes. A technical answer is that the diffusons did create problems, namely, divergent coefficients in the LGW functional, but that for the present problem these difficulties could be dealt with. Nevertheless, one

might wonder what the theory would look like if the critical modes and soft diffusion modes were treated on a more equal footing.

Without realizing it, we have previously addressed the above question. Early work on the metal-insulator transition in disordered interacting electron systems showed that in two-dimensional systems without impurity spin-flip scattering, the triplet interaction scaled to large values under renormalization group iterations.[22] This was interpreted, incorrectly as it turned out later, as a signature of local moment formation in all dimensions.[23] Subsequently, the present authors studied this problem in some detail.[24] We were able to explicitly resum the perturbation theory and show that at a critical value of the interaction strength, or of the disorder, there is a bulk, thermodynamic phase transition. The physical meaning of this phase transition was obscure at the time since no order parameter had been identified, and its description was entirely in terms of soft diffusion modes. However, the critical exponents obtained are identical to those obtained here for the quantum ferromagnetic phase transition, and in both cases logarithmic corrections to scaling are found. Because the exponents in the two cases are identical, we conclude that the transition found earlier by us, whose physical nature was unclear, is actually the ferromagnetic transition. One also concludes that our speculations about the nature of the ordered phase as an ''incompletely frozen spin phase'' with no long-range magnetic order were not correct. On the other hand, the techniques used in Ref. 24 allowed for a determination of the qualitative phase diagram as a function of dimensionality, which our present analysis is not capable of. The theory given here not only explains the nature of the transition, but also explains why the critical behavior at that phase transition could be obtained exactly in three dimensions: The long-range nature of the interactions between the order parameter fluctuations makes the critical phenomena problem exactly soluble. It is also interesting to note that the list of scaling scenarios for soft-mode field theories for disordered interacting fermions given in Sec. IV of Ref. 8 included the present case, namely, a transition to a ferromagnetic state with an order parameter exponent $\beta=2\nu$ for $d<4$.

It should also be pointed out that our earlier theory depended crucially on there being electronic spin conservation. This feature would be lost of there were some type of impurity spin-flip scattering process. In that case, the soft modes that lead to the long-range order parameter interactions acquire a mass or energy gap, and at sufficiently large scales the interactions are effectively of short range. The asymptotic critical phenomena in this case are described by a short-range, local order parameter field theory with a random mass, or temperature, term. In this case the random mass term *is* relevant with respect to the Gaussian fixed point analogous to the one discussed here, which underscores the important role that is played by the order parameter being conserved in our model. The quantum phase transition in a model where it is not, and where the random mass term is therefore relevant with respect to the analog of our Gaussian fixed point (FP), is discussed elsewhere.[25] We also mention that the effect of fermionic soft modes on the ferromagnetic phase transition in *clean* systems has been discussed recently in Ref. 3.

Even though we quote exponent values formally for all $d>2$, it should be mentioned that the region of validity of our analysis shrinks to zero as d approaches 2. As mentioned in Sec. II C, the reference system has all of the characteristics of the system described by the full action S, except that it must not undergo a phase transition lest our separation of modes that is implicit in our singling out $S^t_{\rm int}$ for the decoupling procedure break down. This requirement puts restrictions on the parameter values for which our procedure works. For instance, we cannot go to arbitrarily large disorder (at fixed d) without triggering a metal-insulator transition within the reference system. For $d\to 2$ the metal-insulator transitions occurs at smaller and smaller values of the disorder, and in $d=2$ one obtains a very complicated, and unsolved, situation where various fluctuations compete with each other.

