Localization and mesoscopic persistent current in a disordered metal ring

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A simple model for an electron moving in a disordered one-dimensional system is investigated analytically and the persistent current in this model of a disordered mesoscopic system is studied. We show that persistent currents decrease exponentially with increasing sample circumference. Our analysis is based on comparison with an exact calculation of the average resistance of a disordered one-dimensional chain. [S0163-1829(96)07219-0]

Recently, considerable effort has been made in understanding the nature of electronic states in one dimensional (1D) disordered systems.[1–4] This is partly due to the connection with the problem of two-dimensional electronic systems in a magnetic field and electrons in the presence of the random barriers, especially mesoscopic quantum interference effect contributions to the conductivity.

The investigation of the electrical conductivity of 1D metal with static disorder was initiated by Landauer,[5] who derived a connection between total transmission and resistance. The averaged resistance is $\langle\rho\rangle \sim (e^{L/\xi}-1)$, where ξ is the localization length of the localized states. The electron localization was discussed by Anderson *et al.*[6] in the framework of scaling theory for 1D disorder systems. Thouless[7] has proposed a relation between the density of states and the electron localization length. Since then, several results on the relation between the conductance and the transmission coefficient of the chain have been established numerically and analytically to support the idea of localization.[3,4,8,9] On the other hand, the Kubo formula for the conductivity was used for a tight binding model with random site energies.[10] With the size L of the sample as a scaling parameter, Ahrahams and Stephen[11] examined the changes in conductivity for a small size sample.

The persistent current in 1D disordered mesoscopic ring has been the subject of much interest now. In close analogy to the behavior of localization dependence in the resistance, it was theoretically predicted that the averaged current amplitude would decrease as a function of degrees of disorder for a disordered ring. The current amplitude is found to be $I \sim I_0 e^{-L/\xi}$.[12] This behavior can be understood as the eigenfunctions of electrons in the disordered system, which are exponentically localized. Such a localization dependence has further been discussed by using the concept of one-electron localization[13] and by the single band spinless fermion model[14] in a one-dimension disordered ring. According to these results, the disorder tends to localize the electrons in the lowest energy site in the sample and, consequently, reduces the presistent current.

In this paper, we attempt to examine the issue of exponential decay behavior of persistent current by analytically calculating the averaging persistent current over an ensemble, with a fixed number of electrons for an exactly solvable 1D disorder model. Unlike the previous discussions,[12–14] and in order to show the influence of the localization effect on the persistent current in 1D disordered mesoscopic ring, we solve the generalized Kronig-Penny ring consisting of N arbitrary barriers arranged periodically on the circumference. We thought that it might be of some interest to use the technique developed in the study of resistance fluctuation to show how the electron localization entered when the system is disordered. We consider the one-dimensional ring geometry with a magnetic flux Φ threading through it. Our approach is based on the method used in Refs. 3,4, where the localization length was calculated for the uniformly distributed amplitude of the δ-function potential. We work in terms of a transmission coefficient, the modulus squared of which is inverse proportional to the resistence for the model. We then apply this to calculate averaged persistent current $\bar{I}$. We can obtain an expression for the persistent current in terms of the transmission coefficient of the system. According to the localization theory, the localization length ξ can be expressed in terms of the resistance, which could further be calculated through the transmission coefficient by using Landauer formula. Consequently, the amplitude of persistent current can be estimated through the modulus of the transmission coefficient in terms of localization length, ξ. Hence, we obtain the dependence of persistent current to ξ and ring circumference L. For clarity and convenience, we convert the problem on a ring to a problem on a line with period L. We used an exact solvable model, which differs from the previous ones in literature and this model is solved analytically.