We finally discuss why some of our results are in disagreement with Sachdev's[2] recent general scaling analysis of quantum phase transitions with conserved order parameters. For instance, it follows from our Eqs. (3.10), (3.14) that the Wilson ratio, defined as $W=(m/H)/(C_V/T)$, diverges at criticality rather than being a universal number as predicted in Ref. 2. Also, for $2<d<4$ the function F_γ in Eq. (3.18), for $t=0$ and neglecting corrections to scaling, is a function of T/H, in agreement with Ref. 2, but for $d>4$ this is not the case. The general reason for this breakdown of general scaling is that we work above an upper critical dimensionality, and hence dangerous irrelevant variables have to be considered very carefully, and on a case-by-case basis. This caveat is particularly relevant for quantum phase transitions since they tend to have a low upper critical dimension. It is well known that a given irrelevant variable can be dangerous with respect to some observables but not with respect to others. Specifically, in our case the dangerously irrelevant variable u_4 affects the leading scaling behavior of the magnetization, but not that of the specific heat coefficient, which leads to the divergence of the Wilson ratio. A simple example of this phenomenon is provided by classical ϕ^4 theory in $d>4$, where the dangerous irrelevant variable u (the coefficient of the ϕ^4 term) affects the scale dimension of the magnetization, but not that of the specific heat.[19] In classical ϕ^4 theory this point is obscured by the fact that the saddle-point contribution to the specific heat contains a discontinuity. This is often expressed as $\alpha=0$, with α the specific heat exponent. However, the approach to the discontinuity is described by the α suggested by hyperscaling, namely, $\alpha=2-d/2$; see Chap. VII.4 of Ref. 19. At the quantum FP the situation is clearer, since the saddle-point contribution to C_V is subleading. It is also important to remember that different arguments of a scaling function can be affected in different ways by one and the same dangerous irrelevant variable. Here, the effective scale dimension of H in the specific heat is changed by u_4 [from $(3d-2)/2$ to d in $2<d<4$], but that of T is not, since u_4 imports only the subleading crossover temperature scale into γ_V via the appearance of m in Eq. (3.13).

B. Experimental aspects

In order to apply our theoretical results to experiments, one needs materials that show a transition from a paramagnetic state to a ferromagnetic one at zero temperature as a function of some parameter x. Obvious candidates are mag-

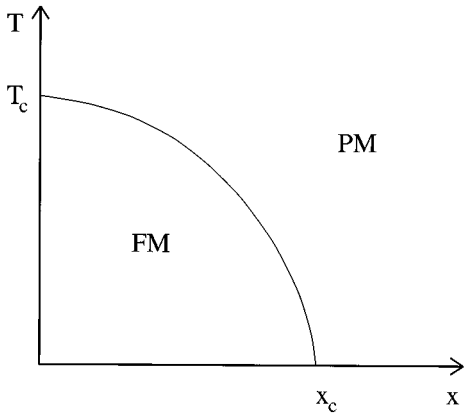

FIG. 2. Schematic phase diagram for an alloy of the form P_xF_{1-x}. T_c is the Curie temperature for the pure ferromagnet F, and x_c is the critical concentration.

netic alloys of the form P_xF_{1-x} with P a paramagnetic metal and F a ferromagnetic one. Such materials show the desired transition as a function of the composition parameter x; examples include $F=\mathrm{Ni}$ and $P=\mathrm{Al, Ga}$.[11] The schematic phase diagram is shown in Fig. 2.

One striking difference between our results for the quantum ferromagnetic phase transition at $(x=x_c, T=0)$ and the classical or thermal transition for Heisenberg ferromagnets is the numerical values of the exponents. For three-dimensional (3D) systems, our Eqs. (3.8), for instance, predict $\beta=2$, while the corresponding value for the thermal transition is[26] $\beta_{\mathrm{class}}\approx 0.37$. The large difference between the classical and the quantum value should be easily observable if it was possible to measure the magnetization at a sufficiently low temperature as a function of x in order to observe the crossover between quantum and classical critical behavior in the vicinity of x_c. One possible way to do such an experiment would involve the preparation of many samples with different values of x over a small region of x. It might also be possible to probe the magnetic phase transition by using the stress tuning technique that was used to study the metal-insulator transition in Si:P.[28] Alternatively, one could prepare a sample with a value of x that is as close as possible to x_c, and measure the magnetic field dependence of the magnetization, extrapolated to $T=0$, to obtain the exponent δ. Again, there is a large difference between our prediction of $\delta=1.5$ in $d=3$, and the classical value $\delta_{\mathrm{class}}\approx 4.86$.

Another possibility, that does not involve an extrapolation to $T=0$, is to measure the zero-field magnetic susceptibility as a function of both $t=|x-x_c|$ and T. Equation (3.10a) predicts

$$\chi_m(t,T)=T^{-1/2}f_\chi(T/t^2). \tag{4.1}$$

Here f_χ is a scaling function that has two branches, f_χ^+ for $x>x_c$ and f_χ^- for $x<x_c$. Both branches approach a constant for large values of their argument, $f_\chi^\pm(y\to\infty)=\mathrm{const}$. For small arguments, we have $f_\chi^+(y\to 0)\sim\sqrt{y}$, while f_χ^- diverges at a nonzero value y^* of its argument, which signalizes the classical transition $f_\chi^-(y\to y^*)\sim(y-y^*)^{-\gamma_{\mathrm{class}}}$, with $\gamma_{\mathrm{class}}\approx 1.39$ the susceptibility exponent for the classical transition. Our prediction is then that a plot of $\chi_m T^{1/2}$ versus

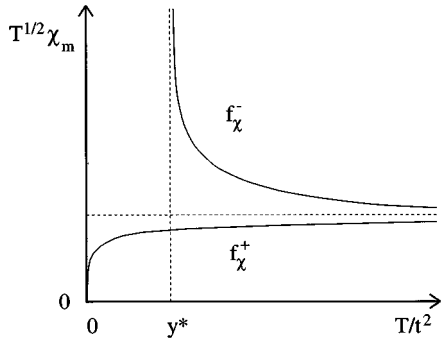

FIG. 3. Schematic prediction for a scaling plot of the magnetic susceptibility.

T/t^2 will yield a universal function the shape of which is schematically shown in Fig. 3. Notice that the exponents are known *exactly*, and so the only adjustable parameter for plotting experimental data will be the position of the critical point. This is in sharp contrast to some other quantum phase transitions, especially metal-insulator transitions, where the exponent values are not even approximately known, which makes scaling plots almost meaningless.[29]

Finally, one can consider the low-temperature behavior of the specific heat. According to Eq. (3.18), as the temperature is lowered for $x\gtrsim x_c$ the leading temperature dependence of the specific heat will be

$$c_V(T)\sim T\ln T. \tag{4.2a}$$

At criticality this behavior will continue to $T=0$, while for $x>x_c$ it will cross over to

$$c_V(T)\sim(\ln t)T. \tag{4.2b}$$

For $x\lesssim x_c$ one will encounter the classical Heisenberg transition where the specific heat shows a finite cusp (i.e., the exponent $\alpha<0$).

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APPENDIX A: WAVE-NUMBER-DEPENDENT SPIN SUSCEPTIBILITY

Here we calculate the wave-number-dependent spin susceptibility in a disordered interacting Fermi system. In terms of the Q-matrix field theory reviewed in Ref. 8, the spin susceptibility reads

$$\chi_s(\mathbf{q},\Omega_n)\sim\sum_{r=0,3}(-)^r T\sum_{m_1,m_2}\langle {}^3_rQ^{\alpha\alpha}_{m_1+n,m_1}(\mathbf{q}) \times {}^3_rQ^{\alpha\alpha}_{m_2-n,m_2}(-\mathbf{q})\rangle. \tag{A1}$$