The Hamiltonian for the system, which consists of uniformly spaced δ-function potential of random strengths, reads

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\sum_{n=0}^{N}\lambda_n\delta(x-na), \tag{1}$$

where m is the bare electron mass and λ_n is potential-strength parameter, which may be considered as an on-site diagonal disorder in the tight binding model, and a is the lattice spacing. We have not written the magnetic flux in the Hamiltonian (1) explicitly, but take the flux into account by requiring a twisted periodic condition for the wave function

$$\psi(x+L)=\exp(i\,2\pi\Phi/\Phi_0)\,\psi(x), \tag{2}$$

where Φ_0 is the flux quantum hc/e and $L=Na$, meaning that a phase would arise when the electron is brought around the ring. The energy for an electron is $E=(\hbar^2/2m)k^2$. The constraint for k would be derived later by the boundary conditions given in Eq. (4) and the quantization rule in Eq. (17).

The electron wave function in the regions without potentials can be written as $\psi(x)=\psi_+(x)+\psi_-(x)$, where

$$\psi_+(x)=A_ne^{ikx},\quad \psi_-(x)=B_ne^{-ikx}. \tag{3}$$

The coefficients A_n and B_n across site $x=na$ are related through the transfer matrix T_n, involving the transmission and reflection amplitudes from the barrier on the site $x=na$, t_n, and r_n. On each site, $x=na$, the wave function acquired a phase $2\pi\Phi/N\Phi_0$ when the evoluation of the electron across the site was due to the requirement of single valueness for the wave function. This can be understood as follows. We start from a point x on the positive side of na then, after going around the ring one returns to the same point from the negetive side of na. The boundary conditions connecting the wave functions across site $x=na$ should be

$$\lim_{\epsilon\to 0}\{\psi_+(na+\epsilon)+\psi_-(na+\epsilon)-\psi_+(na-\epsilon)e^{i(2\pi/N)(\Phi/\Phi_0)}$$
$$-\psi_-(na-\epsilon)e^{(i2\pi/N)(\Phi/\Phi_0)}=0 \tag{4a}$$

and

$$\lim_{\epsilon\to 0}\{\psi'_+(na+\epsilon)+\psi'_-(na+\epsilon)-\psi'_+(na-\epsilon)e^{i(2\pi/N)(\Phi/\Phi_0)}$$
$$-\psi'_-(na-\epsilon)e^{i(2\pi/N)(\Phi/\Phi_0)}\}$$
$$=(2m/\hbar^2)\,\lambda_n\{\psi_+(na)+\psi_-(na)\}e^{i(2\pi/N)(\Phi/\Phi_0)}. \tag{4b}$$

These conditions must be satisfied by the electron crossing the site occupied by the barriers. By performing the transfer matrix procedures to the whole ring successively, the coefficients A and B corresponding to the ends of the line L can be obtained analytically. The twisted periodic condition (2) leads to an equation for these coefficients defined on the ends of the line:

$$\begin{pmatrix}A'\\B'\end{pmatrix}=T\begin{pmatrix}A\\B\end{pmatrix}$$
$$=\exp\left\{i2\pi\frac{\Phi}{\Phi_0}+i[(-1)^M-1]\frac{\pi}{2}\right\}\widetilde{T}\begin{pmatrix}A\\B\end{pmatrix},$$
$$\widetilde{T}=\begin{pmatrix}1/t^* & r/t\\ r^*/t^* & 1/t\end{pmatrix}, \tag{5}$$

where the total tranfer matrix $T=\Pi_{n=1}^{N}T_n$, and we have extended the twisted periodic condition for M electrons on the ring. The second factor in the phase is associated with a translation of an electron along the ring, which would cross other $M-1$ electrons. M is the number of electrons on the ring. The periodic conditions $A'=A$ and $B'=B$ reduce to the quantization rule for k, in the form

$$\mathrm{Re}(1/t)=\cos\{2\pi\Phi/\Phi_0+[(-1)^M-1]\pi/2\}$$
$$=\begin{cases}\cos 2\pi\Phi/\Phi_0 & \text{for } M=\text{odd}\\ \cos 2\pi(\Phi/\Phi_0-\frac{1}{2}) & \text{for } M=\text{even}\end{cases} \tag{6}$$