The classical matrix field Q comprises two fermionic degrees of freedom, and a general matrix element ${}^i_rQ^{\alpha\beta}_{nm}$ has six

indices: i is the spin index (with $i=1,2,3$ the spin-triplet channel), $r=0,3$ denotes the particle-hole channel ($r=1,2$ would be the particle-particle channel), n,m are Matsubata frequency indices, and α,β are replica indices. The matrix elements with $nm<0$ describe the soft modes, i.e., the particle-hole excitations, while those with $nm>0$ are massive. For explicit calculations it is convenient to consider the nonlinear σ-model version of the Q-field theory,[27,8] where the massive modes are integrated out. In the resulting effective model, $Q^2=1$, and a loop expansion can be set up by expanding all Q in terms of $q_{nm}=\Theta(-nm)Q_{nm}$. The contribution of $O(q^2)$ to Eq. (1.1) vanishes for $\Omega_n=0$ and hence does not contribute to $\chi_0(\mathbf{q})=\chi_s(\mathbf{q},\Omega_n=0)$. The one-loop contribution, which is the term of $O(q^4)$, gives

$$\chi_0(\mathbf{q})\sim\frac{1}{V}\sum_{\mathbf{k}}\ T\sum_{n}\ \omega_n\mathscr{D}_n^t(\mathbf{k})\mathscr{D}_n(\mathbf{k})$$
$$\times[\mathscr{D}_n(\mathbf{k}-\mathbf{q})-\mathscr{D}_n^t(\mathbf{k}-\mathbf{q})]. \qquad \text{(A2a)}$$

Here $\mathscr{D}_n$ and $\mathscr{D}_n^t$ are the diffusive propagators of the theory.[8] Their structure is

$$\mathscr{D}_n(\mathbf{k})=1/(k^2+D^{-1}\omega_n), \qquad \text{(A2b)}$$

with D a diffusion coefficient. $\mathscr{D}_n^t$ has the same structure, with D replaced by $D^t\neq D$. Since we are not interested in prefactors, we do not have to specify either D and D^t, or the prefactor in Eq. (A2a). For the reasons discussed at the end of Sec. II C, the one-loop term suffices to calculate the leading infrared wave number dependence of χ_0. Schematically, Eqs. (A2) yield

$$\chi_0(\mathbf{q})\sim\int_q^\Lambda dk k^{d-1}\int_0^\infty d\omega\,\frac{\omega}{(k^2+\omega)^3}, \qquad \text{(A3)}$$

with Λ an ultraviolet cutoff, from which one readily obtains Eq. (2.12a).

APPENDIX B: LOGARITHMIC CORRECTIONS TO SCALING IN $D=4$ AND $D=6$

There are three distinct mechanisms that produce logarithmic corrections to scaling: (1) Marginal operators, (2) Wegner resonance conditions between a set of scale dimensions, and (3) logarithmic corrections to the scale dimension of a dangerous irrelevant operator. The first two mechanism are well known.[20] The third is operative only above an upper critical dimension, and is therefore of particular interest for quantum phase transitions.

In the present case, logarithmic corrections to scaling arise due to all three of these mechanisms. The second one produces corrections to the scaling of the specific heat in all dimensions $2<d<4$, as was discussed in Sec. III D. The first one is operative in $d=4$, where v_4 is a marginal operator; see Eq. (3.5b). If desired, the resulting corrections to scaling can be worked out using standard techniques.[20] Finally, the third mechanism produces corrections to scaling in $d=6$. According to Eq. (2.9), the coefficient $u_4\sim\ln q$ in $d=6$. Via Eq. (3.6) or (3.7) this leads, for instance, to a leading behavior of the spontaneous magnetization,

$$m(t,H=0)\sim\frac{t^{1/2}}{\sqrt{\ln(1/t)}}, \qquad \text{(B1)}$$

and at the critical point we have

$$m(t=0,H)\sim\frac{H^{1/3}}{[\ln(1/H)]^{1/3}}. \qquad \text{(B2)}$$

Other consequences, e.g., for the specific heat in a magnetic field, can be easily worked out.

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