which agrees with previous work.[15] Since $D_N=t^{-1}$, while the amplitudes of transmission t and reflection r are related by $|t|^2+|r|^2=1$, the transfer materix $\widetilde{T}$ can be expressed as

$$\widetilde{T}=\begin{pmatrix}D_N^* & R_N\\ R_N^* & D_N\end{pmatrix}, \tag{7}$$

where

$$D_N=\left(\alpha_N-\frac{\alpha_{N-1}^*\beta_N}{\beta_{N-1}^*}\right)D_{N-1}+\left(\frac{\beta_N}{\beta_{N-1}^*}\right)D_{N-2} \tag{8}$$

and

$$R_N=e^{i2k(N-1/2)a}\frac{\alpha_N^*}{\beta_N}D_N-e^{i2k(N-1/2)a}\frac{1}{\beta_N}D_{N-1}, \tag{9}$$

with the definitions

$$\alpha_n=1-i\frac{1}{2k}V_n,\quad \beta_n=-i\frac{1}{2k}V_ne^{-ika},\quad n>1. \tag{10}$$

$D_0=1$ and $D_1=\alpha_1$, where $k_l=(2mE_l/\hbar^2)^{-1/2}$ and $V_n=2m\lambda_n/\hbar^2$. D_N can be further expressed in a form

$$D_N=\left\{1+\sum_{p}^{N}\sum_{n_p>\cdots>n_1=1}^{N}\frac{(-iV_{n_1})}{2k}\cdots\frac{(-iV_{n_p})}{2k}\right.$$
$$\left.\times\prod_{l=1}^{p-1}\{1-\exp[-i2ka(n_{l+1}-n_l)]\}\right\}, \tag{11}$$

where n runs from 1 to N. In order to make a comparison of orderness to disorderness in the influence on the persistent currents, we at first restrict ourselves to the simple case[3,4] that all the barriers have the same strength $\lambda_n=\lambda$. Then, by setting $V_n=V$ and

$$\cos\beta a = \mathrm{Re}[(1 - i\,V/2k)e^{ika}], \tag{12}$$

the result becomes simple and Eq. (11) can be expressed in the form as

$$D_N = e^{-ikNa}\left[\cos N\beta a - i\left(\frac{V}{2k}\cos ka - \sin ka\right)\frac{\sin N\beta a}{\sin\beta a}\right], \tag{13}$$

and Eq. (9) can be expressed in the form as

$$R_N = e^{ik(N+1)a} iV/2k(\sin N\beta a/\sin\beta a)\,. \tag{14}$$

It is easy to check that $|D_N|^2 - |R_N|^2 = 1$, by using Eq. (12), which is an alternative representation of $|t|^2 + |r|^2 = 1$. It should be mentioned that Eqs. (8) and (13) have a similar form to those in the previous calculation of the resistance of a one-dimensional chain with arbitary δ potentials.[3] But, the wave vector k is different, due to the ring geometric structure. Here, the wave function should satisfy a twisted periodic condition and the boundary conditions on the sites, where δ-function potentials are localized.

For a time-independent flux Φ threaded the ring, the current associated with the lth eigenstate can be derived[12] from Eq. (6) and with help of the definition of the persistent current $I_l = -c\,\partial E_l/\partial\Phi$, the result is

$$I_l = \frac{e}{\hbar}\,\frac{\sin\{2\pi(\Phi/\Phi_0) + [(-1)^M - 1]\pi/2\}}{(\partial \mathrm{Re}(1/t)/\partial E_l)} = \frac{e}{\hbar}\,\frac{\sin\{2\pi(\Phi/\Phi_0) + [(-1)^M - 1]\pi/2\}}{(\partial \mathrm{Re} D_N/\partial E_l)}. \tag{15}$$

An important condition for I_l to be nonzero is that the wave functions of the charged and spin carriers should stay coherent along the circumference Na of the ring.

As pointed out in Ref. 13, the energy quantization rule is determined by the scattering phases, which can be defined through the S matrix. The scattering matrix S is defined as

$$\begin{pmatrix} A' \\ B \end{pmatrix} = S\begin{pmatrix} A \\ B' \end{pmatrix} = D_N^{-1}\begin{pmatrix} \exp\{i2\pi\Phi/\Phi_0 + i[(-1)^M - 1]\pi/2\} & R_N \\ -R_N^* & \exp\{-i2\pi\Phi/\Phi_0 - i[(-1)^M - 1]\pi/2\} \end{pmatrix}\begin{pmatrix} A \\ B' \end{pmatrix}, \tag{16}$$

in derivation of Eq. (16), we have used Eq. (5) and $|D_N|^2 - |R_N|^2 = 1$. The quantization rule acquires the equation

$$\det(S-1) = [\exp(i\varphi_1) - 1][\exp(i\varphi_2) - 1] = 0, \tag{17}$$

where $\varphi_{1,2}$ are the scattering phases, which are determined from the eigenvalues of the S matrix $\exp(i\varphi_{1,2})$. It is evident that one such phase would vanish when the energy of the system is E_F, which gives the Fermi level. It is of interest to note that the eigenvalues are independent of whether the number of electrons is odd or even. If we parametrize the scattering matrix with θ, ϕ and the resistance ρ_N by conveniently defining

$$1/t^* = \sqrt{\rho_N + 1}\,e^{i\theta}, \quad r/t = \sqrt{\rho_N}\,e^{i\phi}, \tag{18}$$

then the eigenvalues for the S matrix in Eq. (16) can be found as

$$\varphi_{1,2} = \theta \pm \arccos[(1/\sqrt{\rho_N + 1})\cos 2\pi\Phi/\Phi_0], \tag{19}$$

where $\theta = -\arctan[(V/2k\cos ka - \sin ka)(\sin\beta a)^{-1}\tan N\beta a]$.

In order to find the equilibrium current for a disordered ring it is sufficient to take into account those energy eigenstates with the energies satisfied in Eq. (17). Also, the total current is given by the sum over all lowest-lying occupied states up to the Fermi level, so for a fixed number of electrons, the ensemble averaged total persistent current $\overline{I}$ is given by

$$\overline{I} = \frac{1}{Na}\left\langle \sum_l \sum_{i=1}^{2} \delta(\varphi_i)\frac{\partial\varphi_i}{\partial E_l}\,\frac{e}{\hbar} \times \frac{\sin\{2\pi(\Phi/\Phi_0) + [(-1)^M - 1]\pi/2\}}{(\partial \mathrm{Re} D_N/\partial E_l)}\right\rangle, \tag{20}$$

where $\langle\ \rangle$ means the ensemble average with the fixed electron number M. Following Ref. 13, the calculation is carried out to the first order in Φ/Φ_0, ($\ll 1$), in the limit of the weak magnetic field. We, therefore, have $\varphi_{1,2} = \theta \pm \arccos(\rho_N + 1)^{-1/2}$. The averaged current is then presented in the form

$$\overline{I} = \pm(M/L)(e/\hbar)(2\pi\Phi/\Phi_0)\langle 1/(\rho_N + 1)\rangle, \tag{21}$$

where the signs "+" and "−" correspond to M being even and odd, respectively. As an expectation, it is of interest to recover the conclusion[16] that, for even M, the current is paramagnetic (i.e., $\overline{I} > 0$ for small $\Phi > 0$), while, for odd M, it is diamagnetic (i.e., $\overline{I} < 0$ for small $\Phi > 0$). In the calculation of averaging over an ensemble of the energy E of the electron on the ring, we restrict ourselves to the situtation of $ka \neq n\pi$ and a weak magnetic field. For the ring with N identical and periodically arranged potentials V, the ρ_N can be expressed as $\rho_N = (V/2k)^2(\sin^2 N\beta a/\sin^2\beta a)$,[3] which does not increase monotonically with N and reduces to zero at $N \to \infty$ for $|\cos\beta a| < 1$. From Eq. (21), we, therefore, know that the ring with N identical and periodically arranged potentials does not have a localization length. The current does not descrease monotonically with the number of barriers on the ring. We are also interested in noting that the resistance that goes to zero corresponds to the situation of no δ scatters on the ring and, therefore, the persistent current approachs to the usual result for N free electrons on the ring.

However, when the δ scatterers are located periodically with random amplitude, the physical quantites have a dependence of localization length on disorderness. For the random δ scatterers arranged periodically on a ring, Gasparian *et al.*[3] had obtained an asymptotic expression of the resistance

$$\rho_N+1=\frac{1}{4\sin^2 ka}\left(\frac{\sin^2 ka}{k^2a^2}\right)^N V^{2N}=\frac{1}{4\sin^2 k_F a}e^{L/\xi}, \quad (22)$$

with $\xi(k)$ being the localization length and given by[3]

$$\xi^{-1}(k_F)=\ln\langle V\rangle^2 a^2\sin^2 k_F a/k_F^2 a^2+\ln V^2/\langle V\rangle^2. \quad (23)$$

By Eq. (22), the average current in Eq. (21) can be concluded in the form

$$\overline{I}=\frac{M}{L}\frac{e}{h}\frac{\Phi}{\Phi_0}\langle 4\sin^2 ka\, e^{-L/\xi}\rangle\simeq\frac{eM}{h}\left(\frac{L}{\xi}\right)\left(\frac{\Phi}{\Phi_0}\right)e^{-L/\xi}, \quad (24)$$

which means that the average current descreases exponentially with increasing sample circumference L. This exponential localization agrees with the results obtained previously in Refs. 12–14.

On the other hand, in the case of barriers with random strength, Gasparian *et al.*[3] obtained

$$\langle\rho\rangle=\sum_{p=1}^{N}\delta(E_F)^p 2^{(p-1)}\times\sum_{1=n_1<\cdots<n_p}^{N}\prod_{l=1}^{p-1}[1-\cos k_F a(n_{l+1}-n_l)],$$

by assuming a uniform distribution of the amplitudes of the δ function λ_n in an interval $[-W/2,W/2]$, where $\delta(E_F)$ is the inverse localization length given by $\delta(E_F)=W^{-1}\int_{-W/2}^{W/2}[V_n^2/4k^2]dV_n=W^2/48k_F^2$. In the weak scattering case, it is easily found

$$\left\langle\frac{1}{\rho_N+1}\right\rangle=1-N\delta(E_F)+\delta(E_F)^2\times\left[1-\frac{\sin(N+1)k_F a\sin(N-1)k_F a}{\sin^2 k_F a}\right]. \quad (25)$$

Therefore, Eq. (21) leads to the conclusion that the average persistent current in the present system has a lower degree of power dependence than the exponential decrease in the circumference of the ring. While for the weak disorder case, $[\delta(E_F)\ll 1]$, the decreasing exponential of persistent current on L is likely happening.

To summarize, a simple model for the electron moving in the disordered one-dimensional system is discussed, which is able to explain the main qualitative feature of the persistent current in a disordered mesoscopic system. Our consideration is based on the hypothesis of localization of the electron in the presence of short-range disorderness and on the local-conductivity formalism. We obtain the dependence of persistent current $\overline{I}$ on L/ξ. A power law dependence of the localization in the amplitude of persistent current would be confered for uniformly distributed strength of the barriers on the ring. Our results are in good agreement with the decreasing exponential with localization in the current literature, if the system is an off resonance transmission and is weakly disordered. However, a power law dependence of circumference and inverse localization length could not be excluded.

